Flat surfaces: computational problems

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Investigate these problems in small groups. Some are delicate and involve a mix of thinking and computing. They are roughly ordered by increasing difficulty. Suggested softwares and libraries: SageMath¹, flipper², snappy³, surface_dynamics⁴, flatsurf⁵.

Most articles in the reference are available for download at http://www.labri.fr/perso/vdelecro/grenoble_school/refs/

1 Where are my commutators?

Recall from Carlos Matheus's lectures that an origami is given by a pair of permutations $(r, u) \in S_n \times S_n$ that together generate a transitive group, and tell, for each square on the surface, which square you find moving right (r) and moving up (u), see figure 1. The stratum $\mathcal{H}(\mu)$ of the origami given by (r, u) can be deduced from the cycle lengths of the commutator $[r, u] = rur^{-1}u^{-1}$.



Figure 1: Gluing pattern of the origami given by permutations r and u.

Picking (r, u) uniformly at random in $S_n \times S_n$, and discarding pairs that generate a non transitive group, how is the genus g of the associated origami distributed? For given n and g, how do these origamis distribute among the different strata?

2 Lyapunov exponents of a periodic windtree model

The periodic windtree model $W_{a,b}$ is the billiard in the plane \mathbb{R}^2 with size $a \times b$ rectangular obstacles at all \mathbb{Z}^2 lattice points. The associated flat surface is a \mathbb{Z}^2 -cover of a genus 5 translation surface called X(a,b) (see [DHL14, §3] and Figure 2)).

Prove that X(1/2, 1/2) is a 12-square origami. Construct it with the package surface_dynamics. Compute approximations of its Lyapunov exponents. Do they match [DHL14, Theorem 7, page 1099]?

3 Arnoux-Yoccoz pseudo-Anosov example

The Arnoux-Yoccoz pseudo-Anosov homeomorphism f of a genus 3 surface is defined in [AY81, Bo13].

- 1. Show that f fixes each singularity but does permute the separatrices.
- 2. Can f be obtained from Rauzy induction?
- 3. Can you decompose f as a product of Dehn twist?

Hint: you might want to use flipper to build the map f from a sequence of triangle flips.

4 World cup special (W. Veech)

A soccer ball is usually sowed from twenty regular hexagons and twelve regular pentagons. The corresponding polyhedron is a buckyball or truncated icosahedron, one of the 13 archimedean solids.

Any polytope in \mathbb{R}^3 gives rise to a flat conical metric on the sphere (ie the metric is flat except at the vertices). When the angles of the polytope are rational multiples of π then there is an appropriate

¹More information on SageMath at https://www.sagemath.org

²More information on flipper at http://flipper.readthedocs.io

³More information on snappy at https://www.math.uic.edu/t3m/SnapPy/

⁴More information on surface_dynamics at http://www.labri.fr/perso/vdelecro/flatsurf_sage.html

 $^{^5\}mathrm{More}$ information on flatsurf at https://github.com/videlec/sage-flatsurf

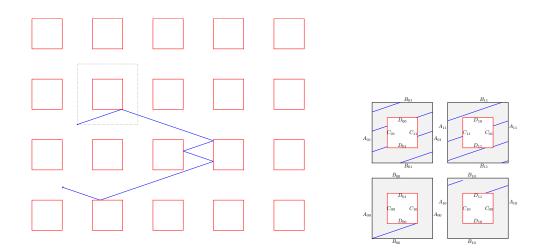


Figure 2: Left: a piece of trajectory in a periodic windtree model of Hardy-Weber. Right: same piece of trajectory, lifted to the associated translation surface, and projected to the surface X(a, b).

ramified cover that is a translation surface. In the case of the buckyball, what are the properties of the translation surface S obtained in this way? In particular...

- 1. What is its genus? its stratum?
- 2. What is its group of translation automorphisms?
- 3. Is it a Veech surface?
- 4. What can be said about its affine group?

Hint: you might want to use flatsurf, which allows dealing with conical metrics and taking covers.

5 Lost in triangles

There are several famous conjectures about billiards in triangles. Namely

- 1. Does every triangular billiard admit a periodic trajectory? (in the rational case, it is known that there are infinitely many of them [Ma86] and the answer is positive for all triangles for which all angles are smaller than 100 degrees [Sc09])
- 2. How does behave the complexity of a billiard sequence? (in the rational case, this is known to be linear [Hu95] while for irrational ones some polynomial upper bounds are kown [GT95])
- 3. Can you find open sets of triangles that admit a periodic orbit? ([HS] for examples)

Investigate these two open problems using a computer program. For example

- Given a triangular billiard, find all possible codings of length *n*. What is the triangle that gives the largest complexity?
- Given a coding such as *ABACCABA* find the set of angles for which this can be realized as a coding of a piece of trajectory.
- What about periodic trajectories?

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