

12

Thermodynamic formalism

In this chapter we develop the ergodic theory of expanding maps on compact metric spaces. This theory evolved from the kind of ideas in statistical mechanics that we discussed in Section 10.3.4 and, for that reason, is often called thermodynamic formalism. We point out, however, that this last expression is much broader, encompassing not only the original setting of mathematical physics but also applications to other mathematical systems, such as the so-called uniformly hyperbolic diffeomorphisms and flows (in this latter regard, see the excellent monograph of Rufus Bowen [Bow75a]).

The main result in this chapter is the following theorem of David Ruelle, which we prove in Section 12.1 (the notion of Gibbs state is also introduced in Section 12.1):

Theorem 12.1 (Ruelle). *Let $f : M \rightarrow M$ be a topologically exact expanding map on a compact metric space and $\varphi : M \rightarrow \mathbb{R}$ be a Hölder function. Then there exists a unique equilibrium state μ for φ . Moreover, the measure μ is exact, it is supported on the whole of M and is a Gibbs state.*

Recall that an expanding map is topologically exact if (and only if) it is topologically mixing (Exercise 11.2.2). Moreover, a topologically exact map is necessarily surjective.

In the particular case when M is a Riemannian manifold and f is differentiable, the equilibrium state of the potential $\varphi = -\log|\det Df|$ coincides with the absolutely continuous invariant measure given by Theorem 11.1.2. In particular, it is the unique physical measure of f . These facts are proved in Section 12.1.8.

The theorem of Livšic that we present in Section 12.2 complements the theorem of Ruelle in a very elegant way. It asserts that two potentials φ and ψ have the same equilibrium state if and only if the difference between them is cohomologous to a constant. In other words, this happens if and only if $\varphi - \psi = c + u \circ f - u$ for some $c \in \mathbb{R}$ and some continuous function u . Moreover, and remarkably, it suffices to check this condition on the periodic orbits of f .

In Section 12.3 we show that the system (f, μ) exhibits exponential decay of correlations in the space of Hölder functions, for every equilibrium state μ of any Hölder potential.

We close this chapter (Section 12.4) with an application of these ideas to a class of geometric and dynamical objects called *conformal repellers*. We prove the *Bowen–Manning formula* according to which the Hausdorff dimension of the repeller is given by the unique zero of the function $t \mapsto P(f, t\varphi_u)$.

12.1 Theorem of Ruelle

Let $f : M \rightarrow M$ be a topologically exact expanding map and φ be a Hölder potential. In what follows, $\rho > 0$ and $\sigma > 1$ are the same constants as in the definition (11.2.1). Recall that we denote by φ_n the orbital sums of φ :

$$\varphi_n(x) = \sum_{j=0}^{n-1} \varphi(f^j(x)) \quad \text{for } x \in M.$$

Before getting into the details of the proof of Theorem 12.1 let us outline the main points. The arguments in the proof turn around the *transfer operator* (or *Ruelle–Perron–Frobenius operator*), the linear operator $\mathcal{L} : C^0(M) \rightarrow C^0(M)$ defined in the Banach space $C^0(M)$ of continuous complex functions by

$$\mathcal{L}g(y) = \sum_{x \in f^{-1}(y)} e^{\varphi(x)} g(x). \tag{12.1.1}$$

Observe that \mathcal{L} is well defined: $\mathcal{L}g \in C^0(M)$ whenever $g \in C^0(M)$. Indeed, as we saw in Lemma 11.2.6, for each $y \in M$ there exist inverse branches

$$h_i : B(y, \rho) \rightarrow M, \quad i = 1, \dots, k$$

of the transformation f such that $\bigcup_{i=1}^k h_i(B(y, \rho))$ coincides with the pre-image of the ball $B(y, \rho)$. Then,

$$\mathcal{L}g = \sum_{i=1}^k (e^\varphi g) \circ h_i \tag{12.1.2}$$

restricted to $B(y, \rho)$ and, clearly, this expression defines a continuous function.

It is clear from the definition that \mathcal{L} is a positive operator: if $g(x) \geq 0$ for every $x \in M$ then $\mathcal{L}g(y) \geq 0$ for every $y \in M$. It is also easy to check that \mathcal{L} is a continuous operator: indeed,

$$\|\mathcal{L}g\| = \sup |\mathcal{L}g| \leq \text{degree}(f) e^{\sup \varphi} \sup |g| = \text{degree}(f) e^{\sup \varphi} \|g\| \tag{12.1.3}$$

for every $g \in C^0(M)$, and that means that $\|\mathcal{L}\| \leq \text{degree}(f) e^{\sup \varphi}$; recall that $\text{degree}(f)$ was defined in (11.2.4).

According to the theorem of Riesz–Markov (Theorem A.3.12), the dual space of the Banach space $C^0(M)$ may be identified with the space $\mathcal{M}(M)$

of all complex Borel measures. Then, the *dual* of the transfer operator is the linear operator $\mathcal{L}^* : \mathcal{M}(M) \rightarrow \mathcal{M}(M)$ defined by

$$\int g d(\mathcal{L}^* \eta) = \int (\mathcal{L}g) d\eta \quad \text{for every } g \in C^0(M) \text{ and } \eta \in \mathcal{M}(M). \quad (12.1.4)$$

This operator is positive, in the sense that if η is a positive measure then $\mathcal{L}^* \eta$ is also a positive measure.

The first step in the proof (Section 12.1.1) is to show that \mathcal{L}^* admits a positive eigenmeasure ν associated with a positive eigenvalue λ . We will see that such a measure admits a positive Jacobian which is Hölder and whose support is the whole space M . Moreover (Section 12.1.2), the eigenmeasure ν is a *Gibbs state*: there exists a constant $P \in \mathbb{R}$ and for each $\varepsilon > 0$ there exists $K \geq 1$ such that

$$K^{-1} \leq \frac{\nu(B(x, n, \varepsilon))}{\exp(\varphi_n(x) - nP)} \leq K \quad \text{for every } x \in M \text{ and every } n \geq 1, \quad (12.1.5)$$

where $B(x, n, \varepsilon)$ is the dynamical ball defined in (9.3.2). Actually, $P = \log \lambda$.

Behind the proof of the Gibbs property are certain results about distortion control that are also crucial to show (Section 12.1.3) that the transfer operator itself \mathcal{L} admits an eigenfunction associated with the eigenvalue λ . This function is strictly positive and Hölder. The measure $\mu = h\nu$ is the equilibrium state we are looking for, although that will take a little while to prove.

It follows easily from the properties of h (Section 12.1.4) that μ is invariant and a Gibbs state, and its support is the whole of M . Moreover, $h_\mu(f) + \int \varphi d\mu = P$. To conclude that μ is indeed an equilibrium state, we need to check that P is equal to the pressure $P(f, \varphi)$. This is done (Section 12.1.5) with the help of the Rokhlin formula (Theorem 9.7.3), which also allows us to conclude that if η is an equilibrium state then η/h is an eigenmeasure of \mathcal{L}^* associated with the eigenvalue $\lambda = \log P(f, \varphi)$. This last result is the key ingredient for proving that the equilibrium state is unique (Section 12.1.6).

The distortion control is, again, crucial for checking (Section 12.1.7) that the system (f, μ) is exact. Finally, in Section 12.1.8 we comment on the special case $\varphi = -\log |\det f|$, when f is an expanding map on a Riemannian manifold. In this case, the reference measure ν is the Lebesgue measure on the manifold itself. Thus, the equilibrium state μ is an invariant measure equivalent to the Lebesgue measure, and so it coincides with the invariant measure constructed in Section 11.1.

Before we start to detail these steps, it is convenient to make a couple of quick comments. First, note that the existence of an equilibrium state follows immediately from Corollary 10.5.9, since Lemma 11.1.4 asserts that every expanding map is expansive. However, this fact is not used in the proof: instead, in Section 12.1.4 we present a much more explicit construction of the equilibrium state.

The other comment concerns the Rokhlin formula. Let \mathcal{P} be any finite partition of M with $\text{diam } \mathcal{P} < \rho$. For each $n \geq 1$, every element of the partition $\mathcal{P}^n = \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$ is contained in the image $h^{n-1}(P)$ of some $P \in \mathcal{P}$ by an inverse branch h^{n-1} of the iterate f^{n-1} . In particular, $\text{diam } \mathcal{P}^n < \sigma^{-n+1}\rho$ for every n . Then, \mathcal{P} satisfies the hypotheses of Theorem 9.7.3 at every point. Hence, the Rokhlin formula holds for every invariant probability measure.

12.1.1 Reference measure

Recall that $C_+^0(M)$ denotes the cone of positive continuous functions. As observed previously, this cone is preserved by the transfer operator \mathcal{L} . The dual cone (recall Example 2.3.3) is defined by

$$C_+^0(M)^* = \{\eta \in C^0(M)^* : \eta(\psi) \geq 0 \text{ for every } \psi \in C_+^0(M)\}$$

and may be seen as the cone of finite positive Borel measures. It follows directly from (12.1.4) that $C_+^0(M)^*$ is preserved by the dual operator \mathcal{L}^* .

Lemma 12.1.1. *Consider the spectral radius $\lambda = \rho(\mathcal{L}^*) = \rho(\mathcal{L})$. Then there exists some probability measure ν on M such that $\mathcal{L}^*\nu = \lambda\nu$.*

Proof. As we saw in Exercise 2.3.3, the cone $C_+^0(M)$ is normal. Hence, we may apply Theorem 2.3.4 with $E = C^0(M)$ and $C = C_+^0(M)$ and $T = \mathcal{L}$. The conclusion of the theorem means that \mathcal{L}^* admits some eigenvector $\nu \in C_+^0(M)^*$ associated with the eigenvalue λ . As we have just explained, ν may be identified with a finite positive measure. Normalizing ν , we may take it to be a probability measure.

In Exercise 12.1.2 we propose an alternative proof of Lemma 12.1.1, based on the Tychonoff–Schauder theorem (Theorem 2.2.3).

Example 12.1.2. Let $f : M \rightarrow M$ be a local diffeomorphism on a compact Riemannian manifold M . Consider the transfer operator \mathcal{L} associated with the potential $\varphi = -\log|\det Df|$. The Lebesgue measure m (that is, the volume measure induced by the Riemannian metric) of M is an eigenmeasure of the transfer operator associated with the eigenvalue $\lambda = 1$:

$$\mathcal{L}^*m = m. \tag{12.1.6}$$

To check this fact, it is enough to show that $\mathcal{L}^*m(E) = m(E)$ for every measurable set E contained in the image of some inverse branch $h_j : B(y, \rho) \rightarrow M$ (because, M being compact, every measurable set may be written as a finite disjoint union of subsets E of this kind). Now, using the expression (12.1.2),

$$\mathcal{L}^*m(E) = \int \mathcal{X}_E d(\mathcal{L}^*m) = \int (\mathcal{L}\mathcal{X}_E) dm = \int \sum_{i=1}^k \frac{\mathcal{X}_E}{|\det Df|} \circ h_i dm.$$

Hence, by the choice of E and the formula of change of variables,

$$\mathcal{L}^*m(E) = \int \sum_{i=1}^k \frac{\mathcal{X}_E}{|\det Df|} \circ h_i dm = \int \mathcal{X}_E dm = m(E).$$

This proves that m is, indeed, a fixed point of \mathcal{L}^* .

Exercise 12.1.3 gives a similar conclusion for Markov measures.

From now on, we always take ν to be a *reference measure*, that is, a probability measure such that $\mathcal{L}^*\nu = \lambda\nu$ for *some* $\lambda > 0$. By the end of the proof of Theorem 12.1 we will find that λ is uniquely determined (in view of Lemma 12.1.1, that means that λ is necessarily equal to the spectral radius of \mathcal{L} and \mathcal{L}^*) and the measure ν itself is also unique.

Initially, we show that f admits a Jacobian with respect to ν , which may be written explicitly in terms of the eigenvalue λ and the potential φ :

Lemma 12.1.3. *The transformation $f : M \rightarrow M$ admits a Jacobian with respect to ν , given by $J_\nu f = \lambda e^{-\varphi}$.*

Proof. Let A be any domain of invertibility of f . Let $(g_n)_n$ be a sequence of continuous functions converging to the characteristic function of A at ν -almost every point and such that $\sup |g_n| \leq 1$ for every n (see Exercise A.3.5). Observe that

$$\mathcal{L}(e^{-\varphi} g_n)(y) = \sum_{x \in f^{-1}(y)} g_n(x).$$

The expression on the right-hand side is bounded by the degree of f , as defined in (11.2.4), and it converges to $\chi_{f(A)}(y)$ at ν -almost every point. Hence, using the dominated convergence theorem, the sequence

$$\int \lambda e^{-\varphi} g_n d\nu = \int e^{-\varphi} g_n d(\mathcal{L}^*\nu) = \int \mathcal{L}(e^{-\varphi} g_n) d\nu$$

converges to $\nu(f(A))$. Since the expression on the left-hand side converges to $\int_A \lambda e^{-\varphi} d\nu$, we conclude that

$$\nu(f(A)) = \int_A \lambda e^{-\varphi} d\nu,$$

which proves the claim.

The next lemma applies, in particular, to the reference measure ν :

Lemma 12.1.4. *Let $f : M \rightarrow M$ be a topologically exact expanding map and η be any Borel probability measure such that there exists Jacobian of f with respect to η . Then η is supported on the whole of M .*

Proof. Suppose, by contradiction, that there exists some open set $U \subset M$ such that $\eta(U) = 0$. Note that f is an open map, since it is a local homeomorphism.

Thus, the image $f(U)$ is also an open set. Moreover, we may write U as a finite disjoint union of domains of invertibility A . For each one of them,

$$\eta(f(A)) = \int_A J_\eta f d\eta = 0.$$

Therefore, $\eta(f(U)) = 0$. By induction, it follows that $\eta(f^n(U)) = 0$ for every $n \geq 0$. Since we take f to be topologically exact, there exists $n \geq 1$ such that $f^n(U) = M$. This contradicts the fact that $\eta(M) = 1$.

12.1.2 Distortion and the Gibbs property

In this section we prove certain distortion bounds that have a central role in the proof of Theorem 12.1. The hypothesis that φ is Hölder is critical at this stage: most of what follows is false, in general, if the potential is only continuous. As a first application of this distortion control, we prove that every reference measure ν is a Gibbs state.

Fix constants $K_0 > 0$ and $\alpha > 0$ such that $|\varphi(z) - \varphi(w)| \leq K_0 d(z, w)^\alpha$ for any $z, w \in M$.

Lemma 12.1.5. *There exists $K_1 > 0$ such that for every $n \geq 1$, every $x \in M$ and every $y \in B(x, n + 1, \rho)$,*

$$|\varphi_n(x) - \varphi_n(y)| \leq K_1 d(f^n(x), f^n(y))^\alpha.$$

Proof. By hypothesis, $d(f^i(x), f^i(y)) < \rho$ for every $0 \leq i \leq n$. Then, for each $j = 1, \dots, n$, the inverse branch $h_j : B(f^n(x), \rho) \rightarrow M$ of f^j at the point $f^{n-j}(x)$, which maps $f^n(x)$ to $f^{n-j}(x)$, also maps $f^n(y)$ to $f^{n-j}(y)$. Hence, recalling (11.2.6), $d(f^{n-j}(x), f^{n-j}(y)) \leq \sigma^{-j} d(f^n(x), f^n(y))$ for every $j = 1, \dots, n$. Then,

$$\begin{aligned} |\varphi_n(x) - \varphi_n(y)| &\leq \sum_{j=1}^n |\varphi(f^{n-j}(x)) - \varphi(f^{n-j}(y))| \\ &\leq \sum_{j=1}^n K_0 \sigma^{-j\alpha} d(f^n(x), f^n(y))^\alpha. \end{aligned}$$

Therefore, we may take any $K_1 \geq K_0 \sum_{j=0}^\infty \sigma^{-j\alpha}$.

As a consequence of Lemma 12.1.5, we obtain the following variation of Proposition 11.1.5 where the usual Jacobian with respect to the Lebesgue measure is replaced by the Jacobian with respect to any reference measure ν :

Corollary 12.1.6. *There exists $K_2 > 0$ such that for every $n \geq 1$, every $x \in M$ and every $y \in B(x, n + 1, \rho)$,*

$$K_2^{-1} \leq \frac{J_\nu f^n(x)}{J_\nu f^n(y)} \leq K_2.$$

Proof. From the expression of the Jacobian in Lemma 12.1.3 it follows that (recall Exercise 9.7.5)

$$J_v f^n(z) = \lambda^n e^{-\varphi_n(z)} \quad \text{for every } z \in M \text{ and every } n \geq 1. \tag{12.1.7}$$

Then, Lemma 12.1.5 yields

$$\left| \log \frac{J_v f^n(x)}{J_v f^n(y)} \right| = \left| \varphi_n(x) - \varphi_n(y) \right| \leq K_1 d(f^n(x), f^n(y))^\alpha \leq K_1 \rho^\alpha.$$

So, it suffices to take $K_2 = \exp(K_1 \rho^\alpha)$.

Now we may show that every reference measure ν is a Gibbs state:

Lemma 12.1.7. *For every small $\varepsilon > 0$, there exists $K_3 = K_3(\varepsilon) > 0$ such that, denoting $P = \log \lambda$,*

$$K_3^{-1} \leq \frac{\nu(B(x, n, \varepsilon))}{\exp(\varphi_n(x) - nP)} \leq K_3 \quad \text{for every } x \in M \text{ and every } n \geq 1.$$

Proof. Consider $\varepsilon < \rho$. Then, $f \mid B(y, \varepsilon)$ is injective for every $y \in M$ and, consequently, $f^n \mid B(x, n, \varepsilon)$ is injective for every $x \in M$ and every n . Then,

$$\nu(f^n(B(x, n, \varepsilon))) = \int_{B(x, n, \varepsilon)} J_v f^n(y) d\nu(y).$$

Up to reducing ε , we may assume that $d(f(x), f(y)) < \rho$ whenever $d(x, y) < \varepsilon$. This implies that $B(x, n, \varepsilon) \subset B(x, n + 1, \rho)$ for every $x \in M$ and $n \geq 1$. Then, by Corollary 12.1.6, the value of $J_v f^n$ at any point $y \in B(x, n, \varepsilon)$ differs from $J_v f^n(x)$ by a factor bounded by the constant K_2 . It follows that

$$K_2^{-1} \nu(f^n(B(x, n, \varepsilon))) \leq J_v f^n(x) \nu(B(x, n, \varepsilon)) \leq K_2 \nu(f^n(B(x, n, \varepsilon))). \tag{12.1.8}$$

Now, $J_v f^n(x) = \lambda^n e^{-\varphi_n(x)} = \exp(nP - \varphi_n(x))$, as we saw in (12.1.7). By Lemma 11.2.7 we also have that $f^n(B(x, n, \varepsilon)) = f(B(f^{n-1}(x), \varepsilon))$, and so

$$\nu(f^n(B(x, n, \varepsilon))) = \int_{B(f^{n-1}(x), \varepsilon)} J_v f dv \tag{12.1.9}$$

for every $x \in M$ and every n . It is clear that the left-hand side of (12.1.9) is bounded above by 1. Moreover, $J_v f = \lambda e^{-\varphi}$ is bounded from zero and (by Exercise 12.1.1 and Lemma 12.1.4) the set $\{\nu(B(y, \varepsilon)) : y \in M\}$ is also bounded from zero. Therefore, the right-hand side of (12.1.9) is bounded below by some number $a > 0$. Using these observations in (12.1.8), we obtain

$$K_2^{-1} a \leq \frac{\nu(B(x, n, \varepsilon))}{\exp(\varphi_n(x) - nP)} \leq K_2.$$

Now it suffices to take $K_3 = \max\{K_2/a, K_2\}$.

