

# 11

## Expanding maps

The distinctive feature of the transformations  $f : M \rightarrow M$  that we study in the last two chapters of this book is that they expand the distance between nearby points: there exists a constant  $\sigma > 1$  such that

$$d(f(x), f(y)) \geq \sigma d(x, y)$$

whenever the distance between  $x$  and  $y$  is small (a precise definition will be given shortly). There is more than one reason why this class of transformations has an important role in ergodic theory.

On the one hand, as we are going to see, expanding maps exhibit very rich dynamical behavior, from the metric and topological point of view as well as from the ergodic point of view. Thus, they provide a natural and interesting context for utilizing many of the ideas and methods that have been introduced so far.

On the other hand, expanding maps lead to paradigms that are useful for understanding many other systems, technically more complex. A good illustration of this is the ergodic theory of uniformly hyperbolic systems, for which an excellent presentation can be found in Bowen [Bow75a].

An important special case of expanding maps are the differentiable transformations on manifolds such that

$$\|Df(x)v\| \geq \sigma \|v\|$$

for every  $x \in M$  and every vector  $v$  tangent to  $M$  at the point  $x$ . We focus on this case in Section 11.1. The main result (Theorem 11.1.2) is that, under the hypothesis that the Jacobian  $\det Df$  is Hölder, the transformation  $f$  admits a unique invariant probability measure absolutely continuous with respect to the Lebesgue measure. Moreover, that probability measure is ergodic and positive on the open subsets of  $M$ .

In Section 11.2 we extend the notion of an expanding map to metric spaces and we give a global description of the topological dynamics of such maps, starting from the study of their periodic points. The main objective is to show that the global dynamics may always be reduced to the topologically exact

case (Theorem 11.2.15). In Section 11.3 we complement this analysis by showing that for these transformations the topological entropy coincides with the growth rate of the number of periodic points.

The study of expanding maps will proceed in Chapter 12, where we will develop the so-called *thermodynamic formalism* for such systems.

## 11.1 Expanding maps on manifolds

Let  $M$  be a compact manifold and  $f : M \rightarrow M$  be a map of class  $C^1$ . We say that  $f$  is *expanding* if there exists  $\sigma > 1$  and some Riemannian metric on  $M$  such that

$$\|Df(x)v\| \geq \sigma \|v\| \quad \text{for every } x \in M \text{ and every } v \in T_x M. \quad (11.1.1)$$

In particular,  $f$  is a local diffeomorphism: the condition (11.1.1) implies that  $Df(x)$  is an isomorphism for every  $x \in M$ . In what follows, we call *Lebesgue measure* on  $M$  the volume measure  $m$  induced by such a Riemannian metric. The precise choice of the metric is not very important, since the volume measures induced by different Riemannian metrics are all equivalent.

**Example 11.1.1.** Let  $f_A : \mathbb{T}^d \rightarrow \mathbb{T}^d$  be the linear endomorphism of the torus induced by some matrix  $A$  with integer coefficients and determinant different from zero. Assume that all the eigenvalues  $\lambda_1, \dots, \lambda_d$  of  $A$  are larger than 1 in absolute value. Then, given any  $1 < \sigma < \inf_i |\lambda_i|$ , there exists an inner product in  $\mathbb{R}^d$  relative to which  $\|Av\| \geq \sigma \|v\|$  for every  $v$ . Indeed, suppose that the eigenvalues are real. Consider any basis of  $\mathbb{R}^d$  that sets  $A$  in canonical Jordan form:  $A = D + \varepsilon N$  where  $N$  is nilpotent and  $D$  is diagonal with respect to that basis. The inner product relative to which such basis is orthonormal has the required property, as long as  $\varepsilon > 0$  is small enough. The reader should have no difficulty extending this argument to the case when there are complex eigenvalues. This shows that the transformation  $f_A$  is expanding.

It is clear from the definition that any map sufficiently close to an expanding one, relative to the  $C^1$  topology, is still expanding. Thus, the observation in Example 11.1.1 provides a whole open set of examples of expanding maps. A classical result of Michael Shub [Shu69] asserts a (much deeper) kind of converse: every expanding map on the torus  $\mathbb{T}^d$  is topologically conjugate to an expanding linear endomorphism  $f_A$ .

Given a probability measure  $\mu$  invariant under a transformation  $f : M \rightarrow M$ , we call the *basin* of  $\mu$  the set  $B(\mu)$  of all points  $x \in M$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu$$

for every continuous function  $\varphi : M \rightarrow \mathbb{R}$ . Note that the basin is always an invariant set. If  $\mu$  is ergodic then  $B(\mu)$  is a full measure set (Exercise 4.1.5).

**Theorem 11.1.2.** *Let  $f : M \rightarrow M$  be an expanding map on a compact (connected) manifold  $M$  and assume that the Jacobian  $x \mapsto \det Df(x)$  is Hölder. Then  $f$  admits a unique invariant probability measure  $\mu$  absolutely continuous with respect to Lebesgue measure  $m$ . Moreover,  $\mu$  is ergodic, its support coincides with  $M$  and its basin has full Lebesgue measure in  $M$ .*

First, let us outline the strategy of the proof of Theorem 11.1.2. The details will be given in the forthcoming sections. The conclusion is generally false if one omits the hypothesis of Hölder continuity: see Quas [Qua99].

It is easy to check (Exercise 11.1.1) that the pre-image under  $f$  of any set with zero Lebesgue measure  $m$  also has zero Lebesgue measure. This means that the image  $f_*\nu$  under  $f$  of any measure  $\nu$  absolutely continuous with respect to  $m$  is also absolutely continuous with respect to  $m$ . In particular, the  $n$ -th image  $f_*^n m$  is always absolutely continuous with respect to  $m$ .

In Proposition 11.1.7 we prove that the *density* (that is, the Radon–Nikodym derivative) of each  $f_*^n m$  with respect to  $m$  is bounded by some constant independent of  $n \geq 1$ . We deduce from this fact that every accumulation point of the sequence

$$\frac{1}{n} \sum_{j=0}^{n-1} f_*^j m,$$

with respect to the weak\* topology, is an invariant probability measure absolutely continuous with respect to Lebesgue measure, with density bounded by that same constant.

An additional argument, using the fact that  $M$  is connected, proves that the accumulation point is unique and has all the properties in the statement of the theorem.

### 11.1.1 Distortion lemma

Starting the proof of Theorem 11.1.2, let us prove the following elementary fact:

**Lemma 11.1.3.** *Let  $f : M \rightarrow M$  be a local diffeomorphism of class  $C^r$ ,  $r \geq 1$  on a compact Riemannian manifold  $M$  and  $\sigma > 0$  be such that  $\|Df(x)v\| \geq \sigma \|v\|$  for every  $x \in M$  and every  $v \in T_x M$ . Then there exists  $\rho > 0$  such that, for any pre-image  $x$  of any point  $y \in M$ , there exists a map  $h : B(y, \rho) \rightarrow M$  of class  $C^r$  such that  $f \circ h = \text{id}$ ,  $h(y) = x$  and*

$$d(h(y_1), h(y_2)) \leq \sigma^{-1} d(y_1, y_2) \quad \text{for every } y_1, y_2 \in B(y, \rho). \tag{11.1.2}$$

*Proof.* By the inverse function theorem, for every  $\zeta \in M$  there exist open neighborhoods  $U(\zeta) \subset M$  of  $\zeta$  and  $V(\zeta) \subset N$  of  $f(\zeta)$  such that  $f$  maps  $U(\zeta)$

diffeomorphically onto  $V(\xi)$ . Since  $M$  is compact, it follows that there exists  $\delta > 0$  such that  $d(\xi, \xi') \geq \delta$  whenever  $f(\xi) = f(\xi')$ . In particular, we may choose these neighborhoods in such a way that  $U(\xi) \cap U(\xi') = \emptyset$  whenever  $f(\xi) = f(\xi')$ . For each  $\eta \in M$ , let

$$W(\eta) = \bigcap_{\xi \in f^{-1}(\eta)} V(\xi).$$

Since  $f^{-1}(\eta)$  is finite (Exercise A.4.6), every  $W(\eta)$  is an open set. Fix  $\rho > 0$  such that  $2\rho$  is a Lebesgue number for the open cover  $\{W(\eta) : \eta \in M\}$  of  $M$ . In particular, for every  $y \in M$  there exists  $\eta \in M$  such that  $B(y, \rho)$  is contained in  $W(\eta)$ , that is, it is contained in  $V(\xi)$  for all  $\xi \in f^{-1}(\eta)$ . Since the  $U(\xi)$  are pairwise disjoint and  $\#f^{-1}(y) = \text{degree}(f) = \#f^{-1}(\eta)$ , given any  $x \in f^{-1}(y)$  there exists exactly one  $\xi \in f^{-1}(\eta)$  such that  $x \in U(\xi)$ . Let  $h$  be the restriction to  $B(y, \rho)$  of the inverse of  $f : U(\xi) \rightarrow V(\xi)$ . By construction,  $f \circ h = \text{id}$  and  $h(y) = x$ . Moreover,  $\|Dh(z)\| = \|Df(h(z))^{-1}\| \leq \sigma^{-1}$  for every  $z$  in the domain of  $h$ . By the mean value theorem, this implies that  $h$  has the property (11.1.2).

Transformations  $h$  as in this statement are called *inverse branches* of the local diffeomorphism  $f$ . Now assume that  $f$  is an expanding map. The condition (11.1.1) means that in this case we may take  $\sigma > 1$  in the hypothesis of the lemma. Then the conclusion (11.1.2) implies that the inverse branches are contractions, with uniform contraction rate.

In particular, we may define inverse branches  $h^n$  of any iterate  $f^n$ ,  $n \geq 1$ , as follows. Given  $y \in M$  and  $x \in f^{-n}(y)$ , let  $h_1, \dots, h_n$  be inverse branches of  $f$  with

$$h_j(f^{n-j+1}(x)) = f^{n-j}(x)$$

for every  $1 \leq j \leq n$ . Since every  $h_j$  is a contraction, its image is contained in a ball around  $f^{n-j}(x)$  with radius smaller than  $\rho$ . Then  $h^n = h_n \circ \dots \circ h_1$  is well defined on the closure of the ball of radius  $\rho$  around  $y$ . It is clear that  $f^n \circ h^n = \text{id}$  and  $h^n(y) = x$ . Moreover, each  $h^n$  is a contraction:

$$d(h^n(y_1), h^n(y_2)) \leq \sigma^{-n} d(y_1, y_2) \quad \text{for every } y_1, y_2 \in B(y, \rho).$$

**Lemma 11.1.4.** *If  $f : M \rightarrow M$  is a  $C^1$  expanding map on a compact manifold then  $f$  is expansive.*

*Proof.* By Lemma 11.1.3, there exists  $\rho > 0$  such that, for any pre-image  $x$  of a point  $y \in M$ , there exists a map  $h : B(y, \rho) \rightarrow M$  of class  $C^1$  such that  $f \circ h = \text{id}$ ,  $h(y) = x$  and

$$d(h(y_1), h(y_2)) \leq \sigma^{-1} d(y_1, y_2) \quad \text{for every } y_1, y_2 \in B(y, \rho).$$

Hence, if  $d(f^n(x), f^n(y)) \leq \rho$  for every  $n \geq 0$  then

$$d(x, y) \leq \sigma^{-n} d(f^n(x), f^n(y)) \leq \sigma^{-n} \rho,$$

which immediately implies that  $x = y$ .

