

Chapter 3

Symmetries

Recall from Section 1.1.5 of in Chapter 1 that a map $f : M \rightarrow M'$ between two translation surfaces is called affine if f in flat charts of M and M' is an invertible \mathbb{R} -affine map. The linear part of this affine map does not depend on the choice of flat charts and give rise to a group morphism $D : \text{Aff}(M) \rightarrow \text{GL}(2, \mathbb{R})$ between the affine automorphism of M and $\text{GL}(2, \mathbb{R})$ called the derivative map. By definition, the image $\Gamma(M) := D(\text{Aff}(M))$ is the Veech group of M and its kernel $\text{Tr}(M) := \ker(D)$ is the group of translations. An important subgroup of $\text{Aff}(M)$ is the subgroup of isometries $\text{Isom}(M) := D^{-1}(\text{SO}(2, \mathbb{R}))$.

By $\text{Aff}^+(M)$, $\text{Isom}^+(M)$ and $\Gamma^+(M)$ we denote the subgroups of the affine, isometry and Veech groups formed by orientation-preserving elements. Note that the indices of $\text{Aff}^+(M)$ in $\text{Aff}(M)$, $\text{Isom}^+(M)$ in $\text{Isom}(M)$ and $\Gamma^+(M)$ in $\Gamma(M)$ are at most 2.

The objective of this chapter is to study $\text{Aff}(M)$ when M is of infinite type. In Section 3.1 we consider $\text{Isom}(M)$ and its subgroup $\text{Tr}(M)$ as groups of holomorphic maps with respect to the underlying Riemann surface structure of M . Next, in Section 3.2 we provide various constructions that show in particular that any subgroup of $\text{SL}(2, \mathbb{R})$ can be realized as the Veech group of an infinite-type translation surface. In Section 3.3 we consider translation coverings $p : \widetilde{M} \rightarrow M$ where M is a finite-type surface. We will show the importance of the action of $\text{Aff}(M)$ on the homology of M in the case of Abelian coverings. We provide in particular a detailed study of the affine automorphisms of the wind-tree model $W_{a,b}$. Finally in Section 3.4 we define the Hooper-Thurston-Veech construction that was mentioned in Section 1.2.6. Finally, using some geometric tools from the previous chapter, we compute the affine groups of baker's surfaces $B_{1/q}$ with $q \in \{2, 3, \dots\}$.

3.1 Analytic affine automorphisms: isometries and translations

Through this section, we adopt the analytic point of view for translation surfaces. That is a translation surface M will be thought as a couple (X, ω) where X is a Riemann surface and ω a non-zero holomorphic one-form (see Definition 1.1.9 and Theorem 1.1.11). We denote $\text{Aut}(X)$ the group of analytic automorphisms of the Riemann surface X . The following elementary lemma whose proof is left to the reader characterizes the intersection between $\text{Aff}(X, \omega)$ and $\text{Aut}(X)$.

Lemma 3.1.1. *An affine homeomorphism $f \in \text{Aff}(X, \omega)$ is analytic if and only if $D(\phi)$ is a similitude, i.e. a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ that corresponds to multiplication by $a + \sqrt{-1}b$ in \mathbb{C} under the identification $\mathbb{R}^2 = \mathbb{C}$.*

In particular orientable isometries, and hence translations, are automorphisms of the underlying Riemann surface: $\text{Tr}(X, \omega) \subset \text{Isom}^+(X, \omega) \subset \text{Aut}(X)$.

A classical result of A. Hurwitz gives upper bounds on $\text{Aut}(X)$ for compact Riemann surfaces X .

Theorem 3.1.2: Hurwitz's automorphism theorem [Hur92]

Let X be a compact Riemann surface of genus $g \geq 2$. Then the group $\text{Aut}(X)$ of analytic automorphism of X is finite and $|\text{Aut}(X)| \leq 84(g - 1)$.

In genus 0 or 1 the automorphism groups are infinite. The following result provides an analogue of Hurwitz theorem for the subgroup of translations $\text{Tr}(X, \omega)$.

Theorem 3.1.3: [SPWS]

Let (X, ω) be a compact translation surface of genus $g \geq 2$. Then $|\text{Tr}(X, \omega)| \leq 4g - 4$.

Moreover, the order is exactly $4g - 4$ if and only if (X, ω) is a normal square-tiled surface^a in the principal stratum $\mathcal{H}(1, \dots, 1)$, that is, when ω has $2g - 2$ simple zeroes. And this could happen only if g is odd or $g - 1$ is divisible by 3.