12.1.3 Invariant density

Next, we show that the transfer operator \mathcal{L} admits some positive eigenfunction h associated with the eigenvalue λ . We are going to find h as a Cesàro accumulation point of the sequence of functions $\lambda^{-n}\mathcal{L}^n 1$. To show that there does exist some accumulation point, we start by proving that this sequence is uniformly bounded and equicontinuous.

Lemma 12.1.8. *There exists $K_4 > 0$ such that*

$$-K_4 d(y_1, y_2)^\alpha \leq \log \frac{\mathcal{L}^n 1(y_1)}{\mathcal{L}^n 1(y_2)} \leq K_4 d(y_1, y_2)^\alpha$$

for every $n \geq 1$ and any $y_1, y_2 \in M$ with $d(y_1, y_2) < \rho$.

Proof. It follows from (12.1.2) that, given any continuous function g ,

$$\mathcal{L}^n g = \sum_i (e^{\varphi_n} g) \circ h_i^n \quad \text{restricted to each ball } B(y, \rho),$$

where the sum is over all inverse branches $h_i^n : B(y, \rho) \rightarrow M$ of the iterate f^n .

In particular,

$$\frac{\mathcal{L}^n 1(y_1)}{\mathcal{L}^n 1(y_2)} = \frac{\sum_i e^{\varphi_n(h_i^n(y_1))}}{\sum_i e^{\varphi_n(h_i^n(y_2))}}.$$

By Lemma 12.1.5, for each of these inverse branches h_i^n one has

$$|\varphi_n(h_i^n(y_1)) - \varphi_n(h_i^n(y_2))| \leq K_1 d(y_1, y_2)^\alpha.$$

Consequently,

$$e^{-K_1 d(y_1, y_2)^\alpha} \leq \frac{\mathcal{L}^n 1(y_1)}{\mathcal{L}^n 1(y_2)} \leq e^{K_1 d(y_1, y_2)^\alpha}.$$

Therefore, one may take any $K_4 \geq K_1$.

It follows that the sequence $\lambda^{-n}\mathcal{L}^n 1$ is bounded from zero and infinity:

Corollary 12.1.9. *There exists $K_5 > 0$ such that $K_5^{-1} \leq \lambda^{-n}\mathcal{L}^n 1(x) \leq K_5$ for every $n \geq 1$ and any $x \in M$.*

Proof. Start by observing that, for every $n \geq 1$,

$$\int \mathcal{L}^n 1 dv = \int 1 d(\mathcal{L}^{*n} v) = \int \lambda^n dv = \lambda^n.$$

In particular, for every $n \geq 1$,

$$\min_{y \in M} \lambda^{-n} \mathcal{L}^n 1(y) \leq 1 \leq \max_{y \in M} \lambda^{-n} \mathcal{L}^n 1(y). \tag{12.1.10}$$

Since f is topologically exact, there exists $N \geq 1$ such that $f^N(B(x, \rho)) = M$ for every $x \in M$ (check Exercise 11.2.3). Now, given any $x, y \in M$, we may find $x' \in B(x, \rho)$ such that $f^N(x') = y$. Then, on the one hand,

$$\mathcal{L}^{n+N} 1(y) = \sum_{z \in f^{-N}(y)} e^{\varphi_N(z)} \mathcal{L}^n 1(z) \geq e^{\varphi_N(x')} \mathcal{L}^n 1(x') \geq e^{-cN} \mathcal{L}^n 1(x').$$

On the other hand, Lemma 12.1.8 gives that $\mathcal{L}^n 1(x') \geq \mathcal{L}^n 1(x) \exp(-K_4 \rho^\alpha)$. Take $c = \sup |\varphi|$ and $K \geq \exp(K_4 \rho^\alpha) e^{cN} \lambda^N$. Combining the previous inequalities, we get that

$$\mathcal{L}^{n+N} 1(y) \geq \exp(-K_4 \rho^\alpha) e^{-cN} \mathcal{L}^n 1(x) \geq K^{-1} \lambda^N \mathcal{L}^n 1(x)$$

for every $x, y \in M$. Therefore, for every $n \geq 1$,

$$\min \lambda^{-(n+N)} \mathcal{L}^{n+N} 1 \geq K^{-1} \max \lambda^{-n} \mathcal{L}^n 1. \tag{12.1.11}$$

Combining (12.1.10) and (12.1.11), we get:

$$\begin{aligned} \max \lambda^{-n} \mathcal{L}^n 1 &\leq K \min \lambda^{-(n+N)} \mathcal{L}^{n+N} 1 \leq K \quad \text{for every } n \geq 1, \\ \min \lambda^{-n} \mathcal{L}^n 1 &\geq K^{-1} \max \lambda^{-n+N} \mathcal{L}^{n-N} 1 \geq K^{-1} \quad \text{for every } n > N. \end{aligned}$$

To conclude the proof, we only have to extend this last estimate to the values $n = 1, \dots, N$. For that, observe that each $\mathcal{L}^n 1$ is a positive continuous function. Since M is compact, it follows that the minimum of $\mathcal{L}^n 1$ is positive for every n . Then, we may take $K_5 \geq K$ such that $\min \lambda^{-n} \mathcal{L}^n 1 \geq K_5^{-1}$ for every $n = 1, \dots, N$.

It follows immediately from Corollary 12.1.9 that the positive eigenvalue λ is uniquely determined. By Lemma 12.1.1, this implies that $\lambda = \rho(\mathcal{L}) = \rho(\mathcal{L}^*)$. We are also going to see, in a while, that $\lambda = e^{P(f, \varphi)}$.

Lemma 12.1.10. *There exists $K_6 > 0$ such that*

$$|\lambda^{-n} \mathcal{L}^n 1(x) - \lambda^{-n} \mathcal{L}^n 1(y)| \leq K_6 d(x, y)^\alpha \quad \text{for any } n \geq 1 \text{ and } x, y \in M.$$

In particular, the sequence $\lambda^{-n} \mathcal{L}^n 1$ is equicontinuous.

Proof. Initially, suppose that $d(x, y) < \rho$. By Lemma 12.1.8,

$$\mathcal{L}^n 1(x) \leq \mathcal{L}^n 1(y) \exp(K_4 d(x, y)^\alpha)$$

and, hence,

$$\lambda^{-n} \mathcal{L}^n 1(x) - \lambda^{-n} \mathcal{L}^n 1(y) \leq [\exp(K_4 d(x, y)^\alpha) - 1] \lambda^{-n} \mathcal{L}^n 1(y).$$

Take $K > 0$ such that $|\exp(K_4 t) - 1| \leq K|t|$ whenever $|t| \leq \rho^\alpha$. Then, using Corollary 12.1.9,

$$\lambda^{-n} \mathcal{L}^n 1(x) - \lambda^{-n} \mathcal{L}^n 1(y) \leq KK_5 d(x, y)^\alpha.$$

Reversing the roles of x and y , we conclude that

$$|\lambda^{-n} \mathcal{L}^n 1(x) - \lambda^{-n} \mathcal{L}^n 1(y)| \leq KK_5 d(x, y)^\alpha \quad \text{whenever } d(x, y) < \rho.$$

When $d(x, y) \geq \rho$, Corollary 12.1.9 gives that

$$|\lambda^{-n} \mathcal{L}^n 1(x) - \lambda^{-n} \mathcal{L}^n 1(y)| \leq 2K_5 \leq 2K_5 \rho^{-\alpha} d(x, y)^\alpha.$$

Hence, it suffices to take $K_6 \geq \max\{KK_5, 2K_5 \rho^{-\alpha}\}$ to get the first part of the statement. The second part is an immediate consequence.

We are ready to show that the transfer operator \mathcal{L} admits some eigenfunction associated with the eigenvalue λ . Corollary 12.1.9 and Lemma 12.1.10 imply that the time average

$$h_n = \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} \mathcal{L}^i 1$$

defines an equicontinuous bounded sequence. Then, by the theorem of Ascoli–Arzelá, there exists some subsequence $(h_{n_i})_i$ converging uniformly to a continuous function h .

Lemma 12.1.11. *The function h satisfies $\mathcal{L}h = \lambda h$. Moreover, $\int h d\nu = 1$ and*

$$K_5^{-1} \leq h(x) \leq K_5 \quad \text{and} \quad |h(x) - h(y)| \leq K_6 d(x, y)^\alpha \quad \text{for every } x, y \in M.$$

Proof. Consider any subsequence $(h_{n_i})_i$ converging to h . As the transfer operator \mathcal{L} is continuous,

$$\begin{aligned} \mathcal{L}h &= \lim_i \mathcal{L}h_{n_i} = \lim_i \frac{1}{n_i} \sum_{k=0}^{n_i-1} \lambda^{-k} \mathcal{L}^{k+1} 1 = \lim_i \frac{\lambda}{n_i} \sum_{k=1}^{n_i} \lambda^{-k} \mathcal{L}^k 1 \\ &= \lim_i \frac{\lambda}{n_i} \sum_{k=0}^{n_i-1} \lambda^{-k} \mathcal{L}^k 1 + \frac{\lambda}{n_i} (\lambda^{-n_i} \mathcal{L}^{n_i} 1 - 1). \end{aligned}$$

The first term on the right-hand side converges to λh whereas the second one converges to zero, because the sequence $\lambda^{-n} \mathcal{L}^n 1$ is uniformly bounded. It follows that $\mathcal{L}h = \lambda h$, as we stated.

Note that $\int \lambda^{-n} \mathcal{L}^n 1 d\nu = \int \lambda^{-n} d(\mathcal{L}^{*n} \nu) = \int 1 d\nu = 1$ for every $n \in \mathbb{N}$, by the definition of ν . It follows that $\int h_n d\nu = 1$ for every n and, using the dominated convergence theorem, $\int h d\nu = 1$. All the other claims in the statement follow, in an entirely analogous way, from Corollary 12.1.9 and Lemma 12.1.10.

12.1.4 Construction of the equilibrium state

Consider the measure defined by $\mu = h\nu$, that is,

$$\mu(A) = \int_A h d\nu \quad \text{for each measurable set } A \subset M.$$

We are going to see that μ is an equilibrium state for the potential φ and satisfies all the other conditions in Theorem 12.1.

From Lemma 12.1.11 we get that $\mu(M) = \int h d\nu = 1$ and so μ is a probability measure. Moreover,

$$K_5^{-1} \nu(A) \leq \mu(A) \leq K_5 \nu(A) \tag{12.1.12}$$

for every measurable set $A \subset M$. In particular, μ is equivalent to the reference measure ν . This fact, together with Lemma 12.1.4, gives that $\text{supp } \mu = M$. It

also follows from the relation (12.1.12), together with Lemma 12.1.7, that μ is a Gibbs state: taking $L = K_5K$, we find that

$$L^{-1} \leq \frac{\mu(B(x, n, \varepsilon))}{\exp(\varphi_n(x) - nP)} \leq L, \tag{12.1.13}$$

for every $x \in M$ and every $n \geq 1$. Recall that $P = \log \lambda$.

Lemma 12.1.12. *The probability measure μ is invariant under f . Moreover, f admits a Jacobian with respect to μ , given by $J_\mu f = \lambda e^{-\varphi} (h \circ f)/h$.*

Proof. Start by noting that $\mathcal{L}((g_1 \circ f)g_2) = g_1 \mathcal{L}g_2$, for any continuous functions $g_1, g_2 : M \rightarrow \mathbb{R}$. Indeed, for every $y \in M$,

$$\begin{aligned} \mathcal{L}((g_1 \circ f)g_2)(y) &= \sum_{x \in f^{-1}(y)} e^{\varphi(x)} g_1(f(x))g_2(x) \\ &= g_1(y) \sum_{x \in f^{-1}(y)} e^{\varphi(x)} g_2(x) = g_1(y) \mathcal{L}g_2(y). \end{aligned} \tag{12.1.14}$$

Thus, for every continuous function $g : M \rightarrow \mathbb{R}$,

$$\begin{aligned} \int (g \circ f) d\mu &= \lambda^{-1} \int (g \circ f)h d(\mathcal{L}^*v) = \lambda^{-1} \int \mathcal{L}((g \circ f)h) dv \\ &= \lambda^{-1} \int g \mathcal{L}h dv = \int gh dv = \int g d\mu. \end{aligned}$$

In view of Proposition A.3.3, this proves that the probability measure μ is invariant under f .

To prove the second claim, consider any domain of invertibility A of f . Then, using Lemma 9.7.4(i),

$$\mu(f(A)) = \int_{f(A)} 1 d\mu = \int_{f(A)} h dv = \int_A J_\nu f (h \circ f) dv = \int_A J_\nu f \frac{h \circ f}{h} d\mu.$$

By Lemma 12.1.3, this means that

$$J_\mu f = J_\nu f \frac{h \circ f}{h} = \lambda e^{-\varphi} \frac{h \circ f}{h},$$

as stated.

Corollary 12.1.13. *The invariant probability measure $\mu = hv$ satisfies*

$$h_\mu(f) + \int \varphi d\mu = P.$$

Proof. Combining the Rokhlin formula (Theorem 9.7.3) with the second part of Lemma 12.1.12,

$$h_\mu(f) = \int \log J_\mu f d\mu = \log \lambda - \int \varphi d\mu + \int (\log h \circ f - \log h) d\mu.$$

Since μ is invariant and $\log h$ is bounded (Corollary 12.1.9), the last term is equal to zero. This shows that $h_\mu(f) = P - \int \varphi d\mu$, as stated.

To complete the proof that $\mu = h\nu$ is an equilibrium state, all that we need to do is to check that $P = \log \lambda$ is equal to the pressure $P(f, \varphi)$. This is done in Corollary 12.1.15 below.

12.1.5 Pressure and eigenvalues

Let η be any probability measure invariant under f and such that

$$h_\eta(f) + \int \varphi d\eta \geq P \tag{12.1.15}$$

(for example: the probability measure μ constructed in the previous section). Let $g_\eta = 1/J_\eta f$ (the Jacobian $J_\eta f$ does exist, by Exercise 9.7.8) and consider also the function $g = \lambda^{-1} e^\varphi h / (h \circ f)$. Observe that

$$\sum_{x \in f^{-1}(y)} g(x) = \frac{1}{\lambda h(y)} \sum_{x \in f^{-1}(y)} e^{\varphi(x)} h(x) = \frac{\mathcal{L}h(y)}{\lambda h(y)} = 1 \tag{12.1.16}$$

for every $y \in M$. Moreover, since η is invariant under f , Exercise 9.7.4 gives that

$$\sum_{x \in f^{-1}(y)} g_\eta(x) = 1 \quad \text{for } \eta\text{-almost every } y \in M. \tag{12.1.17}$$

Using (12.1.15) and the Rokhlin formula (Theorem 9.7.3),

$$0 \leq h_\eta(f) + \int \varphi d\eta - P = \int (-\log g_\eta + \varphi - \log \lambda) d\eta. \tag{12.1.18}$$

By the definition of g and the hypothesis that η is invariant, the integral on the right-hand side of (12.1.18) is equal to

$$\int (-\log g_\eta + \log g + \log h \circ f - \log h) d\eta = \int \log \frac{g}{g_\eta} d\eta. \tag{12.1.19}$$

Recalling the definition of g_η , Exercise 9.7.3 gives that

$$\int \log \frac{g}{g_\eta} d\eta = \int \left(\sum_{x \in f^{-1}(y)} g_\eta(x) \log \frac{g}{g_\eta}(x) \right) d\eta(y). \tag{12.1.20}$$

At this point we need the following elementary fact:

Lemma 12.1.14. *Let $p_i, b_i, i = 1, \dots, k$ be positive real numbers such that $\sum_{i=1}^k p_i = 1$. Then*

$$\sum_{i=1}^k p_i \log b_i \leq \log \left(\sum_{i=1}^k p_i b_i \right),$$

and the identity holds if and only if the numbers b_j are all equal to $\sum_{i=1}^k p_i b_i$.

Proof. Take $a_i = \log(p_i b_i)$ in Lemma 10.4.4. Then the inequality in the conclusion of Lemma 10.4.4 corresponds exactly to the inequality in the

present lemma. Moreover, the identity holds if and only if

$$p_j = \frac{e^{a_j}}{\sum_i e^{a_i}} \Leftrightarrow p_j = \frac{p_j b_j}{\sum_i p_i b_i} \Leftrightarrow b_j = \sum_i p_i b_i$$

for every $j = 1, \dots, n$.

For each $y \in M$, take $p_i = g_\eta(x_i)$ and $b_i = g(x_i)/g_\eta(x_i)$, where the x_i are the pre-images of y . The identity (12.1.17) means that $\sum_i p_i = 1$ for η -almost every y . Then, we may apply Lemma 12.1.14:

$$\begin{aligned} \sum_{x \in f^{-1}(y)} g_\eta(x) \log \frac{g}{g_\eta}(x) &\leq \log \sum_{x \in f^{-1}(y)} g_\eta(x) \frac{g}{g_\eta}(x) \\ &= \log \sum_{x \in f^{-1}(y)} g(x) = 0 \end{aligned} \tag{12.1.21}$$

for η -almost every y ; in the last step we used (12.1.16). Combining the relations (12.1.18) through (12.1.21), we find:

$$h_\eta(f) + \int \varphi d\eta - P = \int \log \frac{g}{g_\eta} d\eta = 0. \tag{12.1.22}$$

Corollary 12.1.15. $P(f, \varphi) = P = \log \rho(\mathcal{L})$.

Proof. By (12.1.22), we have that $h_\eta(f) + \int \varphi d\eta = P$ for every invariant probability measure η such that $h_\eta(f) + \int \varphi d\eta \geq P$. By the variational principle (Theorem 10.4.1), it follows that $P(f, \varphi) = P$. The second identity has been observed before, right after Corollary 12.1.9.

At this point we have completed the proof that the measure $\mu = h\nu$ constructed in the previous section is an equilibrium state for φ . The statement that follows arises from the same kind of ideas and is the basis for proving that this equilibrium state is unique:

Corollary 12.1.16. *If η is an equilibrium state for φ then $\text{supp } \eta = M$ and*

$$J_\eta f = \lambda e^{-\varphi}(h \circ f)/h \quad \text{and} \quad \mathcal{L}^*(\eta/h) = \lambda(\eta/h).$$

Proof. The first claim is an immediate consequence of the second one and Lemma 12.1.4.

Note that the identity in (12.1.22) also implies that the identity in (12.1.21) holds for η -almost every $y \in M$. According to Lemma 12.1.14, that happens if and only if the numbers $b_i = \log(g(x_i)/g_\eta(x_i))$ are all equal. In other words, for η -almost every $y \in M$ there exists a number $c(y)$ such that

$$\frac{g(x)}{g_\eta(x)} = c(y) \quad \text{for every } x \in f^{-1}(y).$$

Moreover, recalling the identities (12.1.18) and (12.1.19),

$$c(y) = \sum_{x \in f^{-1}(y)} c(y)g_\eta(x) = \sum_{x \in f^{-1}(y)} g(x) = 1$$

for η -almost every y . It follows that $g_\eta = g$ at η -almost every point, and so the function $1/g = \lambda e^{-\varphi}(h \circ f)/h$ is a Jacobian of f with respect to η . This proves the second claim.