The next result provides a good control of the distortion of the iterates of  $f$  and their inverse branches, which is crucial for the proof of Theorem 11.1.2. This is the only step of the proof where we use the hypothesis that the Jacobian  $x \mapsto \det Df(x)$  is Hölder. Note that, since  $f$  is a local diffeomorphism and  $M$  is compact, the Jacobian is bounded from zero and infinity. Hence, the logarithm  $\log |\det Df|$  is also Hölder: there exist  $C_0 > 0$  and  $\nu > 0$  such that

$$|\log |\det Df(x)| - \log |\det Df(y)|| \leq C_0 d(x,y)^\nu \quad \text{for any } x, y \in M.$$

**Proposition 11.1.5** (Distortion lemma). *There exists  $C_1 > 0$  such that, given any  $n \geq 1$ , any  $y \in M$  and any inverse branch  $h^n : B(y, \rho) \rightarrow M$  of  $f^n$ ,*

$$\log \frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} \leq C_1 d(y_1, y_2)^\nu \leq C_1 (2\rho)^\nu$$

for every  $y_1, y_2 \in B(y, \rho)$ .

*Proof.* Write  $h^n$  as a composition  $h^n = h_n \circ \dots \circ h_1$  of inverse branches of  $f$ . Analogously,  $h^i = h_i \circ \dots \circ h_1$  for  $1 \leq i < n$  and  $h^0 = \text{id}$ . Then,

$$\log \frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} = \sum_{i=1}^n \log |\det Dh_i(h^{i-1}(y_1))| - \log |\det Dh_i(h^{i-1}(y_2))|.$$

Note that  $\log |\det Dh_i| = -\log |\det Df| \circ h_i$  and recall that every  $h_j$  is a  $\sigma^{-1}$ -contraction. Hence,

$$\log \frac{|\det Dh^n(y_1)|}{|\det Dh^n(y_2)|} \leq \sum_{i=1}^n C_0 d(h^i(y_1), h^i(y_2))^\nu \leq \sum_{i=1}^n C_0 \sigma^{-i\nu} d(y_1, y_2)^\nu.$$

Therefore, to prove the lemma it suffices to take  $C_1 = C_0 \sum_{i=1}^\infty \sigma^{-i\nu}$ .

The geometric meaning of this proposition is made even more transparent by the following corollary:

**Corollary 11.1.6.** *There exists  $C_2 > 0$  such that, for every  $y \in M$  and any measurable sets  $B_1, B_2 \subset B(y, \rho)$ ,*

$$\frac{1}{C_2} \frac{m(B_1)}{m(B_2)} \leq \frac{m(h^n(B_1))}{m(h^n(B_2))} \leq C_2 \frac{m(B_1)}{m(B_2)}.$$

*Proof.* Take  $C_2 = \exp(2C_1(2\rho)^\nu)$ . It follows from the Proposition 11.1.5 that

$$m(h^n(B_1)) = \int_{B_1} |\det Dh^n| \, dm \leq \exp(C_1(2\rho)^\nu) |\det Dh^n(y)| m(B_1) \quad \text{and}$$

$$m(h^n(B_2)) = \int_{B_2} |\det Dh^n| \, dm \geq \exp(-C_1(2\rho)^\nu) |\det Dh^n(y)| m(B_2).$$

Dividing the two inequalities, we get that

$$\frac{m(h^n(B_1))}{m(h^n(B_2))} \leq C_2 \frac{m(B_1)}{m(B_2)}.$$

Inverting the roles of  $B_1$  and  $B_2$  we get the other inequality.

The next result, which is also a consequence of the distortion lemma, asserts that the iterates  $f_*^n m$  of the Lebesgue measure have uniformly bounded densities:

**Proposition 11.1.7.** *There exists  $C_3 > 0$  such that  $(f_*^n m)(B) \leq C_3 m(B)$  for every measurable set  $B \subset M$  and every  $n \geq 1$ .*

*Proof.* It is no restriction to suppose that  $B$  is contained in a ball  $B_0 = B(z, \rho)$  of radius  $\rho$  around some point in the pre-image of  $z \in M$ . Using Corollary 11.1.6, we see that

$$\frac{m(h^n(B))}{m(h^n(B_0))} \leq C_2 \frac{m(B)}{m(B_0)}$$

for every inverse branch  $h^n$  of  $f^n$  at the point  $z$ . Moreover,  $(f_*^n m)(B) = m(f^{-n}(B))$  is the sum of  $m(h^n(B))$  over all the inverse branches, and analogously for  $B_0$ . In this way, we find that

$$\frac{(f_*^n m)(B)}{(f_*^n m)(B_0)} \leq C_2 \frac{m(B)}{m(B_0)}.$$

It is clear that  $(f_*^n m)(B_0) \leq (f_*^n m)(M) = 1$ . Moreover, the Lebesgue measure of the balls with a fixed radius  $\rho$  is bounded from zero for some constant  $\alpha_0 > 0$  that depends only on  $\rho$ . So, to get the conclusion of the proposition it suffices to take  $C_3 = C_2 \alpha_0$ .

We also need the auxiliary result that follows. Recall that, given a function  $\varphi$  and a measure  $\nu$ , we denote by  $\varphi\nu$  the measure defined by  $(\varphi\nu)(B) = \int_B \varphi d\nu$ .

**Lemma 11.1.8.** *Let  $\nu$  be a probability measure on a compact metric space  $X$  and  $\varphi : X \rightarrow [0, +\infty)$  be an integrable function with respect to  $\nu$ . Let  $\mu_i, i \geq 1$ , be a sequence of probability measures on  $X$  converging, in the weak\* topology, to a probability measure  $\mu$ . If  $\mu_i \leq \varphi\nu$  for every  $i \geq 1$  then  $\mu \leq \varphi\nu$ .*

*Proof.* Let  $B$  be any measurable set. For each  $\varepsilon > 0$ , let  $K_\varepsilon$  be a compact subset of  $B$  such that  $\mu(B \setminus K_\varepsilon)$  and  $(\varphi\nu)(B \setminus K_\varepsilon)$  are both less than  $\varepsilon$  (such a compact set does exist, by Proposition A.3.2). Fix  $r > 0$  small enough that the measure of  $A_\varepsilon \setminus K_\varepsilon$  is also less than  $\varepsilon$ , for both  $\mu$  and  $\varphi\nu$ , where  $A_\varepsilon = \{z : d(z, K_\varepsilon) < r\}$ . The set of values of  $r$  for which the boundary of  $A_\varepsilon$  has positive  $\mu$ -measure is at most countable (Exercise A.3.2). Hence, up to changing  $r$  slightly if necessary, we may suppose that the boundary of  $A_\varepsilon$  has measure zero. Then  $\mu = \lim_i \mu_i$  implies that  $\mu(A_\varepsilon) = \lim_i \mu_i(A_\varepsilon) \leq (\varphi\nu)(A_\varepsilon)$ . Making  $\varepsilon \rightarrow 0$ , we conclude that  $\mu(B) \leq (\varphi\nu)(B)$ .

Applying this lemma to our present situation, we obtain

**Corollary 11.1.9.** *Every accumulation point  $\mu$  of the sequence  $n^{-1} \sum_{j=0}^{n-1} f_*^j m$  is an invariant probability measure for  $f$  absolutely continuous with respect to the Lebesgue measure.*

*Proof.* Take  $\varphi$  constant equal to  $C_3$  and let  $\nu = m$ . Choose a subsequence  $(n_i)_i$  such that  $\mu_i = n_i^{-1} \sum_{j=0}^{n_i-1} f_*^j m$  converges to some probability measure  $\mu$ . Proposition 11.1.7 ensures that  $\mu_i \leq \varphi \nu$ . Then, by Lemma 11.1.8, we also have  $\mu \leq \varphi \nu = C_3 m$ . This implies that  $\mu \ll m$  with density bounded by  $C_3$ .

### 11.1.2 Existence of ergodic measures

Next, we show that the measure  $\mu$  we have just constructed is the unique invariant probability measure absolutely continuous with respect to the Lebesgue measure and, moreover, it is ergodic for  $f$ .

Start by fixing a finite partition  $\mathcal{P}_0 = \{U_1, \dots, U_s\}$  of  $M$  into regions with non-empty interior and diameter less than  $\rho$ . Then, for each  $n \geq 1$ , define  $\mathcal{P}_n$  to be the partition of  $M$  into the images of the  $U_i$ ,  $1 \leq i \leq s$ , under the inverse branches of  $f^n$ . The diameter of each  $\mathcal{P}_n$ , that is, the supremum of the diameters of its elements, is less than  $\rho \sigma^{-n}$ .

**Lemma 11.1.10.** *Let  $\mathcal{P}_n$ ,  $n \geq 1$ , be a sequence of partitions in a compact metric space  $M$ , with diameters converging to zero. Let  $\nu$  be a probability measure on  $M$  and  $B$  be any measurable set with  $\nu(B) > 0$ . Then there exist  $V_n \in \mathcal{P}_n$ ,  $n \geq 1$ , such that*

$$\nu(V_n) > 0 \quad \text{and} \quad \frac{\nu(B \cap V_n)}{\nu(V_n)} \rightarrow 1 \quad \text{when } n \rightarrow \infty.$$

*Proof.* Given any  $0 < \varepsilon < \nu(B)$ , let  $K_\varepsilon \subset B$  be a compact set with  $\nu(B \setminus K_\varepsilon) < \varepsilon$ . Let  $K_{\varepsilon,n}$  be the union of all the elements of  $\mathcal{P}_n$  that intersect  $K_\varepsilon$ . Since the diameters of the partitions converge to zero,  $\nu(K_{\varepsilon,n} \setminus K_\varepsilon) < \varepsilon$  for every  $n$  sufficiently large. By contradiction, suppose that

$$\nu(K_\varepsilon \cap V_n) \leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(V_n)$$

for every  $V_n \in \mathcal{P}_n$  that intersects  $K_\varepsilon$ . It would follow that

$$\begin{aligned} \nu(K_\varepsilon) &\leq \sum_{V_n} \nu(K_\varepsilon \cap V_n) \leq \sum_{V_n} \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(V_n) = \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(K_{\varepsilon,n}) \\ &\leq \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} (\nu(K_\varepsilon) + \varepsilon) \leq \nu(B) - \varepsilon < \nu(K_\varepsilon). \end{aligned}$$

This contradiction shows that there must exist some  $V_n \in \mathcal{P}_n$  such that

$$\nu(V_n) \geq \nu(B \cap V_n) \geq \nu(K_\varepsilon \cap V_n) > \frac{\nu(B) - \varepsilon}{\nu(B) + \varepsilon} \nu(V_n)$$

and, consequently,  $\nu(V_n) > 0$ . Making  $\varepsilon \rightarrow 0$  we get the claim.

In the statements that follow, we say that a measurable set  $A \subset M$  is invariant under  $f : M \rightarrow M$  if  $f^{-1}(A) = A$  up to zero Lebesgue measure. According

to Exercise 11.1.1, then we also have that  $f(A) = A$  up to zero Lebesgue measure.

**Lemma 11.1.11.** *If  $A \subset M$  satisfies  $f(A) \subset A$  and has positive Lebesgue measure then  $A$  has full Lebesgue measure inside some  $U_i \in \mathcal{P}_0$ , that is, there exists  $1 \leq i \leq s$  such that  $m(U_i \setminus A) = 0$ .*

*Proof.* By Lemma 11.1.10, we may choose  $V_n \in \mathcal{P}_n$  so that  $m(V_n \setminus A)/m(V_n)$  converges to zero when  $n \rightarrow \infty$ . Let  $U_{i(n)} = f^n(V_n)$ . By Proposition 11.1.5 applied to the inverse branch of  $f^n$  that maps  $U_{i(n)}$  to  $V_n$ , we get that

$$\frac{m(U_{i(n)} \setminus A)}{m(U_{i(n)})} \leq \frac{m(f^n(V_n \setminus A))}{m(f^n(V_n))} \leq \exp(C_1(2\rho)^v) \frac{m(V_n \setminus A)}{m(V_n)}$$

also converges to zero. Since  $\mathcal{P}_0$  is finite, there must be  $1 \leq i \leq s$  such that  $i(n) = i$  for infinitely many values of  $n$ . Then  $m(U_i \setminus A) = 0$ .