^aThat is, a square-tiled surface $p : M \rightarrow \mathbb{T}^2$ defined by a normal subgroup of $\pi_1(\mathbb{T}^2 \setminus \{0\})$, see Example 2.2.9.

As a consequence of Hurwitz theorem, for every compact Riemann surface X of genus $g \geq 2$, the group $\text{Aut}(X)$ acts properly discontinuously on X (see Definition B.0.1). The following result provides the list of surfaces for which automorphisms do not act properly discontinuously (or equivalently, for which the group of automorphism is not countable).

Theorem 3.1.4

Let X be a Riemann surface such that its group $\text{Aut}(X)$ of analytic automorphism does not act properly discontinuously on X . Then X is conformally equivalent to one of the following surfaces:

- the Riemann sphere $\hat{\mathbb{C}}$ for which $\text{Aut}(\hat{\mathbb{C}}) = \text{SL}(2, \mathbb{C})$,
- the plane \mathbb{C} for which $\text{Aut}(\mathbb{C}) = \mathbb{C}^* \rtimes \mathbb{C}$,
- the half-plane \mathbb{H}^2 for which $\text{Aut}(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$,
- an annulus $\{z \in \mathbb{C}; r < |z| < R\}$ with $0 < r < R \leq \infty$ for which $\text{Aut}(X)$ contains a subgroup isomorphic to \mathbb{S}^1 ,
- the punctured plane \mathbb{C}^* for which $\text{Aut}(\mathbb{C}^*) = \mathbb{C}^*$ or
- a torus \mathbb{R}^2/Λ , in which case $\text{Aut}(\mathbb{R}^2/\Lambda)$ contains the group of translations \mathbb{R}^2/Λ .

The main ingredient in the proof is the Poincaré-Koebe's uniformization theorem.

Proof. Let X be a Riemann surface, \tilde{X} its universal cover and $G = \text{Aut}(\tilde{X})$. Let $H \subset G$ be the subgroup acting freely and properly discontinuously on \tilde{X} so that $X = \tilde{X}/H$. In particular the action of H on \tilde{X} has no fixed points.

Let $N_G(H) = \{g \in G \mid gH = Hg\}$ be the normalizer of H in G . We claim that $\text{Aut}(X)$ is in bijection with $N_G(H)/H$. Indeed, every element $\varphi \in \text{Aut}(X)$ can be lifted to an element $\tilde{\varphi}$ which lives in $N_G(H)$ for $\tilde{\varphi}$ must send H -orbits in the universal cover to H -orbits. On the other hand, every element $\tilde{\psi} \in N_G(H)$ descends to an analytic automorphism $\psi \in \text{Aut}(X)$. Moreover $\psi = \text{Id}_X$ if and only if $\tilde{\psi}$ leaves each H -orbits in the universal cover invariant, that is, if $\tilde{\psi} \in H$.

We want to list the possible group H for which $\text{Aut}(X) \simeq N_G(H)/H$ is non-discrete. Using the uniformization theorem, we proceed by cases considering the three different universal covers that X can have:

- $\tilde{X} = \mathbb{S}^2$. Given that every non-trivial analytic automorphism of the Riemann sphere has a fixed point and H cannot have fixed points we have that $X = \tilde{X}$.
- $\tilde{X} = \mathbb{C}$. The only analytic automorphisms of \mathbb{C} without fixed points are the translations. Hence $H = \{1\}$, $H = \mathbb{Z}u$ or $H = \mathbb{Z}u \oplus \mathbb{Z}v$, where u and v are complex numbers different from zero such that $u/v \notin \mathbb{R}$. The resulting quotients are respectively, the plane, a cylinder conformally equivalent to \mathbb{C}^* and a torus.
- $\tilde{X} = \mathbb{H}^2$. Let us first remark that the group $N_G(H)$ is closed in $\text{PSL}(2, \mathbb{R})$. Hence it is either discrete or contains a one-parameter subgroup. The one-parameter subgroups of $\text{PSL}(2, \mathbb{R})$ are well known and are of one of the following type:

– parabolic, i.e. conjugate to $P = \{h_t\}$ where $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, with infinitesimal generator

$$\mathfrak{h} := \left. \frac{d}{dt} \right|_{t=0} h_t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

– hyperbolic, i.e. conjugate to $A = \{g_t\}$ where $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, with infinitesimal generator

$$\mathfrak{a} := \left. \frac{d}{dt} \right|_{t=0} g_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