To prove the third claim, let $\zeta : M \rightarrow \mathbb{R}$ be any continuous function. On the one hand, by the definition of the transfer operator,

$$\int \zeta d\mathcal{L}^*\left(\frac{\eta}{h}\right) = \int \frac{1}{h}(\mathcal{L}\zeta) d\eta = \int \frac{1}{h(y)} \left(\sum_{x \in f^{-1}(y)} e^{\varphi(x)} \zeta(x) \right) d\eta(y). \tag{12.1.23}$$

By the definition of the function g ,

$$\frac{e^{\varphi(x)}}{h(y)} = \frac{\lambda g(x)}{h(x)}.$$

Replacing this identity in (12.1.23), we obtain:

$$\int \zeta d\mathcal{L}^*\left(\frac{\eta}{h}\right) = \int \left(\sum_{x \in f^{-1}(y)} \frac{\lambda g \zeta}{h}(x) \right) d\eta(y). \tag{12.1.24}$$

Then, recalling that $g = g_\eta = 1/J_\eta f$, we may use Exercise 9.7.3 to conclude that

$$\int \zeta d\mathcal{L}^*\left(\frac{\eta}{h}\right) = \int \left(\sum_{x \in f^{-1}(y)} \frac{\lambda g \zeta}{h}(x) \right) d\eta(y) = \int \frac{\lambda \zeta}{h} d\eta.$$

Since the continuous function ζ is arbitrary, this shows that $\mathcal{L}^*(\eta/h) = \lambda(\eta/h)$, as stated.

12.1.6 Uniqueness of the equilibrium state

Let us start by proving the following distortion bound:

Corollary 12.1.17. *There exists $K_7 > 0$ such that for every equilibrium state η , every $n \geq 1$, every $x \in M$ and every $y \in B(x, n + 1, \rho)$,*

$$K_7^{-1} \leq \frac{J_\eta f^n(x)}{J_\eta f^n(y)} \leq K_7.$$

Proof. By Corollary 12.1.16,

$$J_\eta f^n = \lambda e^{-\varphi_n} \frac{h \circ f^n}{h} = J_\nu f^n \frac{h \circ f^n}{h}$$

for each $n \geq 1$. Then, using Corollary 12.1.6 and Lemma 12.1.11,

$$K_2^{-1} K_5^{-4} \leq \frac{J_\eta f^n(x)}{J_\eta f^n(y)} = \frac{J_\nu f^n(x)}{J_\nu f^n(y)} \frac{h(f^n(x))h(y)}{f(f^n(y))h(x)} \leq K_2 K_5^4.$$

Therefore, it suffices to take $K_7 = K_2 K_5^4$.

Lemma 12.1.18. *All the equilibrium states of φ are equivalent measures.*

Proof. Let η_1 and η_2 be equilibrium states. Consider any finite partition \mathcal{P} of M such that every $P \in \mathcal{P}$ has non-empty interior and diameter less than ρ . Since $\text{supp } \eta_1 = \text{supp } \eta_2 = M$ (by Corollary 12.1.16), the set $\{\eta_i(P) : i = 1, 2 \text{ and } P \in \mathcal{P}\}$ is bounded from zero. Consequently, there exists $C_1 > 0$ such that

$$\frac{1}{C_1} \leq \frac{\eta_1(P)}{\eta_2(P)} \leq C_1 \quad \text{for every } P \in \mathcal{P}. \tag{12.1.25}$$

We are going to show that this relation extends to every measurable subset of M , up to replacing C_1 by a convenient constant $C_2 > C_1$.

For each $n \geq 1$, let \mathcal{Q}_n be the partition of M formed by the images $h^n(P)$ of the elements of \mathcal{P} under the inverse branches h^n of the iterate f^n . By the definition of Jacobian, $\eta_i(P) = \int_{h^n(P)} J_{\eta_i} f^n d\eta_i$. Hence, using Corollary 12.1.17,

$$K_7^{-1} J_{\eta_i} f^n(x) \leq \frac{\eta_i(P)}{\eta_i(h^n(P))} \leq K_7 J_{\eta_i} f^n(x)$$

for any $x \in h^n(P)$. Recalling that $J_{\eta_1} f = J_{\eta_2} f$ (Corollary 12.1.16), it follows that

$$K_7^{-2} \leq \frac{\eta_2(P)\eta_1(h^n(P))}{\eta_1(P)\eta_2(h^n(P))} \leq K_7^2. \tag{12.1.26}$$

Combining (12.1.25) and (12.1.26), and taking $C_2 = C_1 K_7^2$, we get that

$$\frac{1}{C_2} \leq \frac{\eta_1(h^n(P))}{\eta_2(h^n(P))} \leq C_2 \tag{12.1.27}$$

for every $P \in \mathcal{P}$, every inverse branch h^n of f^n and every $n \geq 1$. In other words, the property in (12.1.25) holds for every element of \mathcal{Q}_n , with C_2 in the place of C_1 .

Now observe that $\text{diam } \mathcal{Q}_n \leq 2\sigma^{-n}\rho$ for every n . Given any measurable set B and any $\delta > 0$, we may use Proposition A.3.2 to find a compact set $F \subset B$ and an open set $A \supset B$ such that $\eta_i(A \setminus F) < \delta$ for $i = 1, 2$. Let \mathcal{Q}_n be the union of all the elements of the partition \mathcal{Q}_n that intersect F . It is clear that $\mathcal{Q}_n \supset F$ and, assuming that n is large enough, $\mathcal{Q}_n \subset A$. Then,

$$\eta_1(B) \leq \eta_1(A) < \eta_1(\mathcal{Q}_n) + \delta \quad \text{and} \quad \eta_2(B) \geq \eta_2(F) > \eta_2(\mathcal{Q}_n) - \delta.$$

The relation (12.1.27) gives that $\eta_1(\mathcal{Q}_n) \leq C_2 \eta_2(\mathcal{Q}_n)$, since \mathcal{Q}_n is a (disjoint) union of elements of \mathcal{Q}_n . Combining these three inequalities, we obtain

$$\eta_1(B) < C_2(\eta_2(B) + \delta) + \delta.$$

Since δ is arbitrary, we conclude that $\eta_1(B) \leq C_2 \eta_2(B)$ for every measurable set $B \subset M$. Reversing the roles of the two measures, we also get $\eta_2(B) \leq C_2 \eta_1(B)$ for every measurable set $B \subset M$.

These inequalities prove that any two equilibrium states are equivalent measures, with Radon–Nikodym derivatives bounded from zero and infinity.

Combining Lemmas 4.3.3 and 12.1.18 we get that all the *ergodic* equilibrium states are equal. Now, by Proposition 10.5.5, the connected components of any equilibrium state are also equilibrium states (ergodic, of course). It follows that there exists a unique equilibrium state, as stated.

There is an alternative proof of the fact that the equilibrium state is unique that does not use Proposition 10.5.5 and, thus, does not require the theorem of Jacobs. Indeed, the results in the next section imply that the equilibrium state $\mu = h\nu$ in Section 12.1.4 is ergodic. By Lemma 12.1.18, that implies that all the equilibrium states are ergodic. Using Lemma 4.3.3, it follows that all the equilibrium states must coincide.

As a consequence, the reference measure ν is also unique: if there were two distinct reference measures, ν_1 and ν_2 , then $\mu_1 = h\nu_1$ and $\mu_2 = h\nu_2$ would be distinct equilibrium states. Analogously, the positive eigenfunction h is unique up to multiplication by a positive constant.

12.1.7 Exactness

Finally, let us prove that the system (f, μ) is exact. Recall that this means that if $B \subset M$ is such that there exist measurable sets B_n satisfying $B = f^{-n}(B_n)$ for every $n \geq 1$, then B has measure 0 or measure 1.

Let B be such a subset of M and assume that $\mu(B) > 0$. Let \mathcal{P} be a finite partition of M by subsets with non-empty interior and diameter less than ρ . For each n , let \mathcal{Q}_n be the partition of M whose elements are the images $h^n(P)$ of the sets $P \in \mathcal{P}$ under the inverse branches h^n of the iterate f^n .

Lemma 12.1.19. *For every $\varepsilon > 0$ and every $n \geq 1$ sufficiently large there exists some $h^n(P) \in \mathcal{Q}_n$ such that*

$$\mu(B \cap h^n(P)) > (1 - \varepsilon)\mu(h^n(P)). \tag{12.1.28}$$

Proof. Fix $\varepsilon > 0$. Since the measure μ is regular (Proposition A.3.2), given any $\delta > 0$ there exist some compact set $F \subset B$ and some open set $A \supset B$ satisfying $\mu(A \setminus F) < \delta$. Since we assume that $\mu(B) > 0$, this inequality implies that $\mu(F) > (1 - \varepsilon)\mu(A)$, as long as $\delta > 0$ is sufficiently small. Fix δ from now on. Note that $\text{diam } \mathcal{Q}_n < \sigma^{-n}\rho$. Then, for every n sufficiently large, any element $h^n(P)$ of \mathcal{Q}_n that intersects F is contained in A . By contradiction, suppose that (12.1.28) is false for every $h^n(P)$. Then, adding over all the $h^n(P)$ that intersect F ,

$$\begin{aligned} \mu(F) &\leq \sum_{P, h^n} \mu(F \cap h^n(P)) \leq \sum_{P, h^n} \mu(B \cap h^n(P)) \\ &\leq (1 - \varepsilon) \sum_{P, h^n} \mu(h^n(P)) \leq (1 - \varepsilon)\mu(A). \end{aligned}$$

This contradiction proves that (12.1.28) is valid for some $h^n(P) \in \mathcal{Q}_n$.

Consider any $h^n(P) \in \mathcal{Q}_n$ such that (12.1.28). Since $B = f^{-n}(B_n)$ and $f^n \circ h^n = \text{id}$ in its domain, we have that $f^n(h^n(P) \setminus B) = P \setminus B_n$. Then, applying Corollary 12.1.17 to the measure $\eta = \mu$,

$$\mu(P \setminus B_n) = \int_{h^n(P) \setminus B} J_\mu f^n d\mu \leq K_7 \mu(h^n(P) \setminus B) J_\mu f^n(x) \tag{12.1.29}$$

and $\mu(P) = \int_{h^n(P)} J_\mu f^n d\mu \geq K_7^{-1} \mu(h^n(P)) J_\mu f^n(x)$

for any $x \in h^n(P)$. Combining (12.1.28) and (12.1.29),

$$\frac{\mu(P \setminus B_n)}{\mu(P)} \leq K_7^2 \frac{\mu(h^n(P) \setminus B)}{\mu(h^n(P))} \leq K_7^2 \varepsilon.$$

In this way we have shown that, given any $\varepsilon > 0$ and any $n \geq 1$ sufficiently large, there exists some $P \in \mathcal{P}$ such that $\mu(P \setminus B_n) \leq K_7^2 \varepsilon \mu(P)$.

Since the partition \mathcal{P} is finite, it follows that there exist some $P \in \mathcal{P}$ and some sequence $(n_j)_j \rightarrow \infty$ such that

$$\mu(P \setminus B_{n_j}) \rightarrow 0 \quad \text{when } j \rightarrow \infty. \tag{12.1.30}$$

Let P be fixed from now on. Since, by assumption, P has non-empty interior and f is topologically exact, there exists $N \geq 1$ such that $f^N(P) = M$. Let $P = P_1 \cup \dots \cup P_s$ be a finite partition of P into domains of invertibility of f^N . Corollaries 12.1.9 and 12.1.16 give that $J_\mu f^N = \lambda^N e^{-\varphi_N} (h \circ f^N)/f$ is bounded from zero and infinity. Note also that $f^N(P_i \setminus B_{n_j}) = f^N(P_i) \setminus B_{n_j+N}$, because $f^{-n}(B_n) = B$ for every n . Combining these two observations with (12.1.30), we find that, given any $i = 1, \dots, s$, the sequence

$$\mu(f^N(P_i) \setminus B_{n_j+N}) = \mu(f^N(P_i \setminus B_{n_j})) = \int_{P_i \setminus B_{n_j}} J_\mu f^N d\mu$$

converges to zero when $j \rightarrow \infty$. Now, $\{f^N(P_i) : i = 1, \dots, s\}$ is a finite cover of M by measurable sets. Therefore, this last conclusion implies that $\mu(M \setminus B_{n_j+N})$ converges to zero, that is, $\mu(B) = \mu(B_{n_j+N})$ converges to 1 when $j \rightarrow \infty$. That means that $\mu(B) = 1$, of course.

The proof of Theorem 12.1 is complete.

12.1.8 Absolutely continuous measures

In this last section on the theorem of Ruelle we briefly discuss the special case when $f : M \rightarrow M$ is a local diffeomorphism on a compact Riemannian manifold and $\varphi = -\log |\det Df|$. It is assumed that f is such that this potential φ is Hölder. The first goal is to compare the conclusions of the theorem of Ruelle in this case with the results in Section 11.1:

Proposition 12.1.20. *The invariant absolutely continuous probability measure coincides with the equilibrium state μ of the potential $\varphi = -\log |\det Df|$.*

Consequently, it is equivalent to the Lebesgue measure m , with density $d\mu/dm$ Hölder and bounded from zero and infinity, and it is exact.

Proof. We saw in Example 12.1.2 that the Lebesgue measure m is an eigenvector of the dual \mathcal{L}^* of the transfer operator associated with the potential $\varphi = -\log |\det Df|$: more precisely,

$$\mathcal{L}^*m = m.$$

Applying the previous theory (from Lemma 12.1.3 on) with $\lambda = 1$ and $\nu = m$, we find a Hölder function $h : M \rightarrow \mathbb{R}$, bounded from zero and infinity, such that $\mathcal{L}h = h$ and the measure $\mu = hm$ is the equilibrium state of the potential φ . Recalling Corollary 11.1.15, it follows that μ is also the unique probability measure invariant under f and absolutely continuous with respect to m . The fact that h is positive implies that μ and m are equivalent measures. Exactness was proven in Section 12.1.7.

It is worthwhile pointing out that, while the absolutely continuous invariant measure is unique (Theorem 11.1.2), the potential $\varphi = -\log |\det Df|$ depends on the choice of the Riemannian metric on M , because the determinant does. So, Proposition 12.1.20 also implies that all the potentials of this form, corresponding to different choices of the Riemannian metric, have the same equilibrium state. This type of situation is the subject of Section 12.2 and, in particular, Exercise 12.2.3.

It also follows from the proof of Theorem 12.1 that

$$h_\mu(f) - \int \log |\det Df| d\mu = P(f, -\log |\det Df|) = \log \lambda = 0. \tag{12.1.31}$$

Let $\tilde{\varphi}$ be the time average of the function φ , given by the Birkhoff ergodic theorem. Then,

$$\int \log |\det Df| d\mu = \int -\varphi d\mu = \int -\tilde{\varphi} d\mu. \tag{12.1.32}$$

Moreover,

$$-\tilde{\varphi}(x) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \log |\det Df(f^j(x))| = \lim_n \frac{1}{n} \log |\det Df^n(x)| \tag{12.1.33}$$

at μ -almost every point. In the context of our comments about the Oseledets theorem (see the relation (c1) in Section 3.3.5) we mentioned that

$$\lim_n \frac{1}{n} \log |\det Df^n(x)| = \sum_{i=1}^{k(x)} d_i(x) \lambda_i(x), \tag{12.1.34}$$

where $\lambda_1(x), \dots, \lambda_{k(x)}(x)$ are the Lyapunov exponents of the transformation f at the point x and $d_1(x), \dots, d_{k(x)}(x)$ are the corresponding multiplicities.

Combining the relations (12.1.31)–(12.1.34), we find that

$$h_\mu(f) = \int \left(\sum_{i=1}^{k(x)} d_i(x) \lambda_i(x) \right) d\mu(x). \tag{12.1.35}$$

Since these functions are invariant (see the relation (a1) in Section 3.3.5) and the measure μ is ergodic, the functions $k(x)$, $\lambda_i(x)$ and $d_i(x)$ are constant at μ -almost every point. Let us denote by k , λ_i and d_i these constants. Then (12.1.35) translates into the following theorem:

Theorem 12.1.21. *Let $f : M \rightarrow M$ be an expanding map on a compact Riemannian manifold, such that the derivative Df is Hölder. Let μ be the unique invariant probability measure absolutely continuous with respect to the Lebesgue measure on M . Then*

$$h_\mu(f) = \sum_{i=1}^k d_i \lambda_i, \tag{12.1.36}$$

where λ_i , $i = 1, \dots, k$ are the Lyapunov exponents of f at μ -almost every point and d_i , $i = 1, \dots, k$ are the corresponding multiplicities.

As we pointed out before, in Section 9.4.4, this is a special instance of the Pesin entropy formula (Theorem 9.4.5).

12.1.9 Exercises

- 12.1.1. Show that if η is a Borel measure on a compact metric space then for every $\varepsilon > 0$ there exists $b > 0$ such that $\eta(B(y, \varepsilon)) > b$ for every $y \in \text{supp } \eta$.
- 12.1.2. Let $f : M \rightarrow M$ be an expanding map. Consider the non-linear operator $G : \mathcal{M}_1(M) \rightarrow \mathcal{M}_1(M)$ defined in the space $\mathcal{M}_1(M)$ of all Borel probability measures by

$$G(\eta) = \frac{\mathcal{L}^*(\eta)}{\int \mathcal{L}1 d\eta}.$$

Use the Tychonoff–Schauder theorem (Theorem 2.2.3) to show that G admits some fixed point and deduce Lemma 12.1.1.

- 12.1.3. Let $\sigma : \Sigma_A \rightarrow \Sigma_A$ be the one-sided shift of finite type associated with a given transition matrix A (recall Section 10.2.2). Let P be a stochastic matrix such that $P_{ij} = 0$ whenever $A_{ij} = 0$ and p be a probability vector with positive coefficients such that $P^*p = p$. Consider the transfer operator \mathcal{L} associated with the locally constant potential

$$\varphi(i_0, i_1, \dots, i_n, \dots) = -\log \frac{P_{i_1}}{P_{i_0} P_{i_0, i_1}}.$$

Show that the Markov measure μ associated with the matrix P and the vector p satisfies $\mathcal{L}^*\mu = \mu$.

- 12.1.4. Let λ be any positive number and ν be a Borel probability measure such that $\mathcal{L}^*\nu = \lambda\nu$. Show that, given any $u \in L^1(\nu)$ and any continuous

function $v : M \rightarrow \mathbb{R}$,

$$\int (u \circ f)v \, dv = \int u(\lambda^{-1}\mathcal{L}v) \, dv.$$

12.2 Theorem of Livšic

Now we discuss the following issue: when is it the case that two different Hölder potentials ϕ and ψ have the same equilibrium state? Observe that, since these are ergodic measures, the two equilibrium states μ_ϕ and μ_ψ either coincide or are mutually singular (by Lemma 4.3.3).

Recall that two potentials are said to be cohomologous (with respect to f) if the difference between them may be written as $u \circ f - u$ for some continuous function $u : M \rightarrow \mathbb{R}$.

Theorem 12.2.1 (Livšic). *A potential $\varphi : M \rightarrow \mathbb{R}$ is cohomologous to zero if and only if $\varphi_n(x) = 0$ for every $x \in \text{Fix}(f^n)$ and every $n \geq 1$.*

Proof. It is clear that if $\varphi = u \circ f - u$ for some u then

$$\varphi_n(x) = \sum_{j=1}^n u(f^j(x)) - \sum_{j=0}^{n-1} u(f^j(x)) = 0$$

for every $x \in M$ such that $f^n(x) = x$. The converse is a lot more interesting.