**Corollary 11.1.12.** *The transformation  $f : M \rightarrow M$  admits some ergodic invariant probability measure absolutely continuous with respect to the Lebesgue measure.*

*Proof.* It follows from the previous lemma there exist at most  $s = \#\mathcal{P}_0$  pairwise disjoint invariant sets with positive Lebesgue measure. Therefore,  $M$  may be partitioned into a finite number of minimal invariant sets  $A_1, \dots, A_r$ ,  $r \leq s$  with positive Lebesgue measure, where by *minimal* we mean that there are no invariant sets  $B_i \subset A_i$  with  $0 < m(B_i) < m(A_i)$ . Given any absolutely continuous invariant probability measure  $\mu$ , there exists some  $i$  such that  $\mu(A_i) > 0$ . The normalized restriction

$$\mu_i(B) = \frac{\mu(B \cap A_i)}{\mu(A_i)}$$

of  $\mu$  to any such  $A_i$  is invariant and absolutely continuous. Moreover, the assumption that  $A_i$  is minimal implies that  $\mu_i$  is ergodic.

### 11.1.3 Uniqueness and conclusion of the proof

The previous argument also shows that there exist only a finite number of absolutely continuous ergodic probability measures. The last step is to show that, in fact, such a probability measure is unique. For that we use the fact that  $f$  is *topologically exact*:

**Lemma 11.1.13.** *Given any non-empty open set  $U \subset M$ , there exists  $N \geq 1$  such that  $f^N(U) = M$ .*

*Proof.* Let  $x \in U$  and  $r > 0$  be such that the ball of radius  $r$  around  $x$  is contained in  $U$ . Given any  $n \geq 1$ , suppose that  $f^n(U)$  does not cover the whole manifold. Then there exists some curve  $\gamma$  connecting  $f^n(x)$  to a point  $y \in M \setminus f^n(U)$ , and that curve may be taken with length smaller than  $\text{diam} M + 1$ .



Lifting<sup>1</sup>  $\gamma$  by the local diffeomorphism  $f^n$ , we obtain a curve  $\gamma_n$  connecting  $x$  to some point  $y_n \in M \setminus U$ . Then  $r \leq \circ(\gamma_n) \leq \sigma^{-n}(\text{diam}M + 1)$ . This provides an upper bound on the possible value of  $n$ . Hence,  $f^n(U) = M$  for every  $n$  sufficiently large, as claimed.

**Corollary 11.1.14.** *If  $A \subset M$  has positive Lebesgue measure and satisfies  $f(A) \subset A$  then  $A$  has full Lebesgue measure in the whole manifold  $M$ .*

*Proof.* Let  $U$  be the interior of a set  $U_i$  as in Lemma 11.1.11 and  $N \geq 1$  be such that  $f^N(U) = M$ . Then  $m(U \setminus A) = 0$  and, using the fact that  $f$  is a local diffeomorphism, it follows that  $M \setminus A = f^N(U) \setminus f^N(A) \subset f^N(U \setminus A)$  also has Lebesgue measure zero.

The next statement completes the proof of Theorem 11.1.2:

**Corollary 11.1.15.** *Let  $\mu$  be any absolutely continuous invariant probability measure. Then  $\mu$  is ergodic and its basin  $B(\mu)$  has full Lebesgue measure in  $M$ . Consequently,  $\mu$  is unique. Moreover, its support is the whole manifold  $M$ .*

*Proof.* If  $A$  is an invariant set then, by Corollary 11.1.14, either  $A$  or its complement  $A^c$  has Lebesgue measure zero. Since  $\mu$  is absolutely continuous, it follows that either  $\mu(A) = 0$  or  $\mu(A^c) = 0$ . This proves that  $\mu$  is ergodic. Then  $\mu(B(\mu)) = 1$  and, in particular,  $m(B(\mu)) > 0$ . Since  $B(\mu)$  is an invariant set, it follows that it has full Lebesgue measure. Analogously, since the support of  $\mu$  is a compact set with positive Lebesgue measure and  $f(\text{supp } \mu) \subset \text{supp } \mu$ , it must coincide with  $M$ .

Finally, let  $\mu$  and  $\nu$  be any two absolutely continuous invariant probability measures. It follows from what we have just said that the two measures are ergodic and their basins intersect each other. Given any point  $x$  in  $B(\mu) \cap B(\nu)$ , the sequence

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

converges to both  $\mu$  and  $\nu$  in the weak\* topology. Thus,  $\mu = \nu$ .

In general, we say that an invariant probability measure  $\mu$  of a local diffeomorphism  $f : M \rightarrow M$  is a *physical measure* if its basin has positive Lebesgue measure. It follows from Corollary 11.1.15 that in the present context there exists a unique physical measure, which is the absolutely continuous invariant measure  $\mu$ , and its basin has full Lebesgue measure. This last fact may be expressed as follows:

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \rightarrow \mu \quad \text{for Lebesgue almost every } x.$$

<sup>1</sup> Note that any local diffeomorphism from a compact manifold to itself is a covering map.

In Chapter 12 we will find this absolutely continuous invariant probability measure  $\mu$  through a different approach (Proposition 12.1.20) that also shows that the density  $h = d\mu/dm$  is Hölder and bounded away from zero. In particular,  $\mu$  is equivalent to the Lebesgue measure  $m$ , not just absolutely continuous. Moreover (Section 12.1.7), the system  $(f, \mu)$  is exact, not just ergodic. In addition (Lemma 12.1.12), its Jacobian is given by  $J_{\mu}f = |\det Df|(h \circ f)/h$ . Hence, by the Rokhlin formula (Theorem 9.7.3),

$$h_{\mu}(f) = \int \log J_{\mu}f \, d\mu = \int \log |\det Df| \, d\mu + \int \log(h \circ f) \, d\mu - \int \log h \, d\mu.$$

Since  $\mu$  is invariant, this means that

$$h_{\mu}(f) = \int \log |\det Df| \, d\mu.$$

Actually, the facts stated in the previous paragraph can already be proven with the methods available at this point. We invite the reader to do just that (Exercises 11.1.3 through 11.1.6), in the context of expanding maps of the interval, which are technically a bit simpler than expanding maps on a general manifold.

**Example 11.1.16.** We say that a transformation  $f : [0, 1] \rightarrow [0, 1]$  is an *expanding map of the interval* if there exists a countable (possibly finite) family  $\mathcal{P}$  of pairwise disjoint open subintervals whose union has full Lebesgue measure in  $[0, 1]$  and which satisfy:

- (i) The restriction of  $f$  to each  $P \in \mathcal{P}$  is a diffeomorphism onto  $(0, 1)$ ; denote by  $f_P^{-1} : (0, 1) \rightarrow P$  its inverse.
- (ii) There exist  $C > 0$  and  $\theta > 0$  such that, for every  $x, y$  and every  $P \in \mathcal{P}$ ,

$$|\log |D(f_P^{-1})(x)| - \log |D(f_P^{-1})(y)|| \leq C|x - y|^{\theta}.$$

- (iii) There exist  $c > 0$  and  $\sigma > 1$  such that, for every  $n$  and every  $x$ ,

$$|Df^n(x)| \geq c\sigma^n \quad (\text{whenever the derivative is defined.})$$

This class of transformations includes the decimal expansion and the Gauss map as special cases. Its properties are analyzed in Exercises 11.1.3 through 11.1.5.

Exercise 11.1.6 deals with a slightly more general class of transformations, where we replace condition (i) by

- (i') There exists  $\delta > 0$  such that the restriction of  $f$  to each  $P \in \mathcal{P}$  is a diffeomorphism onto some interval  $f(P)$  of length larger than  $\delta$  that contains every element of  $\mathcal{P}$  that it intersects.

### 11.1.4 Exercises

- 11.1.1. Let  $f : M \rightarrow M$  be a local diffeomorphism in a compact manifold and  $m$  be the Lebesgue measure on  $M$ . Check the following facts:

- (a) If  $m(B) = 0$  then  $m(f^{-1}(B)) = 0$ .
- (b) If  $B$  is measurable then  $f(B)$  is measurable.
- (c) If  $m(B) = 0$  then  $m(f(B)) = 0$ .
- (d) If  $A = B$  up to zero Lebesgue measure zero then  $f(A) = f(B)$  and  $f^{-1}(A) = f^{-1}(B)$  up to zero Lebesgue measure.
- (e) If  $A$  is an invariant set then  $f(A) = A$  up to zero Lebesgue measure.
- 11.1.2. Let  $f : M \rightarrow M$  be a transformation of class  $C^1$  such that there exist  $\sigma > 1$  and  $k \geq 1$  satisfying  $\|Df^k(x)v\| \geq \sigma \|v\|$  for every  $x \in M$  and every  $v \in T_x M$ . Show that there exists  $\theta > 1$  and a Riemannian norm  $\langle \cdot \rangle$  equivalent to  $\|\cdot\|$  such that  $\langle Df(x)v \rangle \geq \theta \langle v \rangle$  for every  $x \in M$  and every  $v \in T_x M$ .
- 11.1.3. Show that if  $f : [0, 1] \rightarrow [0, 1]$  is an expanding map of the interval and  $m$  is the Lebesgue measure on  $[0, 1]$  then there exists a function  $\rho : (0, 1) \rightarrow (0, \infty)$  such that  $\log \rho$  is bounded and Hölder and  $\mu = \rho m$  is a probability measure invariant under  $f$ .
- 11.1.4. Show that the measure  $\mu$  in Exercise 11.1.3 is exact and is the unique invariant probability measure of  $f$  absolutely continuous with respect to the Lebesgue measure  $m$ .
- 11.1.5. Show that the measure  $\mu$  in Exercise 11.1.3 satisfies the Rokhlin formula: assuming that  $\log |f'| \in L^1(\mu)$ , we have that  $h_\mu(f) = \int \log |f'| d\mu$ .
- 11.1.6. Prove the following generalization of Exercises 11.1.3 and 11.1.4: if  $f$  satisfies the conditions (i'), (ii) and (iii) in Example 11.1.16 then there exists a finite (non-empty) family of absolutely continuous invariant probability measures ergodic for  $f$  and such that every absolutely continuous invariant probability measure is a convex combination of those ergodic probability measures.

## 11.2 Dynamics of expanding maps

In this section we extend the notion of an expanding map to compact metric spaces and we mention a few interesting examples. In this general setup, an expanding map need not be transitive, let alone topologically exact (compare Lemma 11.1.13). However, Theorems 11.2.14 and 11.2.15 assert that the dynamics may always be reduced to the topologically exact case. This is relevant because for the main results in this section we need the transformation to be topologically exact or, equivalently (Exercise 11.2.2), topologically mixing.

A continuous transformation  $f : M \rightarrow M$  in a compact metric space  $M$  is an *expanding map* if there exist constants  $\sigma > 1$  and  $\rho > 0$  such that for every  $p \in M$  the image of the ball  $B(p, \rho)$  contains a neighborhood of the closure of  $B(f(p), \rho)$  and

$$d(f(x), f(y)) \geq \sigma d(x, y) \quad \text{for every } x, y \in B(p, \rho). \quad (11.2.1)$$

Every expanding map on a manifold, in the sense of Section 11.1, is also expanding in the present sense:

**Example 11.2.1.** Let  $M$  be a compact Riemannian manifold and  $f : M \rightarrow M$  be a map of class  $C^1$  such that  $\|Df(x)v\| \geq \sigma \|v\|$  for every  $x \in M$  and every  $v \in T_x M$ , where  $\sigma$  is a constant larger than 1. Denote  $K = \sup \|Df\|$  (observe that  $K > 1$ ). Fix  $\rho > 0$  small enough that the restriction of  $f$  to every ball  $B(p, 2K\rho)$  is a diffeomorphism onto its image. Consider any  $y \in B(f(p), \sigma\rho)$  and let  $\gamma : [0, 1] \rightarrow B(f(p), \sigma\rho)$  be a minimizing geodesic (that is, such that it realizes the distance between points) with  $\gamma(0) = f(p)$  and  $\gamma(1) = y$ . By the choice of  $\rho$ , there exists a differentiable curve  $\beta : [0, \delta] \rightarrow B(p, \rho)$  such that  $\beta(0) = p$  and  $f(\beta(t)) = \gamma(t)$  for every  $t$ . Observe that (using  $\ell(\cdot)$  to denote the length of a curve),

$$d(p, \beta(t)) \leq \ell(\beta | [0, t]) \leq \sigma^{-1} \ell(\gamma | [0, t]) = \sigma^{-1} td(f(p), y) < t\rho$$

for every  $t$ . This shows that we may take  $\delta = 1$ . Then,  $\beta(1) \in B(p, \rho)$  and  $f(\beta(1)) = \gamma(1) = y$ . In this way, we have shown that  $f(B(p, \rho))$  contains  $B(f(p), \sigma\rho)$ , which is a neighborhood of the closure of  $B(f(p), \rho)$ . Now consider any  $x, y \in B(p, \rho)$ . Note that  $d(f(x), f(y)) < 2K\rho$ . Let  $\gamma : [0, 1] \rightarrow B(f(x), 2K\rho)$  be a minimizing geodesic connecting  $f(x)$  to  $f(y)$ . Arguing as in the previous paragraph, we find a differentiable curve  $\beta : [0, 1] \rightarrow B(x, 2K\rho)$  connecting  $x$  to  $y$  and such that  $f(\beta(t)) = \gamma(t)$  for every  $t$ . Then,

$$d(f(x), f(y)) = \ell(\gamma) \geq \sigma \ell(\beta) \geq \sigma d(x, y).$$

This completes the proof that  $f$  is an expanding map.