- rotational, i.e. conjugate to $R = \{r_\theta\}$ where $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ with infinitesimal generator $\mathfrak{r} := \frac{d}{d\theta}|_{\theta=0} r_\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From now on we assume that $\tilde{X} = \mathbb{H}^2$, that $H \neq \{1\}$ (that corresponds to $X = \mathbb{H}^2$) and that the normalizer $N_G(H)$ contains a one-parameter subgroup $\{s_t\}_{t \in \mathbb{R}}$. For all $m \in H$ and all $t \in \mathbb{R}$, the conjugate $s_t m s_{-t}$ belongs to H . As H is discrete, for t small enough $s_t m s_{-t} = m$ or equivalently $s_t m = m s_t$. Taking the derivative at $t = 0$ we obtain that

$$\mathfrak{s}m = m\mathfrak{s} \quad (3.1)$$

where $\mathfrak{s} = \frac{d}{dt}|_{t=0} s_t$ is the infinitesimal generator of the one-parameter subgroup. We compute the equations obtained in the three cases of subgroups. Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in H . Up to conjugation we can assume that $\{s_t\}$ is one of P , A or R defined above.

In the parabolic case, equation (3.1) writes $\mathfrak{h}m = m\mathfrak{h}$, that is

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}.$$

Hence, $c = 0$ and $a = d$. This implies that either m acts like the identity on \mathbb{H}^2 or $m \in P$. In the former case we conclude that $X = \mathbb{H}^2$. In the later that $H = \langle h_{t_0} \rangle$ for some t_0 (because H is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$). In this case, X is conformally equivalent to the punctured unit disc (which is conformally equivalent to an infinite annulus $\{z \in \mathbb{C}; r < |z| \leq \infty\}$). Moreover, in this case P/H is isomorphic to \mathbb{S}^1 .

Now, assume that $\{s_t\} = A$. Then equation (3.1) gives $\mathfrak{a}m = m\mathfrak{a}$, that is

$$\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

and hence $b = c = 0$. In other words $m \in A$. Hence $H = \langle g_{t_0} \rangle$ for some t_0 and X is conformally equivalent to a finite annulus $\{z \in \mathbb{C}; r < z < R\}$ with $0 < r < R < \infty$. Moreover, in this case P/H is isomorphic to \mathbb{S}^1 .

Finally, let us consider the case $\{s_t\} = R$. Equation (3.1) gives $\mathfrak{r}m = m\mathfrak{r}$ which writes

$$\begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}.$$

Hence, $a = d$ and $b = -c$. In other words $m \in R$. Given that every non-trivial element of R has a fixed point in \mathbb{H}^2 and the action of H on \tilde{X} has no fixed points we conclude that $H = \{1\}$, which was excluded.

□

We will now use Theorem 3.1.4 to classify translation surfaces with uncountable translation group.

Theorem 3.1.5

Let (X, ω) be a translation surface whose translation group does not act properly discontinuously. Then there exists $\lambda \in \mathbb{C}^*$ such that $(X, \lambda\omega)$ is isomorphic as a translation surface to one of the following:

1. the plane (\mathbb{C}, dz) or the half plane (\mathbb{H}^2, dz) ,
2. a torus $(\mathbb{C}/(\mathbb{Z}u \oplus \mathbb{Z}v), \pi_* dz)$, where $\pi : \mathbb{C} \rightarrow \mathbb{C}/(\mathbb{Z}u \oplus \mathbb{Z}v)$ is the natural projection.
3. a strip $(\{z \in \mathbb{C} \mid -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\}, dz)$, isomorphic to $(\mathbb{H}^2, \frac{dz}{z})$,
4. a finite cylinder $(\{z \in \mathbb{C} \mid -\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\}/(z \sim z + t), dz)$, with $0 < t < \infty$.
5. an infinite cylinder $(\{z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \mathrm{Im}(z) \leq \frac{\pi}{2}, R < \mathrm{Re}(z)\}/(z \sim z + i\pi), dz)$, where $-\infty \leq R$.

Proof. The strategy of the proof is the following: we determine first by direct calculation all simply-connected translation surfaces (X, ω) for which $\text{Trans}(X, \omega) < \text{Aut}(X)$ does not act properly discontinuously. Then we use the list of theorem 3.1.4 to get all possible (non-simply connected) quotients.