Suppose that $\varphi_n(x) = 0$ for every $x \in \text{Fix}(f^n)$ and every $n \geq 1$. Consider any point $z \in M$ whose orbit is dense in M ; such a point exists because f is topologically exact and, consequently, transitive. Define the function u on the orbit of z through the following relation:

$$u(f^n(z)) = u(z) + \varphi_n(z), \tag{12.2.1}$$

where $u(z)$ is arbitrary. Observe that

$$u(f^{n+1}(z)) - u(f^n(z)) = \varphi_{n+1}(z) - \varphi_n(z) = \varphi(f^n(z)) \tag{12.2.2}$$

for every $n \geq 0$. In other words, the cohomology relation

$$\phi - \psi = u \circ f - u \tag{12.2.3}$$

holds on the orbit of z . To extend this relation to the whole of M , we use the following fact:

Lemma 12.2.2. *The function u is uniformly continuous on the orbit of z .*

Proof. Given $\varepsilon \in (0, \rho)$, take $\delta > 0$ given by the shadowing lemma (Proposition 11.2.9). Suppose that $k \geq 0$ and $l \geq 1$ are such that $d(f^k(z), f^{k+l}(z)) < \delta$. Then the periodic sequence $(x_n)_n$ of period l given by

$$x_0 = f^k(z), x_1 = f^{k+1}(z), \dots, x_{l-1} = f^{k+l-1}(z), x_l = f^k(z)$$

is a δ -pseudo-orbit. Hence, by Proposition 11.2.9, there exists $x \in \text{Fix}(f^l)$ such that $d(f^j(x), f^{k+j}(z)) < \varepsilon$ for every $j \geq 0$. Since we took $\varepsilon < \rho$, this also implies that $x = h_l(f^l(x))$, where $h_l : B(f^{k+l}(z), \rho) \rightarrow M$ denotes the inverse branch of f^l that maps $f^{k+l}(z)$ to $f^k(z)$. By (11.2.6), it follows that

$$d(f^j(x), f^{k+j}(z)) \leq \sigma^{j-l} d(f^l(x), f^{k+l}(z)) \quad \text{for every } 0 \leq j \leq l. \tag{12.2.4}$$

By the definition (12.2.1),

$$u(f^{k+l}(z)) - u(f^k(z)) = \varphi_{k+l}(z) - \varphi_k(z) = \varphi_l(f^k(z)). \tag{12.2.5}$$

Fix constants $C > 0$ and $\nu > 0$ such that $|\varphi(x) - \varphi(y)| \leq Cd(x, y)^\nu$ for any $x, y \in M$. Then,

$$|\varphi_l(f^k(z)) - \varphi_l(x)| \leq \sum_{j=0}^{j-1} |\varphi(f^{k+j}(z)) - \varphi(f^j(x))| \leq \sum_{j=0}^{j-1} Cd(f^j(x), f^{k+j}(z))^\nu.$$

Using (12.2.4), it follows that

$$|\varphi_l(f^k(z)) - \varphi_l(x)| \leq \sum_{j=0}^{j-1} C\sigma^{\nu(j-l)} d(x, f^{k+l}(z))^\nu \leq C_1 \varepsilon^\nu, \tag{12.2.6}$$

where $C_1 = C \sum_{i=0}^\infty \sigma^{-i\nu}$. Recall that, by assumption, $\psi_l(x) = 0$. Hence, combining (12.2.5) and (12.2.6), we find that $|u(f^{k+l}(z)) - u(f^k(z))| \leq C_1 \varepsilon^\nu$. This completes the proof of the lemma.

It follows from Lemma 12.2.2 that u admits a (unique) continuous extension to the closure of the orbit of z , that is, the ambient space M . Then, by continuity of φ and u , the cohomology relation (12.2.3) extends to the whole M . This proves Theorem 12.2.1.

Theorem 12.2.3. *Let $f : M \rightarrow M$ be a topologically exact expanding map on a compact metric space and ϕ and ψ be two Hölder potentials in M . The following conditions are equivalent:*

- (i) $\mu_\phi = \mu_\psi$;
- (ii) there exist $c \in \mathbb{R}$ and an arbitrary function $u : M \rightarrow \mathbb{R}$ such that $\phi - \psi = c + u \circ f - u$;
- (iii) $\phi - \psi$ is cohomologous to some constant $c \in \mathbb{R}$;
- (iv) there exist $c \in \mathbb{R}$ and a Hölder function $u : M \rightarrow \mathbb{R}$ such that $\phi - \psi = c + u \circ f - u$;
- (v) there exists $c \in \mathbb{R}$ such that $\phi_n(x) - \psi_n(x) = cn$ for every $x \in \text{Fix}(f^n)$ and $n \geq 1$.

Moreover, the constants $c \in \mathbb{R}$ in (ii), (iii), (iv) and (v) coincide; indeed, they are all equal to $P(f, \phi) - P(f, \psi)$.

Proof. It is clear that (iv) implies (iii) and (iii) implies (ii).

If $\phi - \psi = c + u \circ f - u$ for some function u then, given any $x \in \text{Fix}(f^n)$,

$$\phi_n(x) - \psi_n(x) = \sum_{j=0}^{n-1} (\phi - \psi)(f^j(x)) = \sum_{j=0}^{n-1} (c + u(f^{j+1}(x)) - u(f^j(x))).$$

Since $f^n(x) = x$, the sum of the last two terms over every $j = 0, \dots, n - 1$ vanishes. Therefore, $\phi_n(x) - \psi_n(x) = cn$. This proves that (ii) implies (v).

Suppose that $\phi_n(x) - \psi_n(x) = cn$ for every $x \in \text{Fix}(f^n)$ and every $n \geq 0$. That means that the function $\varphi = \phi - \psi - c$ satisfies $\varphi_n(x) = 0$ for every $x \in \text{Fix}(f^n)$ and every $n \geq 0$. Note also that φ is Hölder. Hence, by Theorem 12.2.1, there exists a continuous function $u : M \rightarrow \mathbb{R}$ such that $\varphi = u \circ f - u$. In other words, $\phi - \psi$ is cohomologous to c . This shows that (v) implies (iii).

It follows from (10.3.4) and Proposition 10.3.8 that if ϕ is cohomologous to $\psi + c$ then

$$P(f, \phi) = P(f, \psi + c) = P(f, \psi) + c.$$

On the other hand, given any invariant probability measure ν ,

$$h_\nu(f) + \int \phi \, d\nu = h_\nu(f) + \int (\psi + c) \, d\nu = h_\nu(f) + \int \psi \, d\nu + c.$$

Therefore, ν is an equilibrium state for ϕ if and only if ν is an equilibrium state for ψ . This shows that (iii) implies (i).

If μ_ϕ and μ_ψ coincide then they have the same Jacobian, of course. By Lemma 12.1.12, this means that

$$\lambda_\phi e^{-\phi} \frac{h_\phi \circ f}{h_\phi} = \lambda_\psi e^{-\psi} \frac{h_\psi \circ f}{h_\psi}. \tag{12.2.7}$$

Let $c = \log \lambda_\phi - \log \lambda_\psi$ and $u = \log h_\phi - \log h_\psi$. Both are well defined, since $\lambda_\phi, \lambda_\psi, h_\phi$ and h_ψ are all positive. Moreover, since the functions h_ϕ and h_ψ are Hölder and bounded from zero and infinity (Corollary 12.1.9), the function u is Hölder. Finally, (12.2.7) may be rewritten as follows:

$$\phi - \psi = c + \log u \circ f - u.$$

This shows that (i) implies (iv). The proof of the theorem is complete.

Here is an interesting consequence in the differentiable setting:

Corollary 12.2.4. *Let $f : M \rightarrow M$ be a differentiable expanding map on a compact Riemannian manifold such that the Jacobian $\det Df$ is Hölder. The absolutely continuous invariant probability measure μ coincides with the measure of maximum entropy if and only if there exists $c \in \mathbb{R}$ such that*

$$|\det Df^n(x)| = e^{cn} \quad \text{for every } x \in \text{Fix}(f^n) \text{ and every } n \geq 1.$$

Proof. As we saw in Proposition 12.1.20, μ is the equilibrium state of the potential $\varphi = -\log |\det Df|$. It is clear that the measure of maximum entropy

μ_0 is the equilibrium state of the zero function. Observe that

$$\varphi_n(x) = \sum_{j=0}^{n-1} \log |\det Df(f^j(x))| = \log |\det Df^n(x)|.$$

Therefore, Theorem 12.2.3 gives that $\mu = \mu_0$ if and only if there exists some number $c \in \mathbb{R}$ such that $\log |\det Df^n(x)| = 0 + cn$ for every $x \in \text{Fix}(f^n)$ and every $n \geq 1$.

12.2.1 Exercises

- 12.2.1. Consider the two-sided shift $\sigma : \Sigma \rightarrow \Sigma$ in $\Sigma = \{1, \dots, d\}^{\mathbb{Z}}$. Show that for every Hölder function $\varphi : \Sigma \rightarrow \mathbb{R}$, there exists a Hölder function $\varphi^+ : \Sigma \rightarrow \mathbb{R}$, cohomologous to φ and such that $\varphi^+(x) = \varphi^+(y)$ whenever $x = (x_i)_{i \in \mathbb{Z}}$ and $y = (y_i)_{i \in \mathbb{Z}}$ are such that $x_i = y_i$ for $i \geq 0$.
- 12.2.2. Prove that if the functions $\varphi, \psi : M \rightarrow \mathbb{R}$ are such that there exist constants C, L satisfying $|\varphi_n(x) - \psi_n(x) - nC| \leq L$ for every $x \in M$, then $P(f, \varphi) = P(f, \psi) + C$ and φ is cohomologous to $\psi + C$.
- 12.2.3. Let $f : M \rightarrow M$ be a differentiable expanding map on a compact manifold, with Hölder derivative. Check that any two potentials of the form $\varphi = -\log |\det Df|$, for two different choices of a Riemannian metric on M , are cohomologous. [Observation: In particular, all such potentials have the same equilibrium state, namely, the absolutely continuous invariant probability measure. This was observed before, in Section 12.1.8.]
- 12.2.4. Given $k \geq 2$, let $f : S^1 \rightarrow S^1$ be the (expanding) map given by $f(x) = kx \pmod{\mathbb{Z}}$. Let $g : S^1 \rightarrow S^1$ be a differentiable expanding map of degree k . Show that f and g are topologically conjugate.
- 12.2.5. Given $k \geq 2$, let $f : S^1 \rightarrow S^1$ be the map given by $f(x) = kx \pmod{\mathbb{Z}}$. Let $g : S^1 \rightarrow S^1$ be a differentiable expanding map of degree k , with Hölder derivative. Show that the following conditions are equivalent:
 - (a) f and g are conjugated by some diffeomorphism;
 - (b) f and g are conjugated by some absolutely continuous homeomorphism whose inverse is also absolutely continuous;
 - (c) $(g^n)'(p) = k^n$ for every $p \in \text{Fix}(f^n)$.

12.3 Decay of correlations

Let $f : M \rightarrow M$ be a topologically exact expanding map and $\varphi : M \rightarrow \mathbb{R}$ be a Hölder potential. As before, we denote by ν the reference measure (Section 12.1.1) and by μ the equilibrium state (Section 12.1.4) of the potential φ . Recall that $\mu = h\nu$, where the function h is bounded from zero and infinity (Corollary 12.1.9). In particular, $L^1(\mu) = L^1(\nu)$.

Given $b > 0$ and $\beta > 0$, we say that a function $g : M \rightarrow \mathbb{R}$ is (b, β) -Hölder if

$$|g(x) - g(y)| \leq bd(x, y)^\beta \quad \text{for any } x, y \in M. \tag{12.3.1}$$

We say that g is β -Hölder if it is (b, β) -Hölder for some $b > 0$. Then we denote by $H_\beta(g)$ the smallest of such constants b . Moreover, fixing $\rho > 0$ as in (11.2.1), we denote by $H_{\beta, \rho}(g)$ the smallest constant b such that the inequality in (12.3.1) holds for any $x, y \in M$ with $d(x, y) < \rho$.

The correlations sequence of two functions g_1 and g_2 , with respect to the invariant measure μ , was defined in (7.1.1):

$$C_n(g_1, g_2) = \left| \int (g_1 \circ f^n) g_2 d\mu - \int g_1 d\mu \int g_2 d\mu \right|.$$

We also consider a similar notion for the reference measure ν :

$$B_n(g_1, g_2) = \left| \int (g_1 \circ f^n) g_2 d\nu - \int g_1 d\mu \int g_2 d\nu \right|.$$

In this section we prove that these sequences decay exponentially.

Theorem 12.3.1 (Exponential convergence to equilibrium). *Given $\beta \in (0, \alpha]$, there exists $\Lambda < 1$ and for every β -Hölder function $g_2 : M \rightarrow \mathbb{C}$ there exists $K_1(g_2) > 0$ such that*

$$B_n(g_1, g_2) \leq K_1(g_2) \Lambda^n \int |g_1| d\nu \quad \text{for every } g_1 \in L_1(\nu) \text{ and every } n \geq 1.$$

The proof is presented in Sections 12.3.1 through 12.3.3. It provides an explicit expression for the factor $K_1(g_2)$. Observe also that

$$B_n(g_1, g_2) = \left| \int g_1 d(f_*^n(g_2\nu)) - \int g_1 d\left(\mu \int g_2 d\nu\right) \right|.$$

Then, the conclusion of Theorem 12.3.1 may be interpreted as follows: *the iterates of any measure of the form $g_2\nu$ converge to the invariant measure $\mu \int g_2 d\nu$ exponentially fast.*

Theorem 12.3.2 (Exponential decay of correlations). *For every $\beta \in (0, \alpha]$ there exists $\Lambda < 1$ and for every β -Hölder function $g_2 : M \rightarrow \mathbb{C}$ there exists $K_2(g_2) > 0$ such that*

$$C_n(g_1, g_2) \leq K_2(g_2) \Lambda^n \int |g_1| d\mu \quad \text{for every } g_1 \in L_1(\mu) \text{ and every } n \geq 1.$$

In particular, given any pair g_1 and g_2 of β -Hölder functions, there exists $K(g_1, g_2) > 0$ such that $C_n(g_1, g_2) \leq K(g_1, g_2) \Lambda^n$ for every $n \geq 1$.

Proof. Recall that $\mu = h\nu$ and, according to Corollary 12.1.9, the function h is α -Hölder and satisfies $K_5^{-1} \leq h \leq K_5$ for some $K_5 > 0$. Hence (see Exercise 12.3.5), g_2 is β -Hölder if and only if g_2h is β -Hölder. Moreover,

$$\begin{aligned} C_n(g_1, g_2) &= \int (g_1 \circ f^n) g_2 d\mu - \int g_1 d\mu \int g_2 d\mu \\ &= \int (g_1 \circ f^n)(g_2h) d\nu - \int g_1 d\mu \int (g_2h) d\nu = B_n(g_1, g_2h). \end{aligned}$$

Therefore, it follows from Theorem 12.3.1 that

$$C_n(g_1, g_2) \leq K_1(g_2h)\Lambda^n \int |g_1| dv \leq K_1(g_2h)/K_5 \Lambda^n \int |g_1| d\mu.$$

This proves the first part of the theorem, with $K_2(g_2) = K_1(g_2h)/K_5$. The second part is an immediate consequence: if g_1 is β -Hölder then $g_1 \in L^1(\mu)$ and it suffices to take $K(g_1, g_2) = K_2(g_2) \int |g_1| d\mu$.

Before we move to prove Theorem 12.3.1, let us make a few quick comments. The issue of decay of correlations was already discussed in Section 7.4, from the viewpoint of the spectral gap property. Here we introduce a different approach. The proof of the theorem that we are going to present is based on the notion of *projective distance* associated with a cone, which was introduced by Garret Birkhoff [Bir67]. This tool allows us to obtain exponential convergence to equilibrium (which yields exponential decay of correlations, as we have just shown) without having to analyze the spectrum of the transfer operator. Incidentally, this can also be used to deduce that the spectral gap property does hold in the present context. We will come back to this theme near the end of Section 12.3.

12.3.1 Projective distances

Let E be a Banach space. We call a *cone* any convex subset C of E such that

$$tC \subset C \text{ for every } t > 0 \text{ and } \bar{C} \cap (-\bar{C}) = \{0\}, \tag{12.3.2}$$

where \bar{C} denotes the closure of C (previously we considered only closed cones but at this point it is convenient to loosen that requirement). Given $v_1, v_2 \in C$, define

$$\alpha(v_1, v_2) = \sup\{t > 0 : v_2 - tv_1 \in C\} \text{ and } \beta(v_1, v_2) = \inf\{s > 0 : sv_1 - v_2 \in C\}.$$

Figure 12.1 helps illustrate the geometric meaning of these numbers. By convention, $\alpha(v_1, v_2) = 0$ if $v_2 - tv_1 \notin C$ for every $t > 0$ and $\beta(v_1, v_2) = +\infty$ if $sv_1 - v_2 \notin C$ for every $s > 0$.

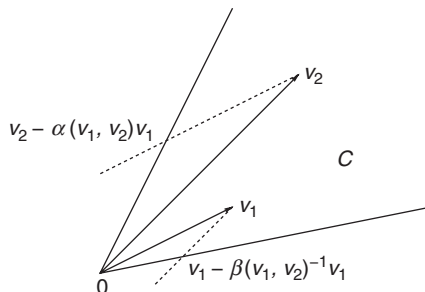


Figure 12.1. Defining the projective distance in a cone C

Note that $\alpha(v_1, v_2)$ is always finite. Indeed, $\alpha(v_1, v_2) = +\infty$ would mean that there exists a sequence $(t_n)_n \rightarrow +\infty$ with $v_2 - t_n v_1 \in C$ for every n . Then, $s_n = 1/t_n$ would be a sequence of positive numbers converging to zero and such that $s_n v_2 - v_1 \in C$ for every n . This would imply that $-v_1 \in \bar{C}$, which would contradict the second condition in (12.3.2). An analogous argument shows that $\beta(v_1, v_2)$ is always positive: $\beta(v_1, v_2) = 0$ would imply $-v_2 \in \bar{C}$.

Given any cone $C \subset E$ and any $v_1, v_2 \in C \setminus \{0\}$, we define

$$\theta(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)}, \tag{12.3.3}$$

with $\theta(v_1, v_2) = +\infty$ whenever $\alpha(v_1, v_2) = 0$ or $\beta(v_1, v_2) = +\infty$. The remarks in the previous paragraph ensure that $\theta(v_1, v_2)$ is always well defined. We call θ the *projective distance* associated with the cone C . This terminology is justified by the proposition that follows, which shows that θ defines a distance in the projective quotient of $C \setminus \{0\}$, that is, in the set of equivalence classes of the relation \sim defined by $v_1 \sim v_2 \Leftrightarrow v_1 = tv_2$ for some $t > 0$.

Proposition 12.3.3. *If C is a cone then*

- (i) $\theta(v_1, v_2) = \theta(v_2, v_1)$ for any $v_1, v_2 \in C$;
- (ii) $\theta(v_1, v_2) + \theta(v_2, v_3) \geq \theta(v_1, v_3)$ for any $v_1, v_2, v_3 \in C$;
- (iii) $\theta(v_1, v_2) \geq 0$ for any $v_1, v_2 \in C$;
- (iv) $\theta(v_1, v_2) = 0$ if and only if there exists $t > 0$ such that $v_1 = tv_2$;
- (v) $\theta(t_1 v_1, t_2 v_2) = \theta(v_1, v_2)$ for any $v_1, v_2 \in C$ and $t_1, t_2 > 0$.

Proof. If $\alpha(v_2, v_1) > 0$ then

$$\begin{aligned} \alpha(v_2, v_1) &= \sup\{t > 0 : v_1 - tv_2 \in C\} = \sup\left\{t > 0 : \frac{1}{t} v_1 - v_2 \in C\right\} \\ &= (\inf\{s > 0 : sv_1 - v_2 \in C\})^{-1} = \beta(v_1, v_2)^{-1}. \end{aligned}$$

Moreover,

$$\begin{aligned} \alpha(v_2, v_1) = 0 &\Leftrightarrow v_1 - tv_2 \notin C \text{ for every } t > 0 \\ &\Leftrightarrow sv_1 - v_2 \notin C \text{ for every } s > 0 \Leftrightarrow \beta(v_1, v_2) = +\infty. \end{aligned}$$

Therefore, $\alpha(v_2, v_1) = \beta(v_1, v_2)^{-1}$ in all cases. Exchanging the roles of v_1 and v_2 , we also get that $\beta(v_2, v_1) = \alpha(v_1, v_2)^{-1}$ for any $v_1, v_2 \in C$. Part (i) of the proposition is an immediate consequence of these observations.