The following fact is useful for constructing further examples:

**Lemma 11.2.2.** *Assume that  $f : M \rightarrow M$  is an expanding map and  $\Lambda \subset M$  is a compact set such that  $f^{-1}(\Lambda) = \Lambda$ . Then the restriction  $f : \Lambda \rightarrow \Lambda$  is also an expanding map.*

*Proof.* It is clear that the condition (11.2.1) remains valid for the restriction. We are left to check that  $f(\Lambda \cap B(p, \rho))$  contains a neighborhood of  $\Lambda \cap \overline{B(f(p), \rho)}$  inside  $\Lambda$ . By assumption,  $f(B(p, \rho))$  contains some neighborhood  $V$  of the closure of  $B(f(p), \rho)$ . Then  $\Lambda \cap V$  is a neighborhood of  $\Lambda \cap \overline{B(f(p), \rho)}$ . Moreover, given any  $y \in \Lambda \cap V$  there exists  $x \in B(p, \rho)$  such that  $f(x) = y$ . Since  $f^{-1}(\Lambda) = \Lambda$ , this point is necessarily in  $\Lambda$ . This proves that  $\Lambda \cap V$  is contained in the image  $f(\Lambda \cap B(p, \rho))$ . Hence, the restriction of  $f$  to the set  $\Lambda$  is an expanding map, as stated.

It is not possible to replace the hypothesis of Lemma 11.2.2 by  $f(\Lambda) = \Lambda$ . See Exercise 11.2.4.

**Example 11.2.3.** Let  $J \subset [0, 1]$  be a finite union of (two or more) pairwise disjoint compact intervals. Consider a map  $f : J \rightarrow [0, 1]$  such that the

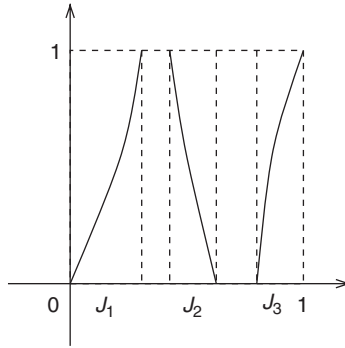


Figure 11.1. Expanding map on a Cantor set

restriction of  $f$  to each connected component of  $J$  is a diffeomorphism onto  $[0, 1]$ . See Figure 11.1. Assume that there exists  $\sigma > 1$  such that

$$|f'(x)| \geq \sigma \quad \text{for every } x \in J. \tag{11.2.2}$$

Denote  $\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(J)$ . In other words,  $\Lambda$  is the set of all points  $x$  whose iterates  $f^n(x)$  are defined for every  $n \geq 0$ . It follows from the definition that  $\Lambda$  is compact (one can show that  $K$  is a Cantor set) and  $f^{-1}(\Lambda) = \Lambda$ . The restriction  $f : \Lambda \rightarrow \Lambda$  is an expanding map. Indeed, fix any  $\rho > 0$  smaller than the distance between any two connected components of  $J$ . Then every ball of radius  $\rho$  inside  $\Lambda$  is contained in a unique connected component of  $J$  and so, by (11.2.2), it is dilated by a factor greater than or equal to  $\sigma$ .

**Example 11.2.4.** Let  $\sigma : \Sigma_A \rightarrow \Sigma_A$  be the one-sided shift of finite type associated with a transition matrix  $A$  (these notions were introduced in Section 10.2.2). Consider in  $\Sigma_A$  the distance defined by

$$d((x_n)_n, (y_n)_n) = 2^{-N}, \quad N = \inf\{n \in \mathbb{N} : x_n \neq y_n\}. \tag{11.2.3}$$

Then  $\sigma_A$  is an expanding map. Indeed, fix  $\rho \in (1/2, 1)$  and  $\sigma = 2$ . The open ball of radius  $\rho$  around any point  $(p_n)_n \in \Sigma_A$  is just the cylinder  $[0; p_0]_A$  that contains the point. The definition (11.2.3) yields

$$d((x_{n+1})_n, (y_{n+1})_n) = 2d((x_n)_n, (y_n)_n)$$

for any  $(x_n)_n$  and  $(y_n)_n$  in the cylinder  $[0; p_0]_A$ . Moreover,  $\sigma_A([0; p_0]_A)$  is the union of all cylinders  $[0; q]$  such that  $A_{p_0, q} = 1$ . In particular, it contains the cylinder  $[0; p_1]_A$ . Since the cylinders are both open and closed in  $\Sigma_A$ , this shows that the image of the ball of radius  $\rho$  around  $(p_n)_n$  contains a neighborhood of the closure of the ball of radius  $\rho$  around  $(p_{n+1})_n$ . This completes the proof that every shift of finite type is an expanding map.

**Example 11.2.5.** Let  $f : S^1 \rightarrow S^1$  be a local diffeomorphism of class  $C^2$  with degree larger than 1. Assume that all the periodic points of  $f$  are hyperbolic,

that is,  $|(f^n)'(x)| \neq 1$  for every  $x \in \text{Fix}(f^n)$  and every  $n \geq 1$ . Let  $\Lambda$  be the complement of the union of the basins of attraction of all the attracting periodic points of  $f$ . Then the restriction  $f : \Lambda \rightarrow \Lambda$  is an expanding map: this is a consequence of a deep theorem of Ricardo Mañé [Mañ85].

For expanding maps  $f : M \rightarrow M$  in a metric space  $M$ , the number of pre-images of a point  $y \in M$  may vary with  $y$  (unless  $M$  is connected; see Exercises 11.2.1 and A.4.6). For example, for a shift of finite type  $\sigma : \Sigma_A \rightarrow \Sigma_A$  (Example 11.2.4) the number of pre-images of a point  $y = (y_n)_n \in \Sigma_A$  is equal to the number of symbols  $i$  such that  $A_{i,y_0} = 1$ ; in general, this number depends on  $y_0$ .

On the other hand, it is easy to see that the number of pre-images is always finite, and even bounded: it suffices to consider a finite cover of  $M$  by balls of radius  $\rho$  and to notice that every point has at most one pre-image in each of those balls. By a slight abuse of language, we call the *degree* of an expanding map  $f : M \rightarrow M$  the maximum number of pre-images of any point, that is,

$$\text{degree}(f) = \max\{\#f^{-1}(y) : y \in M\}. \tag{11.2.4}$$

### 11.2.1 Contracting inverse branches

Let  $f : M \rightarrow M$  be an expanding map. By definition, the restriction of  $f$  to each ball  $B(p, \rho)$  of radius  $\rho$  is injective and its image contains the closure of  $B(f(p), \rho)$ . Thus, the restriction to  $B(p, \rho) \cap f^{-1}(B(f(p), \rho))$  is a homeomorphism onto  $B(f(p), \rho)$ . We denote by

$$h_p : B(f(p), \rho) \rightarrow B(p, \rho)$$

its inverse and call it the *inverse branch of  $f$*  at  $p$ . It is clear that  $h_p(f(p)) = p$  and  $f \circ h_p = \text{id}$ . The condition (11.2.1) implies that  $h_p$  is a  $\sigma^{-1}$ -contraction:

$$d(h_p(z), h_p(w)) \leq \sigma^{-1}d(z, w) \quad \text{for every } z, w \in B(f(p), \rho). \tag{11.2.5}$$

**Lemma 11.2.6.** *If  $f : M \rightarrow M$  is expanding then, for every  $y \in M$ ,*

$$f^{-1}(B(y, \rho)) = \bigcup_{x \in f^{-1}(y)} h_x(B(y, \rho)).$$

*Proof.* The relation  $f \circ h_x = \text{id}$  implies that  $h_x(B(y, \rho))$  is contained in the pre-image of  $B(y, \rho)$  for every  $x \in f^{-1}(y)$ . To prove the other inclusion, let  $z$  be any point such that  $f(z) \in B(y, \rho)$ . By the definition of an expanding map,  $f(B(z, \rho))$  contains  $B(f(z), \rho)$  and, hence, contains  $y$ . Let  $h_z : B(f(z), \rho) \rightarrow M$  be the inverse branch of  $f$  at  $z$  and let  $x = h_z(y)$ . Both  $z$  and  $h_x(f(z))$  are in  $B(x, \rho)$ . Since  $f$  is injective on every ball of radius  $\rho$  and  $f(h_x(f(z))) = f(z)$ , it follows that  $z = h_x(f(z))$ . This completes the proof.

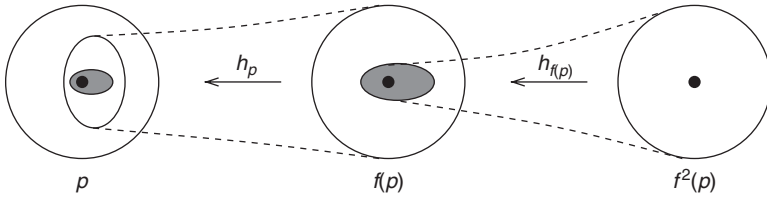


Figure 11.2. Inverse branches of  $f^n$

More generally, for any  $n \geq 1$ , we call the *inverse branch of  $f^n$*  at  $p$  the composition

$$h_p^n = h_p \circ \dots \circ h_{f^{n-1}(p)} : B(f^n(p), \rho) \rightarrow B(p, \rho)$$

of the inverse branches of  $f$  at the iterates of  $p$ . See Figure 11.2. Observe that  $h_p^n(f^n(p)) = p$  and  $f^n \circ h_p^n = \text{id}$ . Moreover,  $f^j \circ h_p^n = h_{f^j(p)}^{n-j}$  for each  $0 \leq j < n$ . Hence,

$$d(f^j \circ h_p^n(z), f^j \circ h_p^n(w)) \leq \sigma^{j-n} d(z, w) \tag{11.2.6}$$

for every  $z, w \in B(f^n(p), \rho)$  and every  $0 \leq j \leq n$ .

**Lemma 11.2.7.** *If  $f : M \rightarrow M$  is an expanding map then  $f^n(B(p, n + 1, \varepsilon)) = B(f^n(p), \varepsilon)$  for every  $p \in M$ ,  $n \geq 0$  and  $\varepsilon \in (0, \rho]$ .*

*Proof.* The inclusion  $f^n(B(p, n + 1, \varepsilon)) \subset B(f^n(p), \varepsilon)$  is an immediate consequence of the definition of a dynamical ball. To prove the converse, consider the inverse branch  $h_p^n : B(f^n(p), \rho) \rightarrow B(p, \rho)$ . Given any  $y \in B(f^n(p), \varepsilon)$ , let  $x = h_p^n(y)$ . Then  $f^n(x) = y$  and, by (11.2.6),

$$d(f^j(x), f^j(p)) \leq \sigma^{j-n} d(f^n(x), f^n(p)) \leq d(y, f^n(p)) < \varepsilon$$

for every  $0 \leq j \leq n$ . This shows that  $x \in B(p, n + 1, \varepsilon)$ .