The sphere is excluded since it supports no Abelian differential. One-parameter subgroups of $\text{Aut}(\mathbb{C})$ are of the form $z \rightarrow z + tw$ or $z \rightarrow e^{t\alpha}z$ for some fixed $\alpha, w \in \mathbb{C}^*$ and $t \in \mathbb{R}$. Let $(\mathbb{C}, \omega = f(z)dz)$, with f an entire function, be an Abelian differential. This form has to be invariant under the one-parameter subgroups of $\text{Aut}(\mathbb{C})$ mentioned before. The first possibility yields:

$$f(z)dz = f(z + tw)dz. \quad (3.2)$$

As this holds for arbitrarily small t , necessarily f is constant so that $\omega = \alpha dz$ for $\alpha \in \mathbb{C}^*$. For the second possibility we get

$$f(z)dz = f(e^{t\alpha}z)e^{t\alpha}dz. \quad (3.3)$$

Taking derivatives with respect to t and setting $t = 0$ we obtain

$$0 = zf'(z) + f(z).$$

The only non-identically zero solutions to this equation are meromorphic functions $f(z) = c/z$ with $c \in \mathbb{C}^*$. This case yields forms which are not Abelian differentials, hence it is excluded.

We now determine all translation surfaces $(\mathbb{H}^2, f(z)dz)$ which are invariant under one-parameter subgroups of $\text{Aut}(\mathbb{H}^2)$. As in the proof of Theorem 3.1.4, there are three one-parameter groups to consider: P (parabolic), A (hyperbolic) and R (rotational). Forms invariant under the parabolic subgroup must be translation invariant, hence must satisfy equation (3.2) and therefore are of the form αdz , for some $\alpha \in \mathbb{C}^*$. Forms invariant under R are also of the form αdz , for some $\alpha \in \mathbb{C}^*$. This is best seen when performing a direct calculation of the invariance equation in the Poincaré disc. Forms invariant under the hyperbolic one-parameter group A must satisfy equation (3.3), for $\alpha = 2$. Hence in this case $\omega = \frac{\alpha dz}{z}$.

In summary: simply connected translation surfaces for which $\text{Trans}(X, \omega) < \text{Aut}(X)$ does not act properly discontinuously are all projectively isomorphic to either: (\mathbb{C}, dz) , (\mathbb{H}^2, dz) or $(\mathbb{H}^2, \frac{dz}{z})$.

We now use theorem (3.1.4). From the list that this results presents we deduce that the only possible (translation) quotients of (\mathbb{C}, dz) are this surface itself, a bi-infinite cylinder of the form $\{z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \text{Im}(z) \leq \frac{\pi}{2}, -\infty < \text{Re}(z)\}/(z \sim z + i\pi)$ or a torus. On the other hand, the only possible (translation) quotients of (\mathbb{H}^2, dz) are this surface itself or a infinite half-cylinder $\{z \in \mathbb{C} \mid -\frac{\pi}{2} \leq \text{Im}(z) \leq \frac{\pi}{2}, R < \text{Re}(z)\}/(z \sim z + i\pi)$, where $-\infty < R$. Finally, the quotients of $(\mathbb{H}^2, \frac{dz}{z})$ are this surface itself (which is isometric to an infinite strip) or finite cylinders of the form $\{z \in \mathbb{C} \mid -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2}\}/(z \sim z + t)$, with $0 < t < \infty$. \square

Exercise 3.1.6

For each case in Corollary 3.1.5 compute $\text{Trans}(X, \omega)$ and $\text{Isom}(X, \omega)$.

Exercise 3.1.7

Give examples which are not in the list of Corollary 3.1.5 of surfaces for which $\text{Trans}(X, \omega)$ acts properly discontinuously on (X, ω) but $\text{Isom}(X, \omega)$ does not.

Only two cases of Theorem 3.1.5 have finite area. Hence we obtain the following:

Corollary 3.1.8. *Except the case of tori and finite cylinders, any finite area translation surface has a finite group of translations.*

3.2 Every group is a Veech group

Recall that the Veech group $\Gamma(M)$ of a translation surface M is the subgroup of $\text{GL}(2, \mathbb{R})$ formed by the derivatives of elements in $\text{Aff}(M)$ and $\Gamma^+(M) = D(\text{Aff}^+(M))$ are the derivatives of orientation-preserving elements. For a finite-type surface M the Veech group $\Gamma(M)$ is Fuchsian, *i.e.*, a discrete subgroup of $\text{SL}(2, \mathbb{R})$. Moreover, this group plays an important role in the dynamics in the dynamics of the translation flows. See C.6 in Appendix C and references therein for a more detailed discussion.

In this section we discuss two simple and naïve questions in the context of infinite-type translation surfaces: are there any restrictions for a Fuchsian group to be Veech? Do Veech groups play a role in the dynamics of the translation flows? As we explain in what follows, the answer to both of these questions is no.