Next, we claim that $\alpha(v_1, v_2)\alpha(v_2, v_3) \leq \alpha(v_1, v_3)$ for any $v_1, v_2, v_3 \in C$. This is obvious if $\alpha(v_1, v_2) = 0$ or $\alpha(v_2, v_3) = 0$; therefore, we may suppose that $\alpha(v_1, v_2) > 0$ and $\alpha(v_2, v_3) > 0$. Then, by definition, there exist increasing sequences of positive numbers $(r_n)_n \rightarrow \alpha(v_1, v_2)$ and $(s_n)_n \rightarrow \alpha(v_2, v_3)$ such that

$$v_2 - r_n v_1 \in C \quad \text{and} \quad v_3 - s_n v_2 \in C \quad \text{for every } n \geq 1.$$

Since C is convex, it follows that $v_3 - s_n r_n v_1 \in C$ and so $s_n r_n \leq \alpha(v_1, v_3)$, for every $n \geq 1$. Passing to the limit as $n \rightarrow +\infty$, we get the claim. An analogous argument shows that $\beta(v_1, v_2)\beta(v_2, v_3) \geq \beta(v_1, v_3)$ for any $v_1, v_2, v_3 \in C$. Part (ii) of the proposition follows immediately from these inequalities.

Part (iii) means, simply, that $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$ for any $v_1, v_2 \in C$. To prove this fact, consider $t > 0$ and $s > 0$ such that $v_2 - tv_1 \in C$ and $sv_1 - v_2 \in C$. Then, by convexity, $(s - t)v_1 \in C$. If $s - t$ were negative, then we would have $-v_1 \in C$, which would contradict the last part of (12.3.2). Therefore, $s \geq t$ for any t and s as above. This implies that $\alpha(v_1, v_2) \leq \beta(v_1, v_2)$.

Let $v_1, v_2 \in C$ be such that $\theta(v_1, v_2) = 0$. Then, $\alpha(v_1, v_2) = \beta(v_1, v_2) = \gamma$ for some $\gamma \in (0, +\infty)$. Hence, there exist an increasing sequence $(t_n)_n \rightarrow \gamma$ and a decreasing sequence $(s_n)_n \rightarrow \gamma$ with

$$v_2 - t_n v_1 \in C \quad \text{and} \quad s_n v_1 - v_2 \in C \quad \text{for every } n \geq 1.$$

Writing $v_2 - t_n v_1 = (v_2 - \gamma v_1) + (\gamma - t_n)v_1$, we conclude that $v_2 - \gamma v_1$ is in the closure \bar{C} of C . Analogously, $\gamma v_1 - v_2 \in \bar{C}$. By the second part of (12.3.2), it follows that $v_2 - \gamma v_1 = 0$. This proves part (iv) of the proposition.

Finally, consider any $t_1, t_2 > 0$ and $v_1, v_2 \in C$. By definition,

$$\alpha(t_1 v_1, t_2 v_2) = \frac{t_2}{t_1} \alpha(v_1, v_2) \quad \text{and} \quad \beta(t_1 v_1, t_2 v_2) = \frac{t_2}{t_1} \beta(v_1, v_2).$$

Hence, $\theta(t_1 v_1, t_2 v_2) = \theta(v_1, v_2)$, as stated in part (v) of the proposition.

Example 12.3.4. Consider the cone $C = \{(x, y) \in E : y > |x|\}$ in $E = \mathbb{R}^2$. The projective quotient of C may be identified with the interval $(-1, 1)$, through $(x, 1) \mapsto x$. Given $-1 < x_1 \leq x_2 < 1$, we have:

$$\begin{aligned} \alpha((x_1, 1), (x_2, 1)) &= \sup\{t > 0 : (x_2, 1) - t(x_1, 1) \in C\} \\ &= \sup\{t > 0 : 1 - t \geq |x_2 - tx_1|\} = \frac{1 - x_2}{1 - x_1}, \end{aligned}$$

$$\text{and} \quad \beta((x_1, 1), (x_2, 1)) = \frac{x_2 + 1}{x_1 + 1}.$$

Therefore,

$$\theta((x_1, 1), (x_2, 1)) = \log R(-1, x_1, x_2, 1), \tag{12.3.4}$$

where

$$R(a, b, c, d) = \frac{(c - a)(d - b)}{(b - a)(d - c)}$$

denotes the *cross-ratio* of four positive numbers $a < b \leq c < d$.

In Exercise 12.3.2 we invite the reader to check a similar fact when the interval is replaced by the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Example 12.3.5. Let $E = C^0(M)$ be the space of continuous functions on a compact metric space M . Consider the cone $C_+ = \{g \in E : g(x) > 0 \text{ for } x \in M\}$. For any $g_1, g_2 \in C_+$,

$$\begin{aligned} \alpha(g_1, g_2) &= \sup \left\{ t > 0 : (g_2 - tg_1)(x) > 0 \text{ for every } x \in M \right\} \\ &= \inf \left\{ \frac{g_2}{g_1}(x) : x \in M \right\} \end{aligned}$$

and $\beta(g_1, g_2) = \sup \left\{ \frac{g_2}{g_1}(x) : x \in M \right\}$.

Therefore,

$$\theta(g_1, g_2) = \log \frac{\sup(g_2/g_1)}{\inf(g_2/g_1)} = \log \sup \left\{ \frac{g_2(x)g_1(y)}{g_1(x)g_2(y)} : x, y \in M \right\}. \tag{12.3.5}$$

This projective distance is complete (Exercise 12.3.3) but that is not always the case (Exercise 12.3.4).

Next, we observe that the projective distance depends monotonically on the cone. Indeed, let C_1 and C_2 be two cones with $C_1 \subset C_2$ and let $\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot), \theta_i(\cdot, \cdot), i = 1, 2$ be the corresponding functions, as defined previously. It is clear from the definitions that, given any $v_1, v_2 \in C_2$,

$$\alpha_1(v_1, v_2) \leq \alpha_2(v_1, v_2) \quad \text{and} \quad \beta_1(v_1, v_2) \geq \beta_2(v_1, v_2)$$

and, consequently, $\theta_1(v_1, v_2) \geq \theta_2(v_1, v_2)$.

More generally, let C_1 and C_2 be cones in Banach spaces E_1 and E_2 , respectively, and let $L : E_1 \rightarrow E_2$ be a linear operator such that $L(C_1) \subset C_2$. Then,

$$\begin{aligned} \alpha_1(v_1, v_2) &= \sup \{ t > 0 : v_2 - tv_1 \in C_1 \} \\ &\leq \sup \{ t > 0 : L(v_2 - tv_1) \in C_2 \} \\ &= \sup \{ t > 0 : L(v_2) - tL(v_1) \in C_2 \} = \alpha_2(L(v_1), L(v_2)) \end{aligned}$$

and, analogously, $\beta_1(v_1, v_2) \geq \beta_2(L(v_1), L(v_2))$. Consequently,

$$\theta_2(L(v_1), L(v_2)) \leq \theta_1(v_1, v_2) \quad \text{for any } v_1, v_2 \in C_1. \tag{12.3.6}$$

Of course, the inequality (12.3.6) is not strict, in general. However, according to the next proposition, one does have a strict inequality whenever $L(C_1)$ has finite θ_2 -diameter in C_2 ; actually, in that case L is a contraction with respect to the projective distances θ_1 and θ_2 . Recall that the hyperbolic tangent is defined by

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}} \quad \text{for every } x \in \mathbb{R}.$$

Keep in mind that the function \tanh takes values in the interval $(0, 1)$.

Proposition 12.3.6. *Let C_1 and C_2 be cones in Banach spaces E_1 and E_2 , respectively, and let $L : E_1 \rightarrow E_2$ be a linear operator such that $L(C_1) \subset C_2$. Suppose that $D = \sup\{\theta_2(L(v_1), L(v_2)) : v_1, v_2 \in C_1\}$ is finite. Then*

$$\theta_2(L(v_1), L(v_2)) \leq \tanh\left(\frac{D}{4}\right)\theta_1(v_1, v_2) \quad \text{for any } v_1, v_2 \in C.$$

Proof. Let $v_1, v_2 \in C_1$. It is no restriction to suppose that $\alpha_1(v_1, v_2) > 0$ and $\beta_1(v_1, v_2) < +\infty$, otherwise $\theta_1(v_1, v_2) = +\infty$ and there is nothing to prove. Then there exist an increasing sequence $(t_n)_n \rightarrow \alpha_1(v_1, v_2)$ and a decreasing sequence $(s_n)_n \rightarrow \beta_1(v_1, v_2)$ such that

$$v_2 - t_n v_1 \in C_1 \quad \text{and} \quad s_n v_1 - v_2 \in C_1.$$

In particular, $\theta_2(L(v_2 - t_n v_1), L(s_n v_1 - v_2)) \leq D$ for every $n \geq 1$. Fix any $D_0 > D$. Then we may choose positive numbers T_n and S_n such that

$$\begin{aligned} L(s_n v_1 - v_2) - T_n L(v_2 - t_n v_1) &\in C_2 \quad \text{and} \\ S_n L(v_2 - t_n v_1) - L(s_n v_1 - v_2) &\in C_2, \end{aligned} \tag{12.3.7}$$

and $\log(S_n/T_n) \leq D_0$ for every $n \geq 1$. The first part of (12.3.7) gives that

$$(s_n + t_n T_n)L(v_1) - (1 + T_n)L(v_2) \in C_2$$

and, by definition of $\beta_2(\cdot, \cdot)$, this implies that

$$\beta_2(L(v_1), L(v_2)) \leq \frac{s_n + t_n T_n}{1 + T_n}.$$

Analogously, the second part of (12.3.7) implies that

$$\alpha_2(L(v_1), L(v_2)) \geq \frac{s_n + t_n S_n}{1 + S_n}.$$

Therefore, $\theta_2(L(v_1), L(v_2))$ cannot exceed

$$\log\left(\frac{s_n + t_n T_n}{1 + T_n} \cdot \frac{1 + S_n}{s_n + t_n S_n}\right) = \log\left(\frac{s_n/t_n + T_n}{1 + T_n} \cdot \frac{1 + S_n}{s_n/t_n + S_n}\right).$$

The last term may be rewritten as

$$\begin{aligned} \log\left(\frac{s_n}{t_n} + T_n\right) - \log(1 + T_n) - \log\left(\frac{s_n}{t_n} + S_n\right) + \log(1 + S_n) &= \\ = \int_0^{\log(s_n/t_n)} \left(\frac{e^x dx}{e^x + T_n} - \frac{e^x dx}{e^x + S_n}\right), \end{aligned}$$

and this expression is less than or equal to

$$\sup_{x>0} \frac{e^x(S_n - T_n)}{(e^x + T_n)(e^x + S_n)} \log\left(\frac{s_n}{t_n}\right).$$

Now we use the following elementary facts:

$$\sup_{y>0} \frac{y(S_n - T_n)}{(y + T_n)(y + S_n)} = \frac{1 - \sqrt{T_n/S_n}}{1 + \sqrt{T_n/S_n}} \leq \frac{1 - e^{-D_0/2}}{1 + e^{-D_0/2}} = \tanh \frac{D_0}{4}.$$

Indeed, the supremum is attained for $y = \sqrt{S_n T_n}$ and the inequality is a consequence of the fact that $\log(S_n/T_n) \leq D_0$. This proves that

$$\theta_2(L(v_1), L(v_2)) \leq \tanh\left(\frac{D_0}{4}\right) \log\left(\frac{s_n}{t_n}\right).$$

Note also that $\theta(v_1, v_2) = \lim_n \log(s_n/t_n)$, due to our choice of s_n and t_n . Hence, taking the limit when $n \rightarrow \infty$ and then making $D_0 \rightarrow D$, we obtain the conclusion of the proposition.

Example 12.3.7. Let C_+ be the cone of positive continuous functions in M . For each $L > 1$, let $C(L) = \{g \in C_+ : \sup |g| \leq L \inf |g|\}$. Then, $C(L)$ has finite diameter in C_+ , for every $L > 1$. Indeed, we have seen in Example 12.3.5 that the projective distance θ associated with C_+ is given by

$$\theta(g_1, g_2) = \log \sup \left\{ \frac{g_2(x)g_1(y)}{g_1(x)g_2(y)} : x, y \in M \right\}.$$

In particular, $\theta(g_1, g_2) \leq 2 \log L$ for any $g_1, g_2 \in C(L)$.

12.3.2 Cones of Hölder functions

Let $f : M \rightarrow M$ be a topologically exact expanding map and $\rho > 0$ and $\sigma > 1$ be the constants in the definition (11.2.1). Let $\mathcal{L} : C^0(M) \rightarrow C^0(M)$ be the transfer operator associated with a Hölder potential $\varphi : M \rightarrow M$. Fix constants $K_0 > 0$ and $\alpha > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq K_0 d(x, y)^\alpha \quad \text{for any } x, y \in M.$$

Given $b > 0$ and $\beta > 0$, we denote by $C(b, \beta)$ the set of positive functions $g \in C^0(M)$ whose logarithm is (b, β) -Hölder on balls of radius ρ , that is, such that

$$|\log g(x) - \log g(y)| \leq b d(x, y)^\beta \quad \text{whenever } d(x, y) < \rho. \tag{12.3.8}$$

Lemma 12.3.8. For any $b > 0$ and $\beta > 0$, the set $C(b, \beta)$ is a cone in the space $E = C^0(M)$ and the corresponding projective distance is given by

$$\theta(g_1, g_2) = \log \frac{\beta(g_1, g_2)}{\alpha(g_1, g_2)},$$

where $\alpha(g_1, g_2)$ is the infimum and $\beta(g_1, g_2)$ is the supremum of the set

$$\left\{ \frac{g_2}{g_1}(x), \frac{\exp(b d(x, y)^\beta) g_2(x) - g_2(y)}{\exp(b d(x, y)^\beta) g_1(x) - g_1(y)} : x \neq y \text{ and } d(x, y) < \rho \right\}.$$

Proof. It is clear that $g \in C$ implies $tg \in C$ for every $t > 0$. Moreover, the closure of C is contained in the set of non-negative functions and so $-\bar{C} \cap \bar{C}$ contains only the zero function. Now, to conclude that C is a cone, we only

have to check that it is convex. Consider any $g_1, g_2 \in C(b, \beta)$. The definition (12.3.8) means that

$$\exp(-bd(x, y)^\beta) \leq \frac{g_i(x)}{g_i(y)} \leq \exp(bd(x, y)^\beta)$$

for $i = 1, 2$ and any $x, y \in M$ with $d(x, y) < \rho$. Then, given $t_1, t_2 > 0$,

$$\exp(-bd(x, y)^\beta) \leq \frac{t_1 g_1(x) + t_2 g_2(x)}{t_1 g_1(y) + t_2 g_2(y)} \leq \exp(bd(x, y)^\beta)$$

for any $x, y \in M$ with $d(x, y) < \rho$. Hence, $t_1 g_1 + t_2 g_2$ is in $C(b, \beta)$.

We proceed to calculate the projective distance. By definition, $\alpha(g_1, g_2)$ is the supremum of all the numbers $t > 0$ satisfying the following three conditions:

$$(g_2 - t g_1)(x) > 0 \Leftrightarrow t < \frac{g_2}{g_1}(x)$$

$$\frac{(g_2 - t g_1)(x)}{(g_2 - t g_1)(y)} \leq \exp(bd(x, y)^\beta) \Leftrightarrow t \leq \frac{\exp(bd(x, y)^\beta) g_2(y) - g_2(x)}{\exp(bd(x, y)^\beta) g_1(y) - g_1(x)}$$

$$\frac{(g_2 - t g_1)(x)}{(g_2 - t g_1)(y)} \geq \exp(-bd(x, y)^\beta) \Leftrightarrow t \leq \frac{\exp(bd(x, y)^\beta) g_2(x) - g_2(y)}{\exp(bd(x, y)^\beta) g_1(x) - g_1(y)}$$

for any $x, y \in M$ with $x \neq y$ and $d(x, y) < \rho$. Hence, $\alpha(g_1, g_2)$ is equal to

$$\inf \left\{ \frac{g_2(x)}{g_1(x)}, \frac{\exp(bd(x, y)^\beta) g_2(x) - g_2(y)}{\exp(bd(x, y)^\beta) g_1(x) - g_1(y)} : x \neq y \text{ and } d(x, y) < \rho \right\}.$$

Analogously, $\beta(g_1, g_2)$ is the supremum of this same set.

The crucial fact that makes the proof of Theorem 12.3.1 work is that the transfer operator tends to improve the regularity of functions or, more precisely, their Hölder constants. The next proposition is a concrete manifestation of this fact:

Lemma 12.3.9. *For each $\beta \in (0, \alpha]$ there exists a constant $\lambda_0 \in (0, 1)$ such that $\mathcal{L}(C(b, \beta)) \subset C(\lambda_0 b, \beta)$ for every b sufficiently large (depending on β).*

Proof. It follows directly from the expression of the transfer operator in (12.1.1) that $\mathcal{L}g$ is positive whenever g is positive. Therefore, we only have to check the second condition in the definition of $C(\lambda_0 b, \beta)$. Consider $y_1, y_2 \in M$ with $d(y_1, y_2) < \rho$. The expression (12.1.2) gives that

$$\mathcal{L}g(y_i) = \sum_{j=1}^k e^{\varphi(x_{i,j})} g(x_{i,j})$$

for $i = 1, 2$, where the points $x_{i,j} \in f^{-1}(y_i)$ satisfy $d(x_{1i}, x_{2i}) \leq \sigma^{-1} d(y_1, y_2)$ for every $1 \leq j \leq k$. By hypothesis, φ is (K_0, α) -Hölder. Since we suppose that

$\beta \leq \alpha$, it follows that φ is (K, β) -Hölder, with $K = K_0(\text{diam } M)^{\alpha - \beta}$. Therefore,

$$\begin{aligned} (\mathcal{L}g)(y_1) &= \sum_{i=1}^k e^{\varphi(x_{1,i})} g(x_{1,i}) = \sum_{i=1}^k e^{\varphi(x_{2,i})} g(x_{2,i}) \frac{g(x_{1,i})e^{\varphi(x_{1,i})}}{g(x_{2,i})e^{\varphi(x_{2,i})}} \\ &\leq \sum_{i=1}^k e^{\varphi(x_{2,i})} g(x_{2,i}) \exp(bd(x_{1,i}, x_{2,i})^\beta + Kd(x_{1,i}, x_{2,i})^\beta) \\ &\leq (\mathcal{L}g)(y_2) \exp((b + K)\sigma^{-\beta} d(y_1, y_2)^\beta) \end{aligned}$$

for every $g \in C(b, \beta)$. Fix $\lambda_0 \in (\sigma^{-\beta}, 1)$. For every b sufficiently large, $(b + K)\sigma^{-\beta} \leq b\lambda_0$. Then the previous relation gives that

$$(\mathcal{L}g)(y_1) \leq (\mathcal{L}g)(y_2) \exp(\lambda_0 b d(y_1, y_2)^\beta),$$

for any $y_1, y_2 \in M$ with $d(y_1, y_2) < \rho$. Exchanging the roles of y_1 and y_2 , we obtain the other inequality.