**Corollary 11.2.8.** *Every expanding map is expansive.*

*Proof.* Assume that  $d(f^n(z), f^n(w)) < \rho$  for every  $n \geq 0$ . This implies that  $z = h_w^n(f^n(z))$  for every  $n \geq 0$ . Then, the property (11.2.6) gives that

$$d(z, w) \leq \sigma^{-n} d(f^n(z), f^n(w)) < \rho \sigma^{-n}.$$

Making  $n \rightarrow \infty$ , we get that  $z = w$ . So,  $\rho$  is a constant of expansivity for  $f$ .

### 11.2.2 Shadowing and periodic points

Given  $\delta > 0$ , we call a  $\delta$ -pseudo-orbit of a transformation  $f : M \rightarrow M$  any sequence  $(x_n)_{n \geq 0}$  such that

$$d(f(x_n), x_{n+1}) < \delta \quad \text{for every } n \geq 0.$$

We say that the  $\delta$ -pseudo-orbit is *periodic* if there is  $\kappa \geq 1$  such that  $x_n = x_{n+\kappa}$  for every  $n \geq 0$ . It is clear that every orbit is a  $\delta$ -pseudo-orbit, for every

$\delta > 0$ . For expanding maps we have a kind of converse: every pseudo-orbit is uniformly close to (we say that it is *shadowed* by) some orbit of the transformation:

**Proposition 11.2.9** (Shadowing lemma). *Assume that  $f : M \rightarrow M$  is an expanding map. Then, given any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every  $\delta$ -pseudo-orbit  $(x_n)_n$  there exists  $x \in M$  such that  $d(f^n(x), x_n) < \varepsilon$  for every  $n \geq 0$ .*

*If  $\varepsilon$  is small enough, so that  $2\varepsilon$  is a constant of expansivity for  $f$ , then the point  $x$  is unique. If, in addition, the pseudo-orbit is periodic then  $x$  is a periodic point.*

*Proof.* It is no restriction to suppose that  $\varepsilon$  is less than  $\rho$ . Fix  $\delta > 0$  so that  $\sigma^{-1}\varepsilon + \delta < \varepsilon$ . For each  $n \geq 0$ , let  $h_n : B(f(x_n), \rho) \rightarrow B(x_n, \rho)$  be the inverse branch of  $f$  at  $x_n$ . The property (11.2.5) ensures that

$$h_n(B(f(x_n), \varepsilon)) \subset B(x_n, \sigma^{-1}\varepsilon) \quad \text{for every } n \geq 1. \tag{11.2.7}$$

Since  $d(x_n, f(x_{n-1})) < \delta$ , it follows that

$$h_n(B(f(x_n), \varepsilon)) \subset B(f(x_{n-1}), \varepsilon) \quad \text{for every } n \geq 1. \tag{11.2.8}$$

Then, we may consider the composition  $h^{n+1} = h_0 \circ \dots \circ h_n$ , and (11.2.8) implies that the sequence of compact sets  $K_{n+1} = h^{n+1}(B(f(x_n), \varepsilon))$  is nested. Take  $x$  in the intersection. For every  $n \geq 0$ , we have that  $x \in K_{n+1}$  and so  $f^n(x)$  belongs to

$$f^n \circ h^{n+1}(B(f(x_n), \varepsilon)) = h_n(B(f(x_n), \varepsilon)).$$

By (11.2.7), this implies that  $d(f^n(x), x_n) \leq \sigma^{-1}\varepsilon < \varepsilon$  for every  $n \geq 0$ .

The other claims in the proposition are simple consequences, as we now explain. If  $x'$  is another point as in the conclusion of the proposition then

$$d(f^n(x), f^n(x')) \leq d(f^n(x), x_n) + d(f^n(x'), x_n) < 2\varepsilon \quad \text{for every } n \geq 0.$$

Since  $2\varepsilon$  is an expansivity constant, it follows that  $x = x'$ . Moreover, if the pseudo-orbit is periodic, with period  $\kappa \geq 1$ , then

$$d(f^n(f^\kappa(x)), x_n) = d(f^{n+\kappa}(x), x_{n+\kappa}) < \varepsilon \quad \text{for every } n \geq 0.$$

By uniqueness, it follows  $f^\kappa(x) = x$ .

It is worthwhile pointing out that  $\delta$  depends linearly on  $\varepsilon$ : the proof of Proposition 11.2.9 shows that we may take  $\delta = c\varepsilon$ , where  $c > 0$  depends only on  $\sigma$ .

We call *pre-orbit* of a point  $x \in M$  any sequence  $(x_{-n})_{n \geq 0}$  such that  $x_0 = x$  and  $f(x_{-n}) = x_{-n+1}$  for every  $n \geq 1$ . If  $x$  is a periodic point, with period  $l \geq 1$ , then it admits a distinguished *periodic pre-orbit*  $(\bar{x}_{-n})_n$ , such that  $\bar{x}_{-kl} = x$  for every integer  $k \geq 1$ .



**Lemma 11.2.10.** *If  $d(x,y) < \rho$  then, given any pre-orbit  $(x_{-n})_n$  of  $x$ , there exists a pre-orbit  $(y_{-n})_n$  of  $y$  asymptotic to  $(x_{-n})_n$ , in the sense that  $d(x_{-n}, y_{-n})$  converges to 0 when  $n \rightarrow \infty$ .*

*Proof.* For each  $n \geq 1$ , let  $h_n : B(x, \rho) \rightarrow M$  be the inverse branch of  $f^n$  with  $h_n(x) = x_{-n}$ . Define  $y_{-n} = h_n(y)$ . It is clear that  $d(x_{-n}, y_{-n}) \leq \sigma^{-n}d(x, y)$ . This implies the claim.

**Theorem 11.2.11.** *Let  $f : M \rightarrow M$  be an expanding map in a compact metric space and  $\Lambda \subset M$  be the closure of the set of all periodic points of  $f$ . Then  $f(\Lambda) = \Lambda$  and the restriction  $f : \Lambda \rightarrow \Lambda$  is an expanding map.*

*Proof.* On the one hand, it is clear that  $f(\Lambda)$  is contained in  $\Lambda$ : if a point  $x$  is accumulated by periodic points  $p_n$  then  $f(x)$  is accumulated by the images  $f(p_n)$ , which are also periodic points. On the other hand, since  $f(\Lambda)$  is a compact set that contains all the periodic points, it must contain  $\Lambda$ . This shows that  $f(\Lambda) = \Lambda$ .

Next, we prove that the restriction  $f : \Lambda \rightarrow \Lambda$  is an expanding map. It is clear that the property (11.2.1) remains valid for the restriction. So, we only have to show that there exists  $r \leq \rho$  (we are going to take  $r = \sigma^{-1}\rho$ ) such that, for every  $x \in \Lambda$ , the image  $f(\Lambda \cap B(x, r))$  contains a neighborhood of  $\Lambda \cap \overline{B(x, r)}$  inside  $\Lambda$ . The main ingredient is the following lemma:

**Lemma 11.2.12.** *Let  $p$  be a periodic point and  $h_p : B(f(p), \rho) \rightarrow B(p, \rho)$  be the inverse branch of  $f$  at  $p$ . If  $y \in B(f(p), \rho)$  is a periodic point then  $h_p(y) \in \Lambda$ .*

*Proof.* Write  $x = h_p(y)$  and  $q = f(p)$ . Consider any  $\varepsilon > 0$  such that  $2\varepsilon$  is a constant of expansivity for  $f$ . Take  $\delta > 0$  given by the shadowing lemma (Proposition 11.2.9). By Lemma 11.2.10, there exists a pre-orbit  $(x_{-n})_n$  of  $x$  asymptotic to the periodic pre-orbit  $(\bar{p}_{-n})_n$  of  $p$ . In particular,

$$d(x_{-k+1}, q) = d(x_{-k+1}, \bar{p}_{-k+1}) < \delta \tag{11.2.9}$$

for every large multiple  $k$  of the period of  $p$ . Analogously, there exists a pre-orbit  $(q_{-n})_n$  of  $q$  asymptotic to the periodic pre-orbit  $(\bar{y}_{-n})_n$  of  $y$ . Fix any multiple  $l$  of the period of  $y$  such that

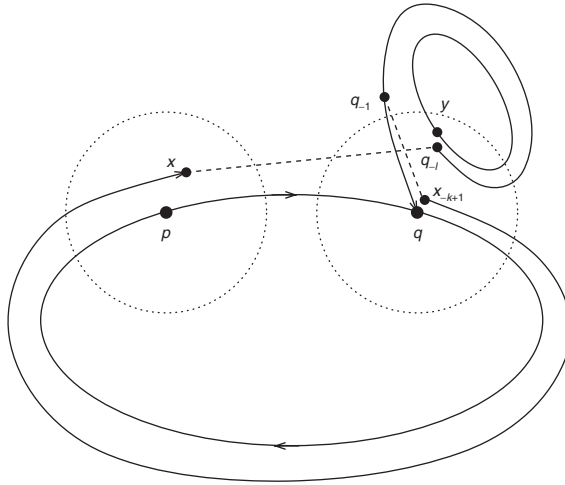
$$d(q_{-l}, f(x)) = d(q_{-l}, y) < \delta. \tag{11.2.10}$$

Now consider the periodic sequence  $(z_n)_n$ , with period  $k + l$ , given by

$$z_0 = x, z_1 = q_{-l}, \dots, z_l = q_{-1}, z_{l+1} = x_{-k+1}, \dots, z_{l+k-1} = x_{-1}, z_{k+l} = x.$$

See Figure 11.3. We claim that  $(z_n)_n$  is a  $\delta$ -pseudo-orbit. Indeed, if  $n$  is a multiple of  $k + l$  then, by (11.2.10),

$$d(f(z_n), z_{n+1}) = d(f(x), q_{-1}) = d(y, q_{-1}) < \delta.$$



**Figure 11.3.** Constructing periodic orbits

If  $n - l$  is a multiple of  $k + l$  then, by (11.2.9),

$$d(f(z_n), z_{n+1}) = d(f(q_{-l}), x_{-k+1}) = d(q, x_{-k+1}) < \delta.$$

In all the other cases,  $f(z_n) = z_{n+1}$ . This proves our claim. Now we may use Proposition 11.2.9 to conclude that there exists a periodic point  $z$  such that  $d(f^n(z), z_n) < \varepsilon$  for every  $n \geq 0$ . In particular,  $d(z, x) < \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, this shows that  $x$  is in the closure of the set of periodic points, as stated.

**Corollary 11.2.13.** *Let  $z \in \Lambda$  and  $h_z : B(f(z), \rho) \rightarrow B(z, \rho)$  be the inverse branch of  $f$  at  $z$ . If  $w \in \Lambda \cap B(f(z), \rho)$  then  $h_z(w) \in \Lambda$ .*

*Proof.* Since  $z \in \Lambda$ , we may find some periodic point  $p$  close enough to  $z$  that  $w \in B(f(p), \rho)$  and  $h_p(w) = h_z(w)$ . Since  $w \in \Lambda$ , we may find periodic points  $y_n \in B(f(p), \rho)$  converging to  $w$ . By Lemma 11.2.12, we have that  $h_p(y_n) \in \Lambda$  for every  $n$ . Passing to the limit, we conclude that  $h_p(w) \in \Lambda$ .

We are ready to conclude the proof of Theorem 11.2.11. Take  $r = \sigma^{-1}\rho$ . The property (11.2.6) implies that  $h_z(B(f(z), \rho))$  is contained in  $B(z, r)$ , for every  $z \in \Lambda$ . Then, Corollary 11.2.13 implies that  $f(\overline{\Lambda \cap B(z, r)})$  contains  $\Lambda \cap B(f(z), \rho)$ , which is a neighborhood of  $\overline{\Lambda \cap B(f(z), r)}$  in  $\Lambda$ . Thus the argument is complete.

