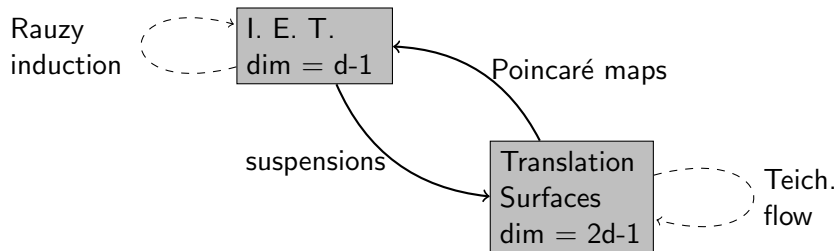


Interval exchange transformations
Part III: Linear recurrence and invariant measures

Vincent Delecroix

November 2015, Salta

Rauzy induction and Teichmüller flow



The Rauzy induction can be viewed as a first return map of the Teichmüller flow.

The set of T -invariant measures

Theorem

Let $T_{\pi,\lambda}$ be an i.e.t. without connection. Let $A_n(\pi, \lambda)$ the matrix products associated to the Rauzy induction (i.e. $\lambda = A_n(\pi, \lambda)\lambda^{(n)}$) and let $C_{\pi,\lambda} := \bigcap_{n \geq 0} A_n(\pi, \lambda)\mathbb{R}_+^{\mathcal{A}}$. The map

$$\begin{aligned} \mathcal{M}_T &\rightarrow \mathbb{R}_+^{\mathcal{A}} \\ \mu &\mapsto (\mu[\alpha])_{\alpha \in \mathcal{A}} \end{aligned}$$

induces an homeomorphism between \mathcal{M}_T and $\mathbb{P}C_{\pi,\lambda}$.

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rk: sharp bound (Katok, Veech, Fickenscher), symbolic systems (Boshernitzan, Monteil), finite topological rank (Bezuglyi-Kwiatkowski-Medynets-Solomyak).

Linear recurrence and Boshernitzan condition

Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a minimal shift with an invariant measure μ . We define

$$\varepsilon_n(X, \mu) = \min_{u \in \mathcal{L}_{X,n}} \mu([u]).$$

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Definition

We say that (X, T, μ)

- ▶ is *linearly recurrent* if $\liminf_{n \rightarrow \infty} n\varepsilon_n(X, \mu) > 0$,
- ▶ satisfies *Boshernitzan condition* if $\limsup_{n \rightarrow \infty} n\varepsilon_n(X, \mu) > 0$.

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(X, T, μ) is *linearly recurrent* iff $\exists C, \forall u, \forall v \in \mathcal{R}_u, |v| \leq C|u|$.

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Theorem

Let $T_{\pi, \lambda} : I \rightarrow I$ be a linearly recurrent iet. Then there exists C_1, C_2 so that for all $x, y \in I$ and $n \geq 1$

$$d(T^n x, x) \geq \frac{C_1}{n} \quad \text{and} \quad \min_{0 \leq k \leq n-1} d(y, T^k x) \leq \frac{C_2}{n}.$$

- ▶ LR \Rightarrow unique ergodicity

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- ▶ for rotations: T_α LR $\Leftrightarrow \alpha$ has bounded partial quotients

Boshernitzan criterion

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Let (X, T, μ) be a shift that satisfies Boshernitzan criterion. Then it is uniquely ergodic.

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Theorem

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rk: the converse is not true! But not far...

Correspondence

(symbolic)		(geometric)	
$n\varepsilon_n$	Rauzy induction	Teichmüller flow	best approximations
$\liminf > 0$ (LR)	finite time positiveness	bounded g_t -orbit	$\liminf q_n \{q_n \alpha\} > 0$
$\limsup > 0$ (Bosh. cond.)	?	non- divergent g_t -orbit	$\limsup q_{n+1} \{q_n \alpha\} > 0$

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Lagrange spectrum (set of $\liminf n\varepsilon_n$) and Dirichlet spectrum (set of $\limsup n\varepsilon_n$). . . (Markov, Hall, Moreira, Ferenczi, Hubert-Marchese-Ulcigrai, Boshernitzan-Delecroix)