Next, we use the family of cones $C(L)$ introduced in Example 12.3.7:

Lemma 12.3.10. *There exists $N \geq 1$ and for every $\beta > 0$ and every $b > 0$ there exists $L > 1$ satisfying $\mathcal{L}^N(C(b, \beta)) \subset C(L)$.*

Proof. By hypothesis, f is topologically exact. Hence, there exists $N \geq 1$ such that $f^N(B(z, \rho)) = M$ for every $z \in M$. Fix N once and for all. Given $g \in C(b, \beta)$, consider any point $z \in M$ such that $g(z) = \sup g$. Consider $y_1, y_2 \in M$. On the one hand,

$$\mathcal{L}^N g(y_1) = \sum_{x \in f^{-N}(y_1)} e^{\varphi_N(x)} g(x) \leq \text{degree}(f^N) e^{N \sup |\varphi|} g(z).$$

On the other hand, by the choice of N , there exists $x \in B(z, \rho)$ such that $f^N(x) = y_2$. Then,

$$\mathcal{L}^N g(y_2) \geq e^{\varphi_N(x)} g(x) \geq e^{-N \sup |\varphi|} e^{-bd(x,z)^\beta} g(z) \geq e^{-N \sup |\varphi| - b\rho^\beta} g(z).$$

Since y_1 and y_2 are arbitrary, this proves that

$$\frac{\sup \mathcal{L}^N g}{\inf \mathcal{L}^N g} \leq \text{degree}(f^N) e^{2N \sup |\varphi| + b\rho^\beta}.$$

Now it suffices to take L equal to the expression on the right-hand side of this inequality.

Combining Lemmas 12.3.9 and 12.3.10, we get that there exists $N \geq 1$ and, given $\beta \in (0, \alpha]$ there exists $\lambda_0 \in (0, 1)$ such that, for every $b > 0$ sufficiently large (depending on N and β) there exists $L > 1$, satisfying

$$\mathcal{L}^N(C(b, \beta)) \subset C(\lambda_0^N b, \beta) \cap C(L). \tag{12.3.9}$$

In what follows, we write $C(c, \beta, R) = C(c, \beta) \cap C(R)$ for any $c > 0, \beta > 0$ and $R > 1$.

Lemma 12.3.11. *For every $c \in (0, b)$ and $R > 1$, the set $C(c, \beta, R) \subset C(b, \beta)$ has finite diameter with respect to the projective distance of the cone $C(b, \beta)$.*

Proof. We use the expression of θ given by Lemma 12.3.8. On the one hand, the hypothesis that $g_1, g_2 \in C(c, \beta)$ ensures that

$$\begin{aligned} \frac{\exp (b d(x, y)^\beta) g_2(x)-g_2(y)}{\exp (b d(x, y)^\beta) g_1(x)-g_1(y)} &= \frac{g_2(x)}{g_1(x)} \frac{1-\exp (-b d(x, y)^\beta)\left(g_2(y) / g_2(x)\right)}{1-\exp (-b d(x, y)^\beta)\left(g_1(y) / g_1(x)\right)} \\ &\geq \frac{g_2(x)}{g_1(x)} \frac{1-\exp (- (b-c) d(x, y)^\beta)}{1-\exp (- (b+c) d(x, y)^\beta)} \\ &\geq \frac{g_2(x)}{g_1(x)} \frac{1-\exp (- (b-c) \rho^\beta)}{1-\exp (- (b+c) \rho^\beta)} \end{aligned}$$

for any $x, y \in M$ with $d(x, y) < \rho$. Denote by r the value of the last fraction on the right-hand side. Then, observing that $r \in (0, 1)$,

$$\alpha\left(g_1, g_2\right) \geq \inf \left\{ \frac{g_2(x)}{g_1(x)}, r \frac{g_2(x)}{g_1(x)} : x \in M \right\} = r \inf \left\{ \frac{g_2(x)}{g_1(x)} : x \in M \right\} \geq r \frac{\inf g_2}{\sup g_1}.$$

Analogously,

$$\beta\left(g_1, g_2\right) \leq \sup \left\{ \frac{g_2(x)}{g_1(x)}, \frac{1}{r} \frac{g_2(x)}{g_1(x)} : x \in M \right\} = \frac{1}{r} \sup \left\{ \frac{g_2(x)}{g_1(x)} : x \in M \right\} \leq \frac{1}{r} \frac{\sup g_2}{\inf g_1}.$$

On the other hand, the hypothesis that $g_1, g_2 \in C(R)$ gives that

$$\frac{\sup g_2}{\inf g_1} \leq R^2 \frac{\inf g_2}{\sup g_1}.$$

Combining these three inequalities, we conclude that $\theta\left(g_1, g_2\right) \leq \log \left(R^2 / r^2\right)$ for any $g_1, g_2 \in C(c, \beta, R)$.

Corollary 12.3.12. *There exists $N \geq 1$ such that for every $\beta \in (0, a]$ and every $b > 0$ sufficiently large there exists $\Lambda_0 < 1$ satisfying*

$$\theta\left(\mathcal{L}^N g_1, \mathcal{L}^N g_2\right) \leq \Lambda_0 \theta\left(g_1, g_2\right) \quad \text{for any } g_1, g_2 \in C(b, \beta).$$

Proof. Take $N \geq 1, \lambda_0 \in (0, 1)$ and $L > 1$ as in (12.3.9) and consider

$$c = \lambda_0^N b \quad \text{and} \quad R = L. \tag{12.3.10}$$

Then $\mathcal{L}^N(C(b, \beta)) \subset C(c, \beta, R)$ and it follows from Lemma 12.3.11 that the diameter D of the image $\mathcal{L}^N(C(b, \beta))$ with respect to the projective distance θ is finite. Take $\Lambda_0 = \tanh(D/4)$. Now the conclusion of the corollary is an immediate application of Proposition 12.3.6.

12.3.3 Exponential convergence

Fix $N \geq 1$, $\beta \in (0, \alpha]$, $b > 0$ and $L > 1$ as in Corollary 12.3.12, and then consider $c > 0$ and $R > 1$ given by (12.3.10). As before, we denote by h the positive eigenfunction (Lemma 12.1.11) and by λ the spectral radius (Corollary 12.1.15) of the transfer operator \mathcal{L} . Recall that h is α -Hölder and bounded from zero and infinity. Therefore, up to increasing the constants b and L , if necessary, we may assume that $h \in C(c, \beta, R)$.

The next lemma follows directly from the previous considerations and is a significant step toward the estimate in Theorem 12.3.1. We continue denoting by $\|\cdot\|$ the norm defined in $C^0(M)$ by $\|\phi\| = \sup\{|\phi(x)| : x \in M\}$.

Lemma 12.3.13. *There exists $C > 0$ and $\Lambda \in (0, 1)$ such that*

$$\|\lambda^{-n} \mathcal{L}^n g - h \int g \, d\nu\| \leq C \Lambda^n \int g \, d\nu \quad \text{for } g \in C(c, \beta, R) \text{ and } n \geq 1.$$

Proof. Let $g \in C(c, \beta, R)$. In particular, $g > 0$ and so $\int g \, d\nu > 0$. The conclusion of the lemma is not affected when we multiply g by any positive number. Hence, it is no restriction to suppose that $\int g \, d\nu = 1$. Then,

$$\int \lambda^{-n} \mathcal{L}^n g \, d\nu = \int \lambda^{-n} g \, d(\mathcal{L}^{*n} \nu) = \int g \, d\nu = 1 = \int h \, d\nu$$

and, hence, $\inf(\lambda^{-n} \mathcal{L}^n g/h) \leq 1 \leq \sup(\lambda^{-n} \mathcal{L}^n g/h)$ for every $n \geq 1$. Now, it follows from the expressions in Lemma 12.3.8 that

$$\begin{aligned} \alpha(\lambda^{-jN} \mathcal{L}^{jN} g, h) &\leq \inf \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h} \leq 1, \\ \beta(\lambda^{-jN} \mathcal{L}^{jN} g, h) &\geq \sup \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h} \geq 1. \end{aligned}$$

Consequently,

$$\begin{aligned} \theta(\lambda^{-jN} \mathcal{L}^{jN} g, h) &\geq \log \beta(\lambda^{-jN} \mathcal{L}^{jN} g, h) \geq \log \sup \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h}, \\ \theta(\lambda^{-jN} \mathcal{L}^{jN} g, h) &\geq -\log \alpha(\lambda^{-jN} \mathcal{L}^{jN} g, h) \geq -\log \inf \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h} \end{aligned}$$

for every $j \geq 0$. Now let D be the diameter of $C(c, \beta, R)$ with respect to the projective distance θ (Lemma 12.3.11). By Proposition 12.3.3 and Corollary 12.3.12,

$$\theta(\lambda^{-jN} \mathcal{L}^{jN} g, h) = \theta(\mathcal{L}^{jN} g, \mathcal{L}^{jN} h) \leq \Lambda_0^j \theta(g, h) \leq \Lambda_0^j D$$

for every $j \geq 0$. Combining this with the previous two inequalities,

$$\exp(-\Lambda_0^j D) \leq \inf \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h} \leq \sup \frac{\lambda^{-jN} \mathcal{L}^{jN} g}{h} \leq \exp(\Lambda_0^j D)$$

for every $j \geq 0$. Fix $C_1 > 0$ such that $|e^x - 1| \leq C_1|x|$ whenever $|x| \leq D$. Then the previous relation implies that

$$|\lambda^{-jN} \mathcal{L}^{jN} g(x) - h(x)| \leq h(x)C_1 \Lambda_0^j D \quad \text{for every } x \in M \text{ and } j \geq 0. \quad (12.3.11)$$

Take $C_2 = C_1 D \sup h$ and $\Lambda = \Lambda_0^{1/N}$. The inequality (12.3.11) means that

$$\|\lambda^{-jN} \mathcal{L}^{jN} g - h\| \leq C_2 \Lambda^{jN} \quad \text{for every } j \geq 1.$$

Given any $n \geq 1$, write $n = jN + r$ with $j \geq 0$ and $0 \leq r < N$. Since the transfer operator $\mathcal{L} : C^0(M) \rightarrow C^0(M)$ is continuous and $\mathcal{L}h = \lambda h$,

$$\|\lambda^{-n} \mathcal{L}^n g - h\| = \|\lambda^{-r} \mathcal{L}^r (\lambda^{-jN} \mathcal{L}^{jN} g - h)\| \leq (\|\mathcal{L}\|/\lambda)^r \|\lambda^{-jN} \mathcal{L}^{jN} g - h\|.$$

Combining the last two inequalities,

$$\|\lambda^{-n} \mathcal{L}^n g - h\| \leq (\|\mathcal{L}\|/\lambda)^r C_2 \Lambda^{n-r}.$$

This proves the conclusion of the lemma, as long as we take $C \geq C_2 (\|\mathcal{L}\|/(\lambda \Lambda))^r$ for every $0 \leq r < N$.

Now we are ready to prove Theorem 12.3.1:

Proof. Start by considering $g_2 \in C(c, \beta, R)$. Using the identity in Exercise 12.1.4 and recalling that $\mu = hv$,

$$\begin{aligned} B_n(g_1, g_2) &= \left| \int g_1 \left(\lambda^{-n} \mathcal{L}^n g_2 - h \int g_2 dv \right) dv \right| \\ &\leq \left\| \lambda^{-n} \mathcal{L}^n g_2 - h \int g_2 dv \right\| \int |g_1| dv. \end{aligned}$$

Therefore, using Lemma 12.3.13,

$$B_n(g_1, g_2) \leq C \Lambda^n \int |g_1| dv \int g_2 dv. \quad (12.3.12)$$

Now let $g_2 : M \rightarrow \mathbb{R}$ be any β -Hölder function and $H = H_\beta(g_2)$. Write $g_2 = g_2^+ - g_2^-$ with

$$g_2^+ = \frac{1}{2}(|g_2| + g_2) + B \quad \text{and} \quad g_2^- = \frac{1}{2}(|g_2| - g_2) + B,$$

with B defined by $B = \max\{H/c, \sup |g_2|/(R-1)\}$. It is clear that the functions g_2^\pm are positive: $g_2^\pm \geq B > 0$. Moreover, they are (H, β) -Hölder:

$$|g_2^\pm(x) - g_2^\pm(y)| \leq |g_2(x) - g_2(y)| \leq Hd(x, y)^\beta,$$

for $x, y \in M$. Hence, using the mean value theorem and the fact that $B \geq H/c$,

$$|\log g_2^\pm(x) - \log g_2^\pm(y)| \leq \frac{|g_2^\pm(x) - g_2^\pm(y)|}{B} \leq \frac{Hd(x, y)^\beta}{B} \leq cd(x, y)^\beta.$$

Moreover, since $B \geq \sup |g_2|/(R-1)$,

$$\sup g_2^\pm \leq \sup |g_2| + B \leq RB \leq R \inf g_2^\pm.$$

Together with the previous relation, this means that $g_2^\pm \in C(c, \beta, R)$. Then, we may apply (12.3.12) to both functions:

$$B_n(g_1, g_2^\pm) \leq C\Lambda^n \int |g_1| dv \int g_2^\pm dv$$

and, consequently,

$$\begin{aligned} B_n(g_1, g_2) &\leq B_n(g_1, g_2^+) + B_n(g_1, g_2^-) \\ &\leq C\Lambda^n \int |g_1| dv \int (g_2^+ + g_2^-) dv. \end{aligned} \tag{12.3.13}$$

Moreover, by the definition of g_2^\pm ,

$$\begin{aligned} \int (g_2^+ + g_2^-) dv &= \int |g_2| dv + 2B \leq \int |g_2| dv + \frac{2H}{c} + \frac{2 \sup |g_2|}{R-1} \\ &\leq \frac{2}{c} H_\beta(g_2) + \frac{R+1}{R-1} \sup |g_2|. \end{aligned} \tag{12.3.14}$$

Take $C_1 = C \max\{2/c, (R+1)/(R-1)\}$ and define

$$K_1(g_2) = 2C_1(\sup |g_2| + H_\beta(g_2)).$$

The relations (12.3.13) and (12.3.14) give that

$$B_n(g_1, g_2) \leq C_1 \Lambda^n \int |g_1| dv (H_\beta(g_2) + \sup |g_2|) \leq \frac{1}{2} K_1(g_2) \Lambda^n \int |g_1| dv.$$

This closes the proof of the theorem in the case when g_2 is a real function.

The general (complex) case follows easily. Note that $K_1(\Re g_2) \leq K_1(g_2)$, because $\sup |\Re g_2| \leq \sup |g_2|$ and $H_\beta(\Re g_2) \leq H_\beta(g_2)$. Analogously, $K_1(\Im g_2) \leq K_1(g_2)$. Therefore, the previous argument yields

$$\begin{aligned} B_n(g_1, g_2) &\leq B_n(g_1, \Re g_2) + B_n(g_1, \Im g_2) \leq \frac{1}{2} (K_1(\Re g_2) + K_1(\Im g_2)) \Lambda^n \int |g_1| dv \\ &\leq K_1(g_2) \Lambda^n \int |g_1| dv. \end{aligned}$$

This completes the proof of Theorem 12.3.1.

We close this section with a few comments about the spectral gap property. Let $C^\beta(M)$ be the vector space of β -Hölder functions $g : M \rightarrow \mathbb{C}$. We leave it to the reader (Exercise 12.3.6) to check the following facts:

- (i) The function $\|g\|_{\beta, \rho} = \sup |g| + H_{\beta, \rho}(g)$ is a complete norm in $C^\beta(M)$.
- (ii) $C^\beta(M)$ is invariant under the transfer operator: $\mathcal{L}(C^\beta(M)) \subset C^\beta(M)$.
- (iii) The restriction $\mathcal{L} : C^\beta(M) \rightarrow C^\beta(M)$ is continuous with respect to the norm $\|\cdot\|_{\beta, \rho}$.

Note that $h \in C^\beta(M)$, since $\beta \leq \alpha$. Define $V = \{g \in C^\beta(M) : \int g dv = 0\}$. Then $C^\beta(M) = V \oplus \mathbb{C}h$, because every function $g \in C^\beta(M)$ may be decomposed, in

a unique way, as the sum of a function in V with a multiple of h :

$$g = (g - h \int g dv) + h \int g dv.$$

Moreover, the direct sum $C^\beta(M) = V \oplus \mathbb{C}h$ is invariant under the transfer operator. Indeed, if $g \in V$ then

$$g \in V \Rightarrow \int \mathcal{L}g dv = \int g d\mathcal{L}^*v = \lambda \int g dv = 0 \Rightarrow \mathcal{L}g \in V.$$

It follows that the spectrum of $\mathcal{L} : C^\beta(M) \rightarrow C^\beta(M)$ is the union of $\{\lambda\}$ with the restriction of \mathcal{L} to the hyperplane V . In Exercise 12.3.8 we invite the reader to show that the spectral radius of $\mathcal{L} | V$ is strictly less than λ . Consequently, $\mathcal{L} : C^\beta(M) \rightarrow C^\beta(M)$ has the spectral gap property.

The book of Viviane Baladi [Bal00] contains an in-depth presentation of the spectral theory of transfer operators and its connections to the issue of decay of correlations, for differentiable (or piecewise differentiable) expanding maps and also for uniformly hyperbolic diffeomorphisms.

12.3.4 Exercises

- 12.3.1. Show that the cross-ratio $R(a, x, y, b)$ is invariant under every Möbius automorphism of the real line, that is, $R(\phi(a), \phi(b), \phi(c), \phi(d)) = R(a, b, c, d)$ for any $a < b \leq c < d$ and every transformation of the form $\phi(x) = (\alpha x + \beta) / (\gamma x + \delta)$ with $\alpha\delta - \beta\gamma \neq 0$.
- 12.3.2. Consider the cone $C = \{(z, s) \in \mathbb{C} \times \mathbb{R} : s > |z|\}$. Its projective quotient may be identified with the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ through $(z, 1) \mapsto z$. Let d be the distance induced in \mathbb{D} , through this identification, by the projective distance of C . Show that d coincides with the Cayley–Klein distance Δ , which is defined by

$$\Delta(p, q) = \log \frac{|aq| |pb|}{|ap| |bq|}, \quad \text{for } p, q \in \mathbb{D},$$
 where a and b are the points where the straight line through p and q intersects the boundary of the disk, denoted in such a way that p is between a and q and q is between p and b . [Observation: The Cayley–Klein distance is related to the Poincaré distance in the disk through the map $z \mapsto (2z)/1 + |z|^2$.]
- 12.3.3. Show that the projective distance associated with the cone C_+ in Example 12.3.7 is complete, in the following sense: with respect to the projective distance, every Cauchy sequence $(g_n)_n$ converges to some element of C_+ . Moreover, if we normalize the functions (for example, fixing any probability measure η on M and requiring that $\int g_n d\eta = 1 = \int g d\eta$ for every n), then $(g_n)_n$ converges uniformly to g .
- 12.3.4. Let M be a compact manifold and C_1 be the cone of positive differentiable functions in M . Show that the corresponding projective distance θ_1 is *not* complete.
- 12.3.5. Check that if $g_1, g_2 : M \rightarrow \mathbb{R}$ are β -Hölder functions, $\theta : M \rightarrow M$ is an L -Lipschitz transformation and η is a probability measure on M then:

- (a) $H_\beta(g_1g_2) \leq \sup |g_1|H_\beta(g_2) + \sup |g_2|H_\beta(g_1)$;
- (b) $\int |g_1| d\eta \leq \sup |g_1| \leq \int |g_1| d\eta + H_\beta(g_1)(\text{diam}M)^\beta$;
- (c) $H_\beta(g \circ \theta) \leq L^\beta H_\beta(g)$.

Moreover, the claim in (a) remains true if we replace H_β by $H_{\beta,\rho}$. The same holds for the claim in (c), as long as $L \leq 1$.