**Theorem 11.2.14.** *Let  $f : M \rightarrow M$  be an expanding map in a compact metric space and  $\Lambda \subset M$  be the closure of the set of periodic points of  $f$ . Then*

$$M = \bigcup_{k=0}^{\infty} f^{-k}(\Lambda).$$

*Proof.* Given any  $x \in M$ , let  $\omega(x)$  denote its  $\omega$ -limit set, that is, the set of accumulation points of the iterates  $f^n(x)$  when  $n \rightarrow \infty$ . First, we show that  $\omega(x) \subset \Lambda$ . Then, we deduce that  $f^k(x) \in \Lambda$  for some  $k \geq 0$ .

Let  $\varepsilon > 0$  be such that  $2\varepsilon$  is a constant of expansivity for  $f$ . Take  $\delta > 0$  given by the shadowing lemma (Proposition 11.2.9) and let  $\alpha \in (0, \delta)$  be such that  $d(f(z), f(w)) < \delta$  whenever  $d(z, w) < \alpha$ . Let  $y$  be any point in  $\omega(x)$ . The definition of the  $\omega$ -limit set implies that there exist  $r \geq 0$  and  $s \geq 1$  such that

$$d(f^r(x), y) < \alpha \quad \text{and} \quad d(f^{r+s}(x), y) < \alpha.$$

Consider the periodic sequence  $(z_n)$ , with period  $s$ , given by

$$z_0 = y, z_1 = f^{r+1}(x), \dots, z_{s-1} = f^{r+s-1}(x), z_s = y.$$

Observe that  $d(f(z_0), z_1) = d(f(y), f^{r+1}(x)) < \delta$  (because  $d(y, f^r(x)) < \alpha$ ),  $d(f(z_{s-1}), z_s) = d(f^{r+s}(x), y) < \alpha < \delta$  and  $f(z_n) = z_{n+1}$  in all the other cases. In particular,  $(z_n)_n$  is a  $\delta$ -pseudo-orbit. Then, by Proposition 11.2.9, there exists some periodic point  $z$  such that  $d(y, z) < \varepsilon$ . Making  $\varepsilon \rightarrow 0$ , we conclude that  $y$  is accumulated by periodic points, that is,  $y \in \Lambda$ .

Let  $\varepsilon > 0$  and  $\delta > 0$  be as before. It is no restriction to suppose that  $\delta < \varepsilon$ . Take  $\beta \in (0, \delta/2)$  such that  $d(f(z), f(w)) < \delta/2$  whenever  $d(z, w) < \beta$ . Since  $\omega(x)$  is contained in  $\Lambda$ , there exist  $k \geq 1$  and points  $w_n \in \Lambda$  such that  $d(f^{n+k}(x), w_n) < \beta$  for every  $n \geq 0$ . Observe that

$$d(f(w_n), w_{n+1}) \leq d(f(w_n), f^{n+k+1}(x)) + d(f^{n+k+1}(x), w_{n+1}) < \delta/2 + \beta < \delta$$

for every  $n \geq 0$ . Therefore,  $(w_n)_n$  is a  $\delta$ -pseudo-orbit in  $\Lambda$ . Since the restriction of  $f$  to  $\Lambda$  is an expanding map (Theorem 11.2.11), it follows from Proposition 11.2.9 *applied to the restriction* that there exists  $w \in \Lambda$  such that  $d(f^n(w), w_n) < \varepsilon$  for every  $n \geq 0$ . Then,

$$d(f^n(f^k(x)), f^n(w)) \leq d(f^{n+k}(x), w_n) + d(w_n, f^n(w)) < \beta + \varepsilon < 2\varepsilon$$

for every  $n \geq 0$ . Then, by expansivity,  $f^k(x) = w$ .

### 11.2.3 Dynamical decomposition

Theorem 11.2.14 shows that for expanding maps the interesting dynamics are localized in the closure  $\Lambda$  of the set of periodic points. In particular,  $\text{supp } \mu \subset \Lambda$  for every invariant probability measure  $f$ . Moreover (Theorem 11.2.11), the restriction of  $f$  to  $\Lambda$  is still an expanding map. Thus, up to replacing  $M$  by  $\Lambda$ , it is no restriction to suppose that the set of periodic points is dense in  $M$ .

**Theorem 11.2.15** (Dynamical decomposition). *Let  $f : M \rightarrow M$  be an expanding map whose set of periodic points is dense in  $M$ . Then there exists a partition of  $M$  into non-empty compact sets  $M_{i,j}$ , with  $1 \leq i \leq k$  and  $1 \leq j \leq m(i)$ , such that:*

- (i)  $M_i = \bigcup_{j=1}^{m(i)} M_{i,j}$  is invariant under  $f$ , for every  $i$ ;
- (ii)  $f(M_{i,j}) = M_{i,j+1}$  if  $j < m(i)$  and  $f(M_{i,m(i)}) = M_{i,1}$ , for every  $i, j$ ;
- (iii) each restriction  $f : M_i \rightarrow M_i$  is a transitive expanding map;
- (iv) each  $f^{m(i)} : M_{i,j} \rightarrow M_{i,j}$  is a topologically exact expanding map.

Moreover, the number  $k$ , the numbers  $m(i)$  and the sets  $M_{i,j}$  are unique up to renumbering.

*Proof.* Consider the relation  $\sim$  defined in the set of periodic points of  $f$  as follows. Given two periodic points  $p$  and  $q$ , let  $(\bar{p}_{-n})_n$  and  $(\bar{q}_{-n})_n$ , respectively, be their periodic pre-orbits. By definition,  $p \sim q$  if and only if there exist pre-orbits  $(p_{-n})_n$  of  $p$  and  $(q_{-n})_n$  of  $q$  such that

$$d(p_{-n}, \bar{q}_{-n}) \rightarrow 0 \quad \text{and} \quad d(\bar{p}_{-n}, q_{-n}) \rightarrow 0. \tag{11.2.11}$$

We claim that  $\sim$  is an equivalence relation. It is clear from the definition that the relation  $\sim$  is reflexive and symmetric. To prove that it is also transitive, suppose that  $p \sim q$  and  $q \sim r$ . Then there exist pre-orbits  $(q_{-n})_n$  of  $q$  and  $(r_{-n})_n$  of  $r$  asymptotic to the periodic pre-orbits  $(\bar{p}_{-n})_n$  of  $p$  and  $(\bar{q}_{-n})_n$  of  $q$ , respectively. Let  $k \geq 1$  be a multiple of the periods of  $p$  and  $q$  such that  $d(r_{-k}, \bar{q}_{-k}) < \rho$ . Note that  $\bar{q}_{-k} = q$ , since  $k$  is a multiple of the period of  $q$ . Then, by Lemma 11.2.10, there exists a pre-orbit  $(r'_{-n})_n$  of the point  $r' = r_{-k}$  such that  $d(r'_{-n}, q_{-n}) \rightarrow 0$ . Then  $d(r'_{-n}, \bar{p}_{-n}) \rightarrow 0$ . Consider the pre-orbit  $(r''_{-n})_n$  of  $r$  defined by

$$r''_{-n} = \begin{cases} r_{-n} & \text{if } n \leq k \\ r'_{-n+k} & \text{if } n > k. \end{cases}$$

Since  $k$  is a multiple of the period of  $p$ , we have  $d(r''_{-n}, \bar{p}_{-n}) = d(r'_{-n+k}, \bar{p}_{-n+k})$  for every  $n > k$ . Therefore,  $(r''_{-n})_n$  is asymptotic to  $(\bar{p}_{-n})_n$ . Analogously, one can find a pre-orbit  $(p''_{-n})_n$  of  $p$  asymptotic to the periodic pre-orbit  $(\bar{r}_{-n})_n$  of  $r$ . Therefore,  $p \sim r$  and this proves that the relation  $\sim$  is indeed transitive.

Next, we claim that  $p \sim q$  if and only if  $f(p) \sim f(q)$ . Start by supposing that  $p \sim q$ , and let  $(p_{-n})_n$  and  $(q_{-n})_n$  be pre-orbits of  $p$  and  $q$  as in (11.2.11). The periodic pre-orbits of  $p' = f(p)$  and  $q' = f(q)$  are given by, respectively,

$$\bar{p}'_{-n} = \begin{cases} f(p) & \text{if } n = 0 \\ \bar{p}_{-n+1} & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \bar{q}'_{-n} = \begin{cases} f(q) & \text{if } n = 0 \\ \bar{q}_{-n+1} & \text{if } n \geq 1. \end{cases}$$

Consider the pre-orbits of  $p$  and  $q$  given by, respectively,

$$p'_{-n} = \begin{cases} f(p) & \text{if } n = 0 \\ p_{-n+1} & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad q'_{-n} = \begin{cases} f(q) & \text{if } n = 0 \\ q_{-n+1} & \text{if } n \geq 1. \end{cases}$$

It is clear that  $(p'_{-n})_n$  is asymptotic to  $(\bar{q}'_{-n})_n$  and  $(q'_{-n})_n$  is asymptotic to  $(\bar{p}'_{-n})_n$ . Hence,  $f(p) \sim f(q)$ . Now suppose that  $f(p) \sim f(q)$ . The previous argument shows that  $f^k(p) \sim f^k(q)$  for every  $k \geq 1$ . When  $k$  is a common multiple of the periods of  $p$  and  $q$  this means that  $p \sim q$ . This completes the proof of our

claim. Note that the statement means that the image and the pre-image of any equivalence class are both equivalence classes.

Observe also that if  $d(p, q) < \rho$  then  $p \sim q$ . Indeed, by Lemma 11.2.10 we may find a pre-orbit of  $q$  asymptotic to the periodic pre-orbit of  $p$  and, analogously, a pre-orbit of  $p$  asymptotic to the periodic pre-orbit of  $q$ . It follows that the equivalence classes are open sets and, since  $M$  is compact, they are finite in number. Moreover, if  $A$  and  $B$  are two different equivalence classes, then their closures  $\bar{A}$  and  $\bar{B}$  are disjoint: the distance between them is at least  $\rho$ . Since  $p \sim q$  if and only if  $f(p) \sim f(q)$ , it follows that the transformation  $f$  permutes the closures of the equivalence classes.

Thus, we may enumerate the closures of the equivalence classes as  $M_{i,j}$ , with  $1 \leq i \leq k$  and  $1 \leq j \leq m(i)$ , in such a way that

$$f(M_{i,j}) = M_{i,j+1} \text{ for } j < m(i) \quad \text{and} \quad f(M_{i,m(i)}) = M_{i,1}. \tag{11.2.12}$$

The properties (i) and (ii) in the statement of the theorem are immediate consequences.

Let us prove property (iii). Since the  $M_i$  are pairwise disjoint, it follows from (11.2.12) that  $f^{-1}(M_i) = M_i$  for every  $i$ . Hence, Lemma 11.2.2 implies that  $f : M_i \rightarrow M_i$  is an expanding map. By Lemma 4.3.4, to show that this map is transitive it suffices to show that given any open subsets  $U$  and  $V$  of  $M_i$  there exists  $n \geq 1$  such that  $f^n(U)$  intersects  $V$ . It is no restriction to assume that  $U \subset M_{i,j}$  for some  $j$ . Moreover, up to replacing  $V$  by some pre-image  $f^{-k}(V)$ , we may suppose that  $V$  is contained in the same  $M_{i,j}$ . Choose periodic points  $p \in U$  and  $q \in V$ . By the definition of equivalence classes, there exists some pre-orbit  $(q_{-n})_n$  of  $q$  asymptotic to the periodic pre-orbit  $(\bar{p}_{-n})_n$  of  $p$ . In particular, we may find  $n$  arbitrarily large such that  $q_{-n} \in U$ . Then  $q \in f^n(U) \cap V$ . Therefore,  $f : M_i \rightarrow M_i$  is transitive.