3.2.1 Veech groups of Loch Ness monsters

Theorem 3.2.1: [PSV11]

Let A be an at most countable group and $F : A \rightarrow \mathrm{GL}^+(2, \mathbb{R})$ a group morphism. Then there exists a translation surface M homeomorphic to the Loch Ness monster and a group isomorphism

$$\begin{aligned} A &\rightarrow \mathrm{Aff}^+(M) \\ a &\mapsto f_a \end{aligned}$$

such that $F(a) = D(f_a)$ where $D : \mathrm{Aff}^+(M) \rightarrow \mathrm{GL}^+(2, \mathbb{R})$ is the derivative map.

Recall from Chapter 2 that the Loch Ness monster is the (homeomorphism class of) topological surface of infinite genus with only one end (see Definition 2.1.9). Loosely speaking, this is the simplest infinite genus surface. It is the homeomorphism type of the translation surface $M(P)$ associated to an irrational polygon P via unfolding (Theorem 1.2.3 proven in Example 2.2.15) as well as the infinite staircase (Exercise 2.2.6).

Proof. The proof we present is constructive and is inspired by [PSV11, Construction 4.6]. We make the construction under the hypothesis that A is not the trivial group. In the case A is the trivial group, one has to construct a translation surface homeomorphic to the Loch Ness monster with trivial affine group.

The idea of the construction is to consider a normal ramified covering $\pi : M \rightarrow \mathbb{R}^2$ with Deck group A . The construction is done so that the deck group action is done via affine elements of M with derivatives prescribed by $F : A \rightarrow \mathrm{GL}^+(2, \mathbb{R})$. Note that unless the image of F is identically Id , the map $\pi : M \rightarrow \mathbb{R}^2$ will not be a translation covering. This construction will ensure that A is contained in the affine group. In order to prevent any additional affine automorphism we perform in a second step some additional surgeries.

We will make use of the slit operation that was introduced in Definition 1.2.16 Section 1.2.5 to describe the Panov planes and that we recall now. Let M_1 and M_2 be two translation surfaces and $\gamma_1 \subset M_1$ and $\gamma_2 \subset M_2$ two closed parallel straight-line segments of equal length which do not encounter any singularities. Then cut the translation surfaces M_1 and M_2 along γ_1 and γ_2 respectively and glue these cuts back so that M_1 becomes attached to M_2 (see the left hand side of Figure 1.15). The translation surface M obtained this way presents two singularities of angle 4π at the extremities of the cuts.

Step 1. We consider \mathbb{R}^2 as a translation surface and construct a normal cover with deck group A . Let us fix a generating set for A with a given ordering $U = \{a_1, a_2, \dots\}$. For each a_j in this generating set let $(u_j, v_j) := F(a_j)^{-1}(1, 0)$ and define

$$x_j := |u_j| + 1 \quad y_j := -(j + |v_1| + |v_2| + \dots + |v_j|).$$

For $i \in \mathbb{N}$, we consider the following segments in \mathbb{R}^2

$$m_i^j := [(2i, j), (2i + 1, j)] \quad m_i^{-j} := [(ix_j, y_j), (ix_j + u_j, y_j + v_j)].$$

See Figure 3.1 for an illustration. Let us remark that these segments are pairwise disjoint.

For each $a \in A$, consider a copy $M'_a \simeq \mathbb{R}^2$ of the Euclidean plane. For each $a \in A$, $a_j \in U$ and $i \in \mathbb{N}$ glue the segment $F(a) \cdot m_i^j$ in M'_a to the segment $F(aa_j) \cdot m_i^{-j}$ in M'_{aa_j} . Note that by definition of m_i^j and m_i^{-j} these segments are parallel and of equal length. We denote by M' the resulting translation surface. It is a covering $\pi : M' \rightarrow \mathbb{R}^2$ ramified over the endpoints of the slits m_i^j and m_i^{-j} and the copies M'_a of the Euclidean plane constitute sheets of this covering.

By construction the group A acts by deck group on M'_a . This action is affine and we denote by $f_a \in \mathrm{Aff}^+(M')$ the corresponding element. We have $D(f_a) = F(a)$.

We now prove that M' is homeomorphic to the Loch Ness monster. The surface M' is first of all connected. Next, the surfaces M'_a and M'_{aa_j} are glued along two infinite families of parallel slits. And these slits do not accumulate in the metric completion of M'_a and M'_{aa_j} . For each slit m_i^j , the boundary