12.3.6. Let $C^\beta(M)$ be the vector space of β -Hölder functions on a compact metric space M . Prove the properties (i), (ii), (iii) stated at the end of Section 12.3.

12.3.7. Endow $C^\beta(M)$ with the norm $\| \cdot \|_{\beta,\rho}$. Let $\mathcal{L} : C^\beta(M) \rightarrow C^\beta(M)$ be the transfer operator associated with an α -Hölder potential $\varphi : M \rightarrow \mathbb{R}$, with $\alpha \geq \beta$. Let λ be the spectral radius, ν be the reference measure, h be the eigenfunction and $\mu = h\nu$ be the equilibrium state of the potential φ . Consider the transfer operator $\mathcal{P} : C^\beta(M) \rightarrow C^\beta(M)$ associated with the potential $\psi = \varphi + \log h - \log h \circ f - \log \lambda$.

- (a) Check that \mathcal{L} is linearly conjugate to $\lambda\mathcal{P}$, and so $\text{spec}(\mathcal{L}) = \lambda \text{spec}(\mathcal{P})$. Moreover, $\mathcal{P}1 = 1$ and $\mathcal{P}^*\mu = \mu$.
- (b) Show that $\int |\mathcal{P}^n g| d\mu \leq \int |g| d\mu$ and $\sup |\mathcal{P}^n g| \leq \sup |g|$ and there exist constants $C > 0$ and $\tau < 1$ such that $H_{\beta,\rho}(\mathcal{P}^n g) \leq \tau^n H_{\beta,\rho}(g) + C \sup |g|$ for every $g \in C^\beta(M)$ and every $n \geq 1$.

12.3.8. The goal of this exercise is to prove that the spectral radius of the restriction of \mathcal{L} to the hyperplane $V = \{g \in C^\beta(M) : \int g d\nu = 0\}$ is strictly less than λ . By part (a) of Exercise 12.3.7, it is enough to consider the case $\mathcal{L} = \mathcal{P}$ (with $\lambda = 1$ and $\nu = \mu$ and $h = 1$). Fix b, β, R as in Corollary 12.3.12.

- (a) Show that there exist $K > 1$ and $r > 0$ such that, for every $v \in V$ with $\|v\|_{\beta,\rho} \leq r$, the function $g = 1 + v$ is in the cone $C(b, \beta, R)$ and satisfies

$$K^{-1} \|v\|_{\beta,\rho} \leq \theta(1, g) \leq K \|v\|_{\beta,\rho}.$$

- (b) Use Corollary 12.3.12 and the previous item to find $C > 0$ and $\tau < 1$ such that $\|\mathcal{P}^n v\|_{\beta,\rho} \leq C\tau^n \|v\|_{\beta,\rho}$ for every $v \in V$. Deduce that the spectral radius of $\mathcal{P}|_V$ is less or equal than $\tau < 1$.

12.4 Dimension of conformal repellers

In this section we present an application of the theory developed previously to the calculation of the Hausdorff dimension of certain invariant sets of expanding maps, that we call conformal repellers. The main result (Theorem 12.4.3) contains a formula for the value of the Hausdorff dimension of the repeller in terms of the pressure of certain potentials.

Detailed presentations of the theory of fractal dimensions and its many applications can be found in the books of Falconer [Fal90], Palis and Takens [PT93, Chapter 4], Pesin [Pes97] and Bonatti, Díaz and Viana [BDV05, Chapter 3].

12.4.1 Hausdorff dimension

Let M be a metric space. In this section, we call a *cover* of M any countable (possibly finite) family of subsets of M , not necessarily open, whose union is the whole of M . The *diameter* of a cover \mathcal{U} is the supremum of the diameters of its elements. For each $d > 0$ and $\delta > 0$, define

$$m_d(M, \delta) = \inf \left\{ \sum_{U \in \mathcal{U}} \text{diam}(U)^d : \mathcal{U} \text{ cover with } \text{diam} \mathcal{U} < \delta \right\}. \tag{12.4.1}$$

That is, we consider all possible covers of M by subsets with diameter less than δ and we try to minimize the sum of the diameters raised to the power d . This number varies with δ in a monotonic fashion: when δ decreases, the class of admissible covers decreases and, thus, the infimum can only increase. We call *Hausdorff measure of M in dimension d* the limit

$$m_d(M) = \lim_{\delta \rightarrow 0} m_d(M, \delta). \tag{12.4.2}$$

Note that $m_d(M) \in [0, \infty]$. Moreover, it follows directly from the definition that

$$m_{d_1}(M, \delta) \leq \delta^{d_1 - d_2} m_{d_2}(M, \delta) \quad \text{for every } \delta > 0 \text{ and any } d_1 > d_2 > 0.$$

Making $\delta \rightarrow 0$, it follows that $m_{d_1}(M) = 0$ or $m_{d_2}(M) = \infty$ or both. Therefore, there exists a unique $d(M) \in [0, \infty]$ such that $m_d(M) = \infty$ for every $d < d(M)$ and $m_d(M) = 0$ for every $d > d(M)$. We call $d(M)$ the *Hausdorff dimension* of the metric space M .

Example 12.4.1. Consider the usual Cantor set K in the real line. That is,

$$K = \bigcap_{n=0}^{\infty} K^n$$

where $K^0 = [0, 1]$ and every K^n , $n \geq 1$ is obtained by removing from each connected component of K^{n-1} the central open subinterval with relative length $1/3$. Let $d_0 = \log 2 / \log 3$. We are going to show that $m_{d_0}(M) = 1$, which implies that $d(M) = d_0$.

To prove the upper bound, consider, for each $n \geq 0$, the cover \mathcal{V}^n of K whose elements are the intersections of K with each of the connected components of K^n . It is clear that the sequence $(\mathcal{V}^n)_n$ is increasing: $\mathcal{V}^{n-1} \prec \mathcal{V}^n$ for every $n \geq 1$. Note also that \mathcal{V}^n has exactly 2^n elements, all with diameter equal to 3^{-n} . Therefore,

$$\sum_{V \in \mathcal{V}^n} (\text{diam } V)^{d_0} = 2^n 3^{-nd_0} = 1 \quad \text{for every } n. \tag{12.4.3}$$

Since $\text{diam } \mathcal{V}^n \rightarrow 0$ when $n \rightarrow \infty$, it follows that $m_{d_0}(M) \leq 1$ and so $d(M) \leq d_0$.

The lower bound is a bit more difficult, because one needs to deal with arbitrary covers. We are going to show that, given any cover \mathcal{U} of M ,

$$\sum_{U \in \mathcal{U}} (\text{diam } U)^{d_0} \geq 1. \tag{12.4.4}$$

Clearly, this implies that $m_{d_0}(M) \geq 1$ and so $d(M) \geq d_0$.

Let us call an *open segment* the intersection of K with any interval of the real line whose endpoints are not in K . It is clear that every subset of K is contained in some open segment whose diameter is only slightly larger. Hence, given any cover \mathcal{U} , we can always find covers \mathcal{U}' whose elements are open segments such that $\sum_{U' \in \mathcal{U}'} (\text{diam } U')^{d_0}$ is as close to $\sum_{U \in \mathcal{U}} (\text{diam } U)^{d_0}$ as we want. So, it is no restriction to assume from the start that the elements of \mathcal{U} are open segments. Then, since K is compact, we may also assume that \mathcal{U} is finite. For any open segment U there exists $n \geq 0$ such that every element of \mathcal{V}^n , $m \geq n$ that intersects U is contained in U . Since \mathcal{U} is finite, we may choose the same n for all its elements. We claim that

$$\sum_{U \in \mathcal{U}} (\text{diam } U)^{d_0} \geq \sum_{V \in \mathcal{V}^n} (\text{diam } V)^{d_0}. \tag{12.4.5}$$

Clearly, (12.4.3) and (12.4.5) imply (12.4.4). We are left to prove (12.4.5).

The strategy is to modify the cover \mathcal{U} successively, in such a way that the expression on the left-hand side of (12.4.5) never increases and one reaches the cover \mathcal{V}^n after finitely many modifications. For each $U \in \mathcal{U}$, let $k \geq 0$ be minimum such that U intersects a unique element V of \mathcal{V}^k . The choice of n implies that $k \leq n$: for $k > n$, if U intersects an element of \mathcal{V}^k then U contains all the 2^{k-n} elements of \mathcal{V}^k inside the same element of \mathcal{V}^n . Suppose that $k < n$. By the choice of k , the set U intersects exactly two elements of \mathcal{V}^{k+1} . Let them be denoted V_1 and V_2 and let U_1 and U_2 be their intersections with U . Then

$$\text{diam } U_i \leq \text{diam } V_i = 3^{-k-1} \quad \text{and} \quad \text{diam } U = \text{diam } U_1 + 3^{-k-1} + \text{diam } U_2.$$

Hence (Exercise 12.4.1),

$$(\text{diam } U)^{d_0} \geq (\text{diam } U_1)^{d_0} + (\text{diam } U_2)^{d_0}.$$

This means that the value on the left-hand side of (12.4.5) does not increase when we replace U by U_1 and U_2 in the cover \mathcal{U} . On the one hand, the new cover satisfies the same conditions as the original: U_1 and U_2 are open segments (because V_1 , V_2 and U are open segments) and they contain every element of \mathcal{V}^n that they intersect. On the other hand, by construction, each one of them intersects a unique element of \mathcal{V}^{k+1} . Therefore, after finitely many repetitions of this procedure we reduce the initial situation to the case where $k = n$ for every $U \in \mathcal{U}$. Now, the choice of n implies that in that case each $U \in \mathcal{U}$ contains the unique $V \in \mathcal{V}^n$ that it intersects. Observe that this means that $U = V$. In particular, any elements of \mathcal{U} that correspond to the same

$V \in \mathcal{V}^m$ must coincide. Eliminating such repetitions we obtain the cover \mathcal{V}^n . This completes the calculation.

In general, the Hausdorff measure of a metric space M in its dimension $d(M)$ may take any value. In many interesting situations, including Example 12.4.1 and the much more general construction that we treat next, this measure is positive and finite. But there are many other cases where it is either zero or infinity.

12.4.2 Conformal repellers

Let D, D_1, \dots, D_N be compact convex subsets of an Euclidean space \mathbb{R}^ℓ such that $D_i \subset D$ for every i and $D_i \cap D_j = \emptyset$ whenever $i \neq j$. Assume that

$$\text{vol}(D \setminus D_*) > 0, \tag{12.4.6}$$

where $D_* = D_1 \cup \dots \cup D_N$ and vol denote the volume measure on \mathbb{R}^ℓ . Assume also that there exists a map $f : D_* \rightarrow D$ such that the restriction to each D_i is a homeomorphism onto D . See Figure 12.2. Note that the sequence of pre-images $f^{-n}(D)$ is decreasing. Their intersection

$$\Lambda = \bigcap_{n=0}^{\infty} f^{-n}(D) \tag{12.4.7}$$

is called a *repeller* of f . In other words, Λ is the set of points x whose iterates $f^n(x)$ are defined for every $n \geq 1$. It is clear that Λ is compact and $f^{-1}(\Lambda) = \Lambda$.

Example 12.4.2. The Cantor set K in Example 12.4.1 is the repeller of the transformation $f : [0, 1/3] \cup [2/3, 1] \rightarrow [0, 1]$ given by $f(x) = 3x$ if $x \in [0, 1/3]$ and $f(x) = 3x - 2$ if $x \in [2/3, 1]$. A more general class of examples in dimension 1 was introduced in Example 11.2.3.

In what follows we take the map $f : D_* \rightarrow D$ to be of class C^1 ; for points on the boundary of the domain this just means that f admits a C^1 extension to some neighborhood. We also make the following additional hypotheses.

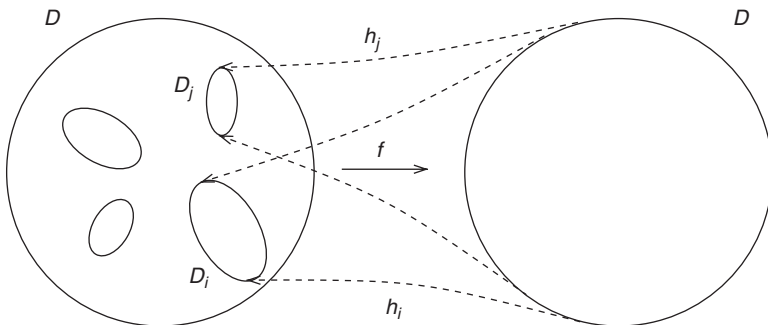


Figure 12.2. A repeller

The first hypothesis is that the map f is expanding: there exists $\sigma > 1$ such that

$$\|Df(x)v\| \geq \sigma \quad \text{for every } x \in D_* \text{ and every } v \in \mathbb{R}^\ell. \tag{12.4.8}$$

Then it is not difficult to check that the restriction $f : \Lambda \rightarrow \Lambda$ to the repeller is an expanding map in the sense of Section 11.2.

The second hypothesis is that the logarithm of the Jacobian of f is Hölder: there exist $C > 0$ and $\theta > 0$ such that

$$\log \frac{|\det Df(x)|}{|\det Df(y)|} \leq C\|x - y\|^\theta \quad \text{for every } x, y \in D_*. \tag{12.4.9}$$

Up to choosing C sufficiently large, the inequality is automatically satisfied when x and y belong to distinct subdomains D_i and D_j , because $d(D_i, D_j) > 0$.

The third and last hypothesis is that the map f is conformal:

$$\|Df(x)\| \|Df(x)^{-1}\| = 1 \quad \text{for every } x \in D_*. \tag{12.4.10}$$

It is important to note that this condition is automatic when $\ell = 1$. For $\ell = 2$, it holds if and only if the map f is analytic.

All these conditions are satisfied in the case of the Cantor set (Examples 12.4.1 and 12.4.2). They are also satisfied in Example 11.2.3, as long as we take the derivative of the corresponding map f to be Hölder.

Theorem 12.4.3 (Bowen–Manning formula). *Suppose that $f : D_* \rightarrow D$ satisfies the conditions (12.4.8), (12.4.9) and (12.4.10). Then the Hausdorff dimension of the repeller is given by*

$$d(\Lambda) = d_0\ell,$$

where $d_0 \in (0, 1)$ is the unique number such that $P(f, -d_0 \log |\det Df|) = 0$.

The reader should be warned that we allow ourselves a slight abuse of language, in order not to overload the notations: throughout this section, $P(f, \psi)$ always denotes the pressure of a potential $\psi : \Lambda \rightarrow \mathbb{R}$ with respect to the restriction $f : \Lambda \rightarrow \Lambda$ to the repeller, even if at other points of the arguments we consider the map f defined on the whole domain D_* .

Before we start proving the theorem, let us mention the following interesting special case:

Example 12.4.4. Let $f : J \rightarrow [0, 1]$ be a map as in Example 11.2.3 and assume that the restriction of f to each connected component J_i of J is affine: the absolute value of the derivative is constant, equal to the inverse of the length $|J_i|$. Then the Hausdorff dimension of the repeller K of the map f is the unique number τ such that

$$\sum_i |J_i|^\tau = 1. \tag{12.4.11}$$

To obtain this conclusion from Theorem 12.4.3 it suffices to note that

$$P(f, -t \log |f'|) = \log \sum_i |J_i|^t \quad \text{for every } t. \tag{12.4.12}$$

We let the reader check this last claim (Exercise 12.4.6).

12.4.3 Distortion and conformality

Let us call an inverse branch of f the inverse $h_i : D \rightarrow D_i$ of the restriction of f to each domain D_i . More generally, we call an inverse branch of f^n any composition

$$h^n = h_{i_0} \circ \dots \circ h_{i_{n-1}} \tag{12.4.13}$$

with $i_0, \dots, i_{n-1} \in \{1, \dots, N\}$. For each $n \geq 1$, denote by \mathcal{I}^n the family of all inverse branches h^n of f^n . By construction, the images $h^n(D)$, $h^n \in \mathcal{I}^n$ are pairwise disjoint and their union contains Λ .

The principal goal in this section is to prove the following geometric estimate, which is at the heart of the proof of Theorem 12.4.3:

Proposition 12.4.5. *There exists $C_0 > 1$ such that for every $n \geq 1$, every $h^n \in \mathcal{I}^n$, every $E \subset h^n(D)$ and every $x \in h^n(D)$:*

$$\frac{1}{C_0} [\text{diam} f^n(E)]^\ell \leq [\text{diam} E]^\ell |\det Df^n(x)| \leq C_0 [\text{diam} f^n(E)]^\ell. \tag{12.4.14}$$

Starting the proof of this proposition, observe that our hypotheses imply that every inverse branch h_i of f is a diffeomorphism with $\|Dh_i\| \leq \sigma^{-1}$. Then, since D is convex, we may use the mean value theorem to conclude that

$$\|h_i(z) - h_i(w)\| \leq \sigma^{-1} \|z - w\| \quad \text{for every } z, w \in D. \tag{12.4.15}$$

For each inverse branch h^n as in (12.4.13), let us consider the sequence of inverse branches

$$h^{n-k} = h_{i_k} \circ \dots \circ h_{i_{n-1}}, \quad k = 0, \dots, n - 1. \tag{12.4.16}$$

Note that $h^{n-k}(D) \subset D_{i_k}$ for each k . It follows from (12.4.15) that each h^{n-k} is a σ^{k-n} -contraction. In particular,

$$\text{diam} h^{n-k}(D) \leq \sigma^{k-n} \text{diam} D \quad \text{for every } k = 0, \dots, n - 1. \tag{12.4.17}$$

Recall that the *convex hull* of a set $X \subset \mathbb{R}^\ell$ is the union of all the line segments whose endpoints are in X . It is clear that the convex hull has the same diameter as the set itself. Since D_i is convex for every i , the convex hull of each $h^{n-k}(D)$ is contained in D_{i_k} . In particular, the derivative Df is defined at every point in the convex hull of every $h^{n-k}(D)$.

Lemma 12.4.6. *There exists $C_1 > 1$ such that, for every $n \geq 1$ and every inverse branch $h^n \in \mathcal{T}^n$,*

$$\prod_{k=0}^{n-1} \frac{|\det Df(z_k)|}{|\det Df(w_k)|} \leq C_1$$

for any z_k, w_k in the convex hull of $h^{n-k}(D)$ for $k = 0, \dots, n - 1$.

Proof. The condition (12.4.9) gives that

$$\log \frac{|\det Df(z_k)|}{|\det Df(w_k)|} \leq C \|z_k - w_k\|^\theta \leq C [\text{diam } h^{n-k}(D)]^\theta$$

for each $k = 0, \dots, n - 1$. Then, using (12.4.17),

$$\log \prod_{k=0}^{n-1} \frac{|\det Df(z_k)|}{|\det Df(w_k)|} \leq \sum_{k=0}^{n-1} C [\text{diam } h^{n-k}(D)]^\theta \leq C [\text{diam } D]^\theta \sum_{k=0}^{n-1} \sigma^{(k-n)\theta}.$$

Therefore, it suffices to take $C_1 = \exp(C(\text{diam } D)^\theta \sum_{j=1}^\infty \sigma^{-j\theta})$.