Next, we prove property (iv). Since the  $M_{i,j}$  are pairwise disjoint, it follows from (11.2.12) that  $f^{-m(i)}(M_{i,j}) = M_{i,j}$  for every  $i$ . Hence (Lemma 11.2.2),  $g = f^{m(i)} : M_{i,j} \rightarrow M_{i,j}$  is an expanding map. We also want to prove that  $g$  is topologically exact. Let  $U$  be a non-empty open subset of  $M_{i,j}$  and  $p$  be a periodic point of  $f$  in  $U$ . By (11.2.12), the period  $\kappa$  is a multiple of  $m(i)$ , say  $\kappa = sm(i)$ . Let  $q$  be any periodic point of  $f$  in  $M_{i,j}$ . By the definition of the equivalence relation  $\sim$ , there exists some pre-orbit  $(q_{-n})_n$  of  $q$  asymptotic to the periodic pre-orbit  $(\bar{p}_{-n})_n$  of  $p$ . In particular,  $d(q_{-kn}, p) \rightarrow 0$  when  $n \rightarrow \infty$ . Then  $h_q^{\kappa n}(B(q, \rho))$  is contained in  $U$  for every  $n$  sufficiently large. This implies that  $g^{sn}(U) = f^{\kappa n}(U)$  contains  $B(q, \rho)$  for every  $n$  sufficiently large. Since  $M_{i,j}$  is compact, we may find a finite cover by balls of radius  $\rho$  around periodic points. Applying the previous argument to each of those periodic points, we deduce that  $g^{sn}(U)$  contains  $M_{i,j}$  for every  $n$  sufficiently large. Therefore,  $g$  is topologically exact.

We are left to prove that  $k$ , the  $m(i)$  and the  $M_{i,j}$  are unique. Let  $N_{r,s}$ , with  $1 \leq r \leq l$  and  $1 \leq s \leq n(r)$ , be another partition as in the statement. Initially,

let us consider the partitions  $\mathcal{M} = \{M_i : 1 \leq i \leq k\}$  and  $\mathcal{N} = \{N_r : 1 \leq r \leq l\}$ , where  $N_r = \bigcup_{s=1}^{n(r)} N_{r,s}$ . Given any  $i$  and  $r$ , the sets  $M_i$  and  $N_r$  are open, closed, invariant and transitive. We claim that either  $M_i \cap N_r = \emptyset$  or  $M_i = N_r$ . Indeed, since the intersection is open, if it is non-empty then it intersects any orbit that is dense in  $M_i$  (or in  $N_r$ ). Since the intersection is also closed and invariant, it follows that it contains  $M_i$  (and  $N_r$ ). In other words,  $M_i = N_r$ . This proves our claim. It follows that the partitions  $\mathcal{M}$  and  $\mathcal{N}$  coincide, that is,  $k = l$  and  $M_i = N_i$  up to renumbering. Now, fix  $i$ . The transformation  $f$  permutes the  $M_{i,j}$  and the  $N_{i,s}$  cyclically, with periods  $m(i)$  and  $n(i)$ . Since  $f^{m(i)n(i)}$  is transitive on each  $M_{i,j}$  and each  $N_{i,s}$ , the same argument as in the first part of this paragraph shows that, given any  $j$  and  $s$ , either  $M_{i,j} \cap N_{i,s} = \emptyset$  or  $M_{i,j} = N_{i,s}$ . It follows that  $m(i) = n(i)$  and the families  $M_{i,j}$  and  $N_{i,s}$  coincide, up to cyclic renumbering.

The following consequence of the theorem contains Lemma 11.1.13:

**Corollary 11.2.16.** *If  $M$  is connected and  $f : M \rightarrow M$  is an expanding map then the set of periodic points is dense in  $M$  and  $f$  is topologically exact.*

*Proof.* We claim that  $\Lambda$  is an open subset of  $f^{-1}(\Lambda)$ . Indeed, consider  $\delta \in (0, \rho)$  such that  $d(x, y) < \delta$  implies  $d(f(x), f(y)) < \rho$ . Assume that  $x \in f^{-1}(\Lambda)$  is such that  $d(x, \Lambda) < \delta$ . Then there exists  $z \in \Lambda$  such that  $d(x, z) < \delta < \rho$  and so  $d(f(x), f(z)) < \rho$ . Applying Corollary 11.2.13 with  $w = f(x)$ , we get that  $x = h_z(w) \in \Lambda$ . Therefore,  $\Lambda$  contains its  $\delta$ -neighborhood inside  $f^{-1}(\Lambda)$ . This implies our claim.

Then, the set  $S = f^{-1}(\Lambda) \setminus \Lambda$  is closed in  $f^{-1}(\Lambda)$  and, consequently, it is closed in  $M$ , so  $f^{-n}(S)$  is closed in  $M$  for every  $n \geq 0$ . By Theorem 11.2.14, the space  $M$  is a countable pairwise disjoint union of closed sets  $\Lambda$  and  $f^{-n}(S)$ ,  $n \geq 0$ . By the Baire theorem, some of these closed sets have a non-empty interior. Since  $f$  is an open map, it follows that  $\Lambda$  has a non-empty interior.

Now consider the restriction  $f : \Lambda \rightarrow \Lambda$ . By Theorem 11.2.11, this is an expanding map. Let  $\{\Lambda_{i,j} : 1 \leq i \leq k, 1 \leq j \leq m(i)\}$  be the partition of the domain  $\Lambda$  given by Theorem 11.2.15. Then some  $\Lambda_{i,j}$  contains an open subset  $V$  of  $M$ . Since  $f^{m(i)}$  is topologically exact,  $f^{nm(i)}(V) = \Lambda_{i,j}$  for some  $n \geq 1$ . Using once more the fact that  $f$  is an open map, it follows that the compact set  $M_{i,j}$  is an open subset of  $M$ . By connectivity, it follows that  $M = \Lambda_{i,j}$ . This implies that  $\Lambda = M$  and  $f : M \rightarrow M$  is topologically exact.

### 11.2.4 Exercises

- 11.2.1. Show that if  $f : M \rightarrow M$  is a local homeomorphism in a compact connected metric space then the number of pre-images  $\#f^{-1}(y)$  is the same for every  $y \in M$ .
- 11.2.2. Show that if an expanding map is topologically mixing then it is topologically exact.

- 11.2.3. Let  $f : M \rightarrow M$  be a topologically exact transformation in a compact metric space. Show that for every  $r > 0$  there exists  $N \geq 1$  such that  $f^N(B(x,r)) = M$  for every  $x \in M$ .
- 11.2.4. Consider the expanding map  $f : S^1 \rightarrow S^1$  given by  $f(x) = 2x \pmod{\mathbb{Z}}$ . Give an example of a compact set  $\Lambda \subset S^1$  such that  $f(\Lambda) = \Lambda$  but the restriction  $f : \Lambda \rightarrow \Lambda$  is not an expanding map.
- 11.2.5. Let  $f : M \rightarrow M$  be an expanding map and  $\Lambda$  be the closure of the set of periodic points of  $f$ . Show that  $h(f) = h(f \upharpoonright \Lambda)$ .
- 11.2.6. Let  $f : M \rightarrow M$  be an expanding map such that the set of periodic points is dense in  $M$  and let  $M_i, M_{i,j}$  be the compact subsets given by Theorem 11.2.15. Show that  $h(f) = \max_i h(f \upharpoonright M_i)$  and

$$h(f \upharpoonright M_i) = \frac{1}{m(i)} h(f^{m(i)} \upharpoonright M_{i,j}) \quad \text{for any } i, j.$$

- 11.2.7. Let  $\sigma_A : \Sigma_A \rightarrow \Sigma_A$  be a shift of finite type. Interpret the decomposition given by Theorem 11.2.15 in terms of the matrix  $A$ .

### 11.3 Entropy and periodic points

In this section we analyze the distribution of periodic points of an expanding map  $f : M \rightarrow M$  from a quantitative point of view. We show (Section 11.3.1) that the rate of growth of the number of periodic points is equal to the topological entropy; compare this statement with the discussion in Section 10.2.1. Another interesting conclusion (Section 11.3.2) is that every invariant probability measure may be approximated, in the weak\* topology, by invariant probability measures supported on periodic orbits. These results are based on the following property:

**Proposition 11.3.1.** *Let  $f : M \rightarrow M$  be a topologically exact expanding map. Then, given any  $\varepsilon > 0$  there exists  $\kappa \geq 1$  such that, given any  $x_1, \dots, x_s \in M$ , any  $n_1, \dots, n_s \geq 1$  and any  $k_1, \dots, k_s \geq \kappa$ , there exists a point  $p \in M$  such that, denoting  $m_j = \sum_{i=1}^j n_i + k_i$  for  $j = 1, \dots, s$  and  $m_0 = 0$ ,*

- (i)  $d(f^{m_{j-1}+i}(p), f^i(x_j)) < \varepsilon$  for  $0 \leq i < n_j$  and  $1 \leq j \leq s$ , and
- (ii)  $f^{m_s}(p) = p$ .

*Proof.* Given  $\varepsilon > 0$ , take  $\delta > 0$  as in the shadowing lemma (Proposition 11.2.9). Without loss of generality, we may suppose that  $\delta < \varepsilon$  and  $2\varepsilon$  is a constant of expansivity for  $f$  (recall Corollary 11.2.8). Since  $f$  is topologically exact, given any  $z \in M$  there exists  $\kappa \geq 1$  such that  $f^k(B(z, \delta)) = M$  for every  $k \geq \kappa$ . Moreover (see Exercise 11.2.3), since  $M$  is compact, we may choose  $\kappa$  depending only on  $\delta$ . Let  $x_j, n_j, k_j \geq \kappa, j = 1, \dots, s$  be as in the statement. In particular, for each  $j = 1, \dots, s - 1$  there exists  $y_j \in B(f^{n_j}(x_j), \delta)$  such that  $f^{k_j}(y_j) = x_{j+1}$ . Analogously, there exists  $y_s \in B(f^{n_s}(x_s), \delta)$  such that  $f^{k_s}(y_s) = x_1$ . Consider the

periodic  $\delta$ -pseudo-orbit  $(z_n)_{n \geq 0}$  defined by

$$z_n = \begin{cases} f^{n-m_{j-1}}(x_j) & \text{for } 0 \leq n - m_{j-1} < n_j \text{ and } j = 1, \dots, s \\ f^{n-m_{j-1}-n_j}(y_j) & \text{for } 0 \leq n - m_{j-1} - n_j < k_j \text{ and } j = 1, \dots, s \\ z_{n-m_s} & \text{for } n \geq m_s. \end{cases}$$

By the second part of the shadowing lemma, there exists some periodic point  $p \in M$ , with period  $m_s$ , whose trajectory  $\varepsilon$ -shadows this periodic pseudo-orbit  $(z_n)_n$ . In particular, the conditions (i) and (ii) in the statement hold.

The property in the conclusion of Proposition 11.3.1 was introduced by Rufus Bowen [Bow71] and is called *specification by periodic points*. When condition (i) holds, but the point  $p$  is not necessarily periodic, we say that  $f$  has the property of *specification*.

### 11.3.1 Rate growth of periodic points

Let  $f : M \rightarrow M$  be an expanding map. Then  $f$  is expansive (by Lemma 11.1.4) and so it follows from Proposition 10.2.2 that the rate of growth of the number of periodic points is bounded above by the topological entropy:

$$\limsup_n \frac{1}{n} \log \# \text{Fix}(f^n) \leq h(f). \tag{11.3.1}$$

In this section we prove that, in fact, the identity holds in (11.3.1). We start with the topologically exact case, where one may even replace the limit superior by a limit:

**Proposition 11.3.2.** *For any topologically exact expanding map  $f : M \rightarrow M$ ,*

$$\lim_n \frac{1}{n} \log \# \text{Fix}(f^n) = h(f).$$

*Proof.* Given  $\varepsilon > 0$ , fix  $\kappa \geq 1$  satisfying the conclusion of Proposition 11.3.1 with  $\varepsilon/2$  instead of  $\varepsilon$ . For each  $n \geq 1$ , let  $E$  be a maximal  $(n, \varepsilon)$ -separated set. According to Proposition 11.3.1, for each  $x \in E$  there exists  $p(x) \in B(x, n, \varepsilon/2)$  with  $f^{n+\kappa}(p(x)) = p(x)$ . We claim that the map  $x \mapsto p(x)$  is injective. Indeed, consider any  $y \in E \setminus \{x\}$ . Since the set  $E$  was chosen to be  $(n, \varepsilon)$ -separated,  $B(x, n, \varepsilon/2) \cap B(y, n, \varepsilon/2) = \emptyset$ . This implies that  $p(x) \neq p(y)$ , which proves our claim. In particular, it follows that

$$\# \text{Fix}(f^{n+\kappa}) \geq \#E = s_n(f, \varepsilon, M) \quad \text{for every } n \geq 1$$

(recall the definition (10.1.9) in Section 10.2.1). Then,

$$\liminf_n \frac{1}{n} \log \# \text{Fix}(f^{n+\kappa}) \geq \liminf_n \frac{1}{n} \log s_n(f, \varepsilon, M).$$

Making  $\varepsilon \rightarrow 0$  and using Corollary 10.1.8, we find that



$$\liminf_n \frac{1}{n} \log \# \text{Fix}(f^{n+\kappa}) \geq \lim_{\varepsilon \rightarrow 0} \liminf_n \frac{1}{n} \log s_n(f, \varepsilon, M) = h(f). \tag{11.3.2}$$

Together with (11.3.1), this implies the claim in the proposition.

Proposition 11.3.2 is not true, in general, if  $f$  is not topologically exact. For example, given an arbitrary expanding map  $g : M \rightarrow M$ , consider  $f : M \times \{0, 1\} \rightarrow M \times \{0, 1\}$  defined by  $f(x, i) = (g(x), 1 - i)$ . Then  $f$  is an expanding map but all its periodic points have even period. In particular, in this case,

$$\liminf_n \frac{1}{n} \log \# \text{Fix}(f^n) = 0.$$

However, the next proposition shows that this type of example is the worst that can happen. The proof also makes it clear when and how the limit may fail to exist.

**Proposition 11.3.3.** *For any expanding map  $f : M \rightarrow M$ ,*

$$\limsup_n \frac{1}{n} \log \# \text{Fix}(f^n) = h(f).$$

*Proof.* By Theorem 11.2.11, the restriction  $f$  to the set of periodic points is an expanding map. According to Exercise 11.2.5 this restriction has the same entropy as  $f$ . Obviously, the two transformations have the same periodic points. Therefore, up to replacing  $f$  by this restriction, we may suppose that the set of periodic points is dense in  $M$ . Then, by the theorem of dynamical decomposition (Theorem 11.2.11), one may write  $M$  as a finite union of compact sets  $M_{i,j}$ , with  $1 \leq i \leq k$  and  $1 \leq j \leq m(i)$ , such that each  $f^{m(i)} : M_{i,j} \rightarrow M_{i,j}$  is a topologically exact expanding map. According to Exercise 11.2.6, there exists  $1 \leq i \leq k$  such that

$$h(f) = \frac{1}{m(i)} h(f^{m(i)} | M_{i,1}). \tag{11.3.3}$$

It is clear that

$$\begin{aligned} \limsup_n \frac{1}{n} \log \# \text{Fix}(f^n) &\geq \limsup_n \frac{1}{nm(i)} \log \# \text{Fix}(f^{nm(i)}) \\ &\geq \frac{1}{m(i)} \limsup_n \frac{1}{n} \log \# \text{Fix}((f^{m(i)} | M_{i,1})^n). \end{aligned} \tag{11.3.4}$$

Moreover, Proposition 11.3.2 applied to  $f^{m(i)} : M_{i,1} \rightarrow M_{i,1}$  yields

$$\lim_n \frac{1}{n} \log \# \text{Fix}((f^{m(i)} | M_{i,1})^n) = h(f^{m(i)} | M_{i,1}). \tag{11.3.5}$$

Combining (11.3.3)–(11.3.5), we find that

$$\limsup_n \frac{1}{n} \log \# \text{Fix}(f^n) \geq h(f), \tag{11.3.6}$$

as we wanted to prove.

### 11.3.2 Approximation by atomic measures

Given a periodic point  $p$ , with period  $n \geq 1$ , consider the probability measure  $\mu_p$  defined by

$$\mu_p = \frac{1}{n}(\delta_p + \delta_{f(p)} + \dots + \delta_{f^{n-1}(p)}).$$

Clearly,  $\mu_p$  is invariant and ergodic for  $f$ . We are going to show that if  $f$  is expanding then the set of measures of this form is dense in the space  $\mathcal{M}_1(f)$  of all invariant probability measures:

**Theorem 11.3.4.** *Let  $f : M \rightarrow M$  be a topologically exact expanding map. Then every probability measure  $\mu$  invariant under  $f$  can be approximated, in the weak\* topology, by invariant probability measures supported on periodic orbits.*

*Proof.* Let  $\varepsilon > 0$  and  $\Phi = \{\phi_1, \dots, \phi_N\}$  be a finite family of continuous functions in  $M$ . We want to show that the neighborhood  $V(\mu, \Phi, \varepsilon)$  defined in (2.1.1) contains some measure  $\mu_p$  supported on a periodic orbit. By the theorem of Birkhoff, for  $\mu$ -almost every point  $x \in M$ ,

$$\tilde{\phi}_i(x) = \lim_n \frac{1}{n} \sum_{t=0}^{n-1} \phi_i(f^t(x)) \quad \text{exists for every } i. \tag{11.3.7}$$

Fix  $C > \sup |\phi_i| \geq \sup |\tilde{\phi}_i|$  and take  $\delta > 0$  such that

$$d(x, y) < \delta \implies |\phi_i(x) - \phi_i(y)| < \frac{\varepsilon}{5} \quad \text{for every } i. \tag{11.3.8}$$

Fix  $\kappa = \kappa(\delta) \geq 1$  given by the property of specification (Proposition 11.3.1). Choose points  $x_j \in M$ ,  $1 \leq j \leq s$  satisfying (11.3.7) and positive numbers  $\alpha_j$ ,  $1 \leq j \leq s$  such that  $\sum_j \alpha_j = 1$  and

$$\left| \int \tilde{\phi}_i d\mu - \sum_{j=1}^s \alpha_j \tilde{\phi}_i(x_j) \right| < \frac{\varepsilon}{5} \quad \text{for every } i \tag{11.3.9}$$

(use Exercise A.2.6). Take  $k_j \equiv \kappa$  and choose integer numbers  $n_j$  much bigger than  $\kappa$ , in such a way that

$$\left| \frac{n_j}{m_s} - \alpha_j \right| < \frac{\varepsilon}{5Cs} \tag{11.3.10}$$

(recall that  $m_s = \sum_j (n_j + k_j) = s\kappa + \sum_j n_j$ ) and, using (11.3.8),

$$\left| \sum_{t=0}^{n_j-1} \phi_i(f^t(x_j)) - n_j \tilde{\phi}_i(x_j) \right| < \frac{\varepsilon}{5} n_j \quad \text{for } 1 \leq i \leq N. \tag{11.3.11}$$

Combining (11.3.9) and (11.3.10) with the fact that  $\int \tilde{\phi}_i d\mu = \int \phi_i d\mu$ , we get

$$\left| \int \phi_i d\mu - \sum_{j=1}^s \frac{n_j}{m_s} \tilde{\phi}_i(x_j) \right| < \frac{\varepsilon}{5} + \frac{\varepsilon}{5Cs} s \sup |\tilde{\phi}_i| < \frac{2\varepsilon}{5}. \tag{11.3.12}$$

By Proposition 11.3.1, there exists some periodic point  $p \in M$ , with period  $m_s$ , such that  $d(f^{m_j-1+t}(p), f^t(x_j)) < \delta$  for  $0 \leq t < n_j$  and  $1 \leq j \leq s$ . Then, the property (11.3.8) implies that

$$\left| \sum_{t=0}^{n_j-1} \phi_i(f^{m_j-1+t}(p)) - \sum_{t=0}^{n_j-1} \phi_i(f^t(x_j)) \right| < \frac{\varepsilon}{5} n_j \quad \text{for } 1 \leq j \leq s.$$

Combining this relation with (11.3.11), we obtain

$$\left| \sum_{t=0}^{n_j-1} \phi_i(f^{m_j-1+t}(p)) - n_j \tilde{\phi}_i(x_j) \right| < \frac{2\varepsilon}{5} n_j \quad \text{for } 1 \leq j \leq s. \tag{11.3.13}$$

Since  $\sum_j \alpha_j = 1$ , the condition (11.3.10) implies that

$$s\kappa = m_s - \sum_{j=1}^s n_j < \frac{\varepsilon}{5C} m_s.$$

Then (11.3.13) implies that

$$\left| \sum_{t=0}^{m_s-1} \phi_i(f^t(p)) - \sum_{j=1}^s n_j \tilde{\phi}_i(x_j) \right| < \frac{2\varepsilon}{5} \sum_{j=1}^s n_j + s\kappa \sup |\tilde{\phi}_i| < \frac{3\varepsilon}{5} m_s. \tag{11.3.14}$$

Let  $\mu_p$  be the invariant probability measure supported on the orbit of  $p$ . The first term in (11.3.14) coincides with  $m_s \int \phi_i d\mu_p$ . Therefore, adding the inequalities (11.3.12) and (11.3.14), we conclude that

$$\left| \int \phi_i d\mu_p - \int \phi_i d\mu \right| < \frac{2\varepsilon}{5} + \frac{3\varepsilon}{5} = \varepsilon \quad \text{for every } 1 \leq i \leq N.$$

This means that  $\mu_p \in V(\mu, \Phi, \varepsilon)$ , as we wanted to prove.

### 11.3.3 Exercises

- 11.3.1. Let  $f : M \rightarrow M$  be a continuous transformation in a compact metric space  $M$ . Check that if some iterate  $f^l$ ,  $l \geq 1$  has the property of specification, or specification by periodic points, then so does  $f$ .
- 11.3.2. Let  $f : M \rightarrow M$  be a continuous transformation in a metric space with the property of specification. Show that  $f$  is topologically mixing.
- 11.3.3. Let  $f : M \rightarrow M$  be a topologically mixing expanding map and  $\varphi : M \rightarrow \mathbb{R}$  be a continuous function. Assume that there exist probability measures  $\mu_1, \mu_2$  invariant under  $f$  and such that  $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$ . Show that there exists  $x \in M$  such that the time average of  $\varphi$  on the orbit of  $x$  does not converge. [Observation: One can show (see [BS00]) that the set  $M_\varphi$  of points where the time average of  $\varphi$  does not converge has full entropy and full Hausdorff dimension.]
- 11.3.4. Prove the following generalization of Proposition 11.3.2: if  $f : M \rightarrow M$  is a topologically exact expanding map then

$$P(f, \phi) = \lim_k \frac{1}{k} \log \sum_{p \in \text{Fix}(f^k)} e^{\phi_k(p)} \quad \text{for every H\"older function } \phi : M \rightarrow \mathbb{R}.$$

11.3.5. Let  $f : M \rightarrow M$  be an expanding map of class  $C^1$  on a compact manifold  $M$ . Show that  $f$  admits:

- (a) A neighborhood  $\mathcal{U}_0$  in the  $C^0$  topology (that is, the topology of uniform convergence) such that  $f$  is a topological factor of every  $g \in \mathcal{U}_0$ . In particular,  $h(g) \geq h(f)$  for every  $g \in \mathcal{U}_0$ .
- (b) A neighborhood  $\mathcal{U}_1$  in the  $C^1$  topology such that every  $g \in \mathcal{U}_1$  is topologically conjugate to  $f$ . In particular,  $g \mapsto h(g)$  is constant on  $\mathcal{U}_1$ .