The time has come for us to exploit the conformality hypothesis (12.4.10). Given any linear isomorphism $L : \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$, it is clear that $|\det L| \leq \|L\|^\ell$, and analogously for the inverse. Therefore,

$$1 = |\det L| |\det L^{-1}| \leq (\|L\| \|L^{-1}\|)^\ell.$$

Hence, $\|L\| \|L^{-1}\| = 1$ implies that $|\det L| = \|L\|^\ell$, and analogously for the inverse. Therefore, (12.4.10) implies that

$$|\det Df(y)| = \|Df(y)\|^\ell \quad \text{for every } y \in D_*. \tag{12.4.18}$$

Now we are ready to prove Proposition 12.4.5:

Proof of Proposition 12.4.5. Let n, h^n, E and x be as in the statement. Let w be a point of maximum for the norm of Dh^n in the domain D . By the mean value theorem,

$$\|x_1 - x_2\| \leq \|Dh^n(w)\| \|f^n(x_1) - f^n(x_2)\| \tag{12.4.19}$$

for any x_1, x_2 in E . Observe that $Dh^n(w)$ is the inverse of $Df^n(z)$, with $z = h^n(w)$. Hence, by conformality, $\|Dh^n(w)\| = \|Df^n(z)\|^{-1}$. Moreover, using Lemma 12.4.6 and (12.4.18),

$$|\det Df^n(x)| \leq C_1 |\det Df^n(z)| = C_1 \|Df^n(z)\|^\ell. \tag{12.4.20}$$

Combining (12.4.19) and (12.4.20), we obtain

$$\|x_1 - x_2\|^\ell \leq C_1 |\det Df^n(x)|^{-1} \|f^n(x_1) - f^n(x_2)\|^\ell.$$

Varying $x_1, x_2 \in E$, it follows that

$$[\text{diam } E]^\ell \leq C_1 |\det Df^n(x)|^{-1} [\text{diam } f^n(E)]^\ell.$$

This proves the second inequality in (12.4.14), as long as we take $C_0 \geq C_1$.

The proof of the other inequality is similar. For each $k = 0, \dots, n - 1$, let z_k be a point of maximum for the norm of Df restricted to the convex hull of $h^{n-k}(D)$. Then,

$$\begin{aligned} \|f^{k+1}(x_1) - f^{k+1}(x_2)\| &\leq \|Df(z_k)\| \|f^k(x_1) - f^k(x_2)\| \\ &= |\det Df(z_k)|^{1/\ell} \|f^k(x_1) - f^k(x_2)\| \end{aligned}$$

for every k and any $x_1, x_2 \in E$. Hence,

$$\|f^n(x_1) - f^n(x_2)\|^\ell \leq \prod_{k=0}^{n-1} |\det Df(z_k)| \|x_1 - x_2\|^\ell. \tag{12.4.21}$$

By Lemma 12.4.6,

$$\prod_{k=0}^{n-1} |\det Df(z_k)| \leq C_1 |\det Df^n(x)|. \tag{12.4.22}$$

Combining (12.4.21) and (12.4.22), we obtain

$$\|y_1 - y_2\|^\ell \leq C_1 |\det Df^n(x)| \|x_1 - x_2\|^\ell.$$

Varying y_1, y_2 , we conclude that

$$[\text{diam} f^n(E)]^\ell \leq C_1 |\det Df^n(x)| [\text{diam} E]^\ell.$$

This proves the first inequality in (12.4.14), for any $C_0 \geq C_1$.

12.4.4 Existence and uniqueness of d_0

Now we prove the existence and uniqueness of the number d_0 in the statement of Theorem 12.4.3. Denote $\phi = -\log |\det Df|$ and consider the function

$$\Psi : \mathbb{R} \rightarrow \mathbb{R}, \quad \Psi(t) = P(f, t\phi).$$

We want to show that there exists a unique d_0 such that $\Psi(d_0) = 0$.

Uniqueness is easy to prove. Indeed, the hypotheses (12.4.8) and (12.4.10) imply that

$$\phi = \log |\det Df^{-1} \circ f| = \ell \log \|Df^{-1} \circ f\| \leq -\ell \log \sigma.$$

Then, given any $s < t$, we have $t\phi \leq s\phi - (t - s)\ell \log \sigma$. Using (10.3.4) and (10.3.5), it follows that

$$P(f, t\phi) \leq P(f, s\phi) - (t - s)\ell \log \sigma < P(f, s\phi).$$

This proves that Ψ is strictly decreasing, and so there exists at most one $d_0 \in \mathbb{R}$ such that $\Psi(d_0) = 0$.

On the other hand, it follows from Proposition 10.3.6 that Ψ is continuous. Hence, to prove the existence of d_0 it is enough to show that $\Psi(0) > 0 > \Psi(1)$. This may be done as follows.

Let \mathcal{L} be the open cover of Λ whose elements are the images $h(\Lambda)$ of Λ under all the inverse branches of f . For each $n \geq 1$, the iterated sum \mathcal{L}^n is formed by the images $h^n(\Lambda)$ of Λ under the inverse branches of f^n . It follows from (12.4.17) that $\text{diam } \mathcal{L}^n \leq \sigma^{-n} \text{diam } D$ for every n , and so $\text{diam } \mathcal{L}^n \rightarrow 0$. Then, since the elements of \mathcal{L} are pairwise disjoint, we may use Exercise 10.3.3 to conclude that

$$P(f, \psi) = P(f, \psi, \mathcal{L}) \quad \text{for every potential } \psi. \tag{12.4.23}$$

In particular, $\Psi(0) = P(f, 0, \mathcal{L}) = h(f, \mathcal{L})$. Note that each family \mathcal{L}^n is a minimal cover of the repeller, that is, no proper subfamily covers Λ . Therefore, $H(\mathcal{L}^n) = \log \# \mathcal{L}^n = n \log N$ for every n and, consequently, $h(f, \mathcal{L}) = \log N$. This proves that $\Psi(0)$ is positive.

Proposition 12.4.7. $\Psi(1) = \lim_n \frac{1}{n} \log \text{vol}(f^{-n}(D)) < 0$.

Proof. By (12.4.23), we have that $\Psi(1) = P(f, \phi, \mathcal{L})$. In other words,

$$\Psi(1) = \lim_n \frac{1}{n} \log P_n(f, \phi, \mathcal{L}) = \lim_n \frac{1}{n} \log \sum_{h^n \in \mathcal{I}^n} e^{\phi_n(h^n(\Lambda))}.$$

Since $\phi = -\log |\det Df|$, this means that

$$\Psi(1) = \lim_n \frac{1}{n} \log \sum_{h^n \in \mathcal{I}^n} \sup_{h^n(D)} \frac{1}{|\det Df^n|}. \tag{12.4.24}$$

On the other hand, by the formula of change of variables,

$$\text{vol}(f^{-n}(D)) = \sum_{h^n \in \mathcal{I}^n} \text{vol}(h^n(D)) = \sum_{h^n \in \mathcal{I}^n} \int_D \frac{1}{|\det Df^n|} \circ h^n dx.$$

It follows from Lemma 12.4.6 that

$$\inf_{h^n(D)} |\det Df^n| \leq |\det Df^n|(h^n(z)) \leq C_1 \inf_{h^n(D)} |\det Df^n|$$

for every $z \in h^n(D)$ and every $h^n \in \mathcal{I}^n$. Consequently,

$$\text{vol}(f^{-n}(D)) \leq \text{vol}(D) \sum_{h^n \in \mathcal{I}^n} \sup_{h^n(D)} \frac{1}{|\det Df^n|} \leq C_1 \text{vol}(f^{-n}(D)).$$

Combining these inequalities with (12.4.24), we conclude that

$$\limsup_n \frac{1}{n} \log \text{vol}(f^{-n}(D)) \leq \Psi(1) \leq \liminf_n \frac{1}{n} \log \text{vol}(f^{-n}(D)).$$

This proves the identity in the statement of the proposition.

It remains to prove that the volume of the pre-images $f^{-n}(D)$ decays exponentially fast. For that, observe that $f^{-(n+1)}(D) = f^{-n}(D_*)$ is the disjoint union of the images $h^n(D_*)$, with $h^n \in \mathcal{I}^n$. Therefore,

$$\frac{\text{vol}(f^{-(n+1)}(D))}{\text{vol}(f^{-n}(D))} = \frac{\sum_{h^n \in \mathcal{I}^n} \text{vol}(h^n(D_*))}{\sum_{h^n \in \mathcal{I}^n} \text{vol}(h^n(D))} \leq \max_{h^n \in \mathcal{I}^n} \frac{\text{vol}(h^n(D_*))}{\text{vol}(h^n(D))}. \tag{12.4.25}$$

By the formula of change of variables,

$$\begin{aligned} \text{vol}(h^n(D)) &= \int_D \frac{1}{|\det Df^n|} \circ h^n dx \quad \text{and} \\ \text{vol}(h^n(D \setminus D_*)) &= \int_{D \setminus D_*} \frac{1}{|\det Df^n|} \circ h^n dx. \end{aligned}$$

Hence, using Lemma 12.4.6,

$$\frac{\text{vol}(h^n(D \setminus D_*))}{\text{vol}(h^n(D))} \geq \frac{1}{C_1} \frac{\text{vol}(D \setminus D_*)}{\text{vol}(D)} \tag{12.4.26}$$

for every $h^n \in \mathcal{I}^n$. By the hypothesis (12.4.6), the expression on the right-hand side of (12.4.26) is positive. Fix $\beta > 0$ close enough to zero that $1 - e^{-\beta}$ is smaller than that expression. Then

$$\frac{\text{vol}(h^n(D) \setminus h^n(D_*))}{\text{vol}(h^n(D))} \geq 1 - e^{-\beta}$$

for every $h^n \in \mathcal{I}^n$. Combining this inequality with (12.4.25) and the fact that $\text{vol}(h^n(D) \setminus h^n(D_*)) = \text{vol}h^n(D) - \text{vol}h^n(D_*)$, we obtain that

$$\frac{\text{vol}(f^{-(n+1)}(D))}{\text{vol}(f^{-n}(D))} \leq e^{-\beta} \quad \text{for every } n \geq 0$$

(the case $n = 0$ follows directly from the hypothesis (12.4.6)). Hence,

$$\lim_n \frac{1}{n} \log \text{vol}(f^{-n}(D)) \leq -\beta < 0.$$

This concludes the proof of the proposition.

Figure 12.3 summarizes the conclusions in this section. Recall that the function defined by $\Psi(t) = P(f, -t \log |\det Df|)$ is convex, by Proposition 10.3.5.

12.4.5 Upper bound

Here we show that $d(\Lambda) \leq b\ell$ for every $b > 0$ such that $P(f, b\phi) < 0$. In view of the observations in the previous section, this proves that $d(\Lambda) \leq d_0\ell$.

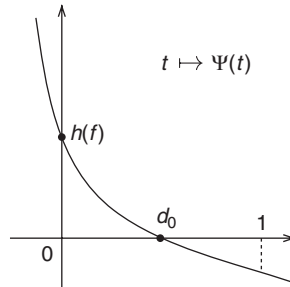


Figure 12.3. Pressure and Hausdorff dimension

Let \mathcal{L} be the open cover of Λ introduced in the previous section and let $b > 0$ be such that $P(f, b\phi) < 0$. The property (12.4.23) implies that

$$P(f, b\phi, \mathcal{L}) = P(f, b\phi) < -\kappa$$

for some $\kappa > 0$. By the definition (10.3.2), it follows that

$$P_n(f, b\phi, \mathcal{L}) \leq e^{-\kappa n} \quad \text{for every } n \text{ sufficiently large.} \tag{12.4.27}$$

It is clear that \mathcal{L}^n is a minimal cover of Λ : no proper subfamily covers Λ . Hence, recalling the definition (10.3.1), the inequality (12.4.27) implies that

$$\sum_{L \in \mathcal{L}^n} e^{b\phi_n(L)} \leq e^{-\kappa n} \quad \text{for every } n \text{ sufficiently large.} \tag{12.4.28}$$

It is clear that every $L \in \mathcal{L}^n$ is compact. Hence, by continuity of the Jacobian,

$$e^{\phi_n(L)} = \sup_L |\det Df^n|^{-1} = |\det Df^n(x)|^{-1}$$

for some $x \in L$. It is also clear that $f^n(L) = \Lambda$ for every $L \in \mathcal{L}^n$. Then, taking $E = L$ in Proposition 12.4.5,

$$[\text{diam } L]^\ell e^{-\phi_n(L)} \leq C_0 [\text{diam } \Lambda]^\ell.$$

Combining this inequality with (12.4.28), we obtain that

$$\sum_{L \in \mathcal{L}^n} [\text{diam } L]^{b\ell} \leq C_0^b [\text{diam } \Lambda]^{b\ell} \sum_{L \in \mathcal{L}^n} e^{b\phi_n(L)} \leq C_0^b [\text{diam } \Lambda]^{b\ell} e^{-\kappa n}$$

for every n sufficiently large. Since the expression on the right-hand side converges to zero, and the diameter of the covers \mathcal{L}^n also converges to zero, it follows that $m_{bc}(M) = 0$. Therefore, $d(M) \leq b\ell$.

12.4.6 Lower bound

Now we show that $d(\Lambda) \geq a\ell$ for every a such that $P(f, a\phi) > 0$. This implies that $d(\Lambda) \geq d_0\ell$, which completes the proof of Theorem 12.4.3.

As observed in the previous section, the cover \mathcal{L} realizes the pressure and all its iterated sums \mathcal{L}^n are minimal covers of Λ . Hence, the choice of a implies that there exists $\kappa > 0$ such that

$$P_n(f, a\phi, \mathcal{L}) = \sum_{L \in \mathcal{L}^n} e^{a\phi_n(L)} \geq e^{\kappa n} \quad \text{for every } n \text{ sufficiently large.} \tag{12.4.29}$$

Fix such an n . Let $\varepsilon > 0$ be a lower bound for the distance between any two elements of \mathcal{L}^n : a lower bound does exist because the elements of \mathcal{L}^n are compact and pairwise disjoint. Fix $\rho \in (0, \varepsilon^{a\ell})$. The reason for this choice will be clear soon. We claim that

$$\sum_{U \in \mathcal{U}} [\text{diam } U]^{a\ell} \geq 2^{-a\ell} \rho \tag{12.4.30}$$

for every cover \mathcal{U} of Λ . By definition, this implies that $m_{al}(\Lambda) \geq 2^{-al}\rho > 0$ and, consequently, $d(\Lambda) \geq al$. Therefore, to end the proof of Theorem 12.4.3 it suffices to prove this claim.

Let us suppose that there exists some open cover of Λ which does not satisfy (12.4.30). Then, using Exercise 12.4.3, there exists some open cover \mathcal{U} of Λ with

$$\sum_{U \in \mathcal{U}} [\text{diam } U]^{al} < \rho < \varepsilon^{al}. \tag{12.4.31}$$

By compactness, we may suppose that this open cover \mathcal{U} is finite. The relation (12.4.31) implies that every $U \in \mathcal{U}$ has diameter less than ε . Hence, each $U \in \mathcal{U}$ intersects at most one $L \in \mathcal{L}^n$. Since \mathcal{L}^n covers Λ and U is a non-empty subset of Λ , we also have that U intersects some $L \in \mathcal{L}^n$. This means that \mathcal{U} is the disjoint union of the families

$$\mathcal{U}_L = \{U \in \mathcal{U} : U \cap L \neq \emptyset\}, \quad L \in \mathcal{L}^n.$$

If $U \in \mathcal{U}_L$ then $U \subset L$. Let us consider the families $f^n(\mathcal{U}_L) = \{f^n(U) : U \in \mathcal{U}_L\}$. Observe that each one of them is a cover of Λ . Moreover, using Proposition 12.4.5,

$$\sum_{V \in f^n(\mathcal{U}_L)} [\text{diam } V]^{al} = \sum_{U \in \mathcal{U}_L} [\text{diam } f^n(U)]^{al} \leq C_0 e^{-a\phi_n(L)} \sum_{U \in \mathcal{U}_L} [\text{diam } U]^{al}. \tag{12.4.32}$$

Therefore,

$$\sum_{U \in \mathcal{U}} [\text{diam } U]^{al} = \sum_{L \in \mathcal{L}^n} \sum_{U \in \mathcal{U}_L} [\text{diam } U]^{al} \geq \sum_{L \in \mathcal{L}^n} C_0^{-1} e^{a\phi_n(L)} \sum_{V \in f^n(\mathcal{U}_L)} [\text{diam } V]^{al}.$$

Let us suppose that

$$\sum_{V \in f^n(\mathcal{U}_L)} [\text{diam } V]^{al} \geq \sum_{U \in \mathcal{U}} [\text{diam } U]^{al} \quad \text{for every } L \in \mathcal{L}^n.$$

Then, the previous inequality implies

$$\sum_{U \in \mathcal{U}} [\text{diam } U]^{al} \geq \sum_{L \in \mathcal{L}^n} C_0^{-1} e^{a\phi_n(L)} \sum_{U \in \mathcal{U}} [\text{diam } U]^{al} \geq C_0^{-1} e^{kn} \sum_{U \in \mathcal{U}} [\text{diam } U]^{al}.$$

This is a contradiction, because $e^{kn} > C_0$. Hence, there exists $L \in \mathcal{L}^n$ such that

$$\sum_{V \in f^n(\mathcal{U}_L)} [\text{diam } V]^{al} \leq \sum_{U \in \mathcal{U}} [\text{diam } U]^{al} < \rho.$$

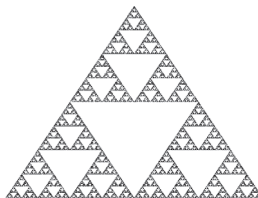


Figure 12.4. Sierpinski triangle

Thus, we may repeat the previous procedure with $f^n(\mathcal{U}_L)$ in the place of \mathcal{U} . Observe, however, that $\#f^n(\mathcal{U}_L) = \#\mathcal{U}_L$ is strictly less than $\#\mathcal{U}$. Therefore, this process must stop after a finite number of steps. This contradiction proves the claim 12.4.30.

The proof of Theorem 12.4.3 is complete. However, it is possible to prove an even stronger result: in the conditions of the theorem, the Hausdorff measure of Λ in dimension $d(M)$ is positive and finite. We leave this statement as a special challenge (Exercise 12.4.7) for the reader who remained with us till the end of this book!

12.4.7 Exercises

12.4.1. Let $d = \log 2 / \log 3$. Show that $(x_1 + 1 + x_2)^d \geq x_1^d + x_2^d$ for every $x_1, x_2 \in [0, 1]$. Moreover, the identity holds if and only if $x_1 = x_2 = 1$.

12.4.2. Let $f : M \rightarrow N$ be a Lipschitz map, with Lipschitz constant L . Show that

$$m_d(f(A)) \leq L^d m_d(A)$$

for any $d \in (0, \infty)$ and any $A \subset M$. Use this fact to show that if $A \subset \mathbb{R}^n$ and $t > 0$, then $m_d(tA) = t^d m_d(A)$, where $tA = \{tx : x \in A\}$.

12.4.3. Represent by $m_d^o(M)$ and $m_d^c(M)$ the numbers defined in the same ways as the Hausdorff measure $m_d(M)$ but considering only covers by open sets and covers by closed sets, respectively. Show that $m_d^o(M) = m_d^c(M) = m_d(M)$.

12.4.4. (Mass distribution principle) Let μ be a finite measure on a compact metric space M and assume that there exist numbers $d, K, \rho > 0$ such that $\mu(B) \leq K(\text{diam } B)^d$ for every set $B \subset M$ with diameter less than ρ . Show that if $A \subset M$ is such that $\mu(A) > 0$ then $m_d(A) > 0$ and so $d(A) \geq d$.

12.4.5. Use the mass distribution principle to show that the Hausdorff dimension of the *Sierpinski triangle* (Figure 12.4) is equal to $d_0 = \log 3 / \log 2$ and the Hausdorff measure in dimension d_0 is positive and finite.

12.4.6. Check the pressure formula (12.4.12).

12.4.7. Adapting arguments from Exercise 12.4.5, show that in the conditions of Theorem 12.4.3 one has $0 < m_{d(\Lambda)}(\Lambda) < \infty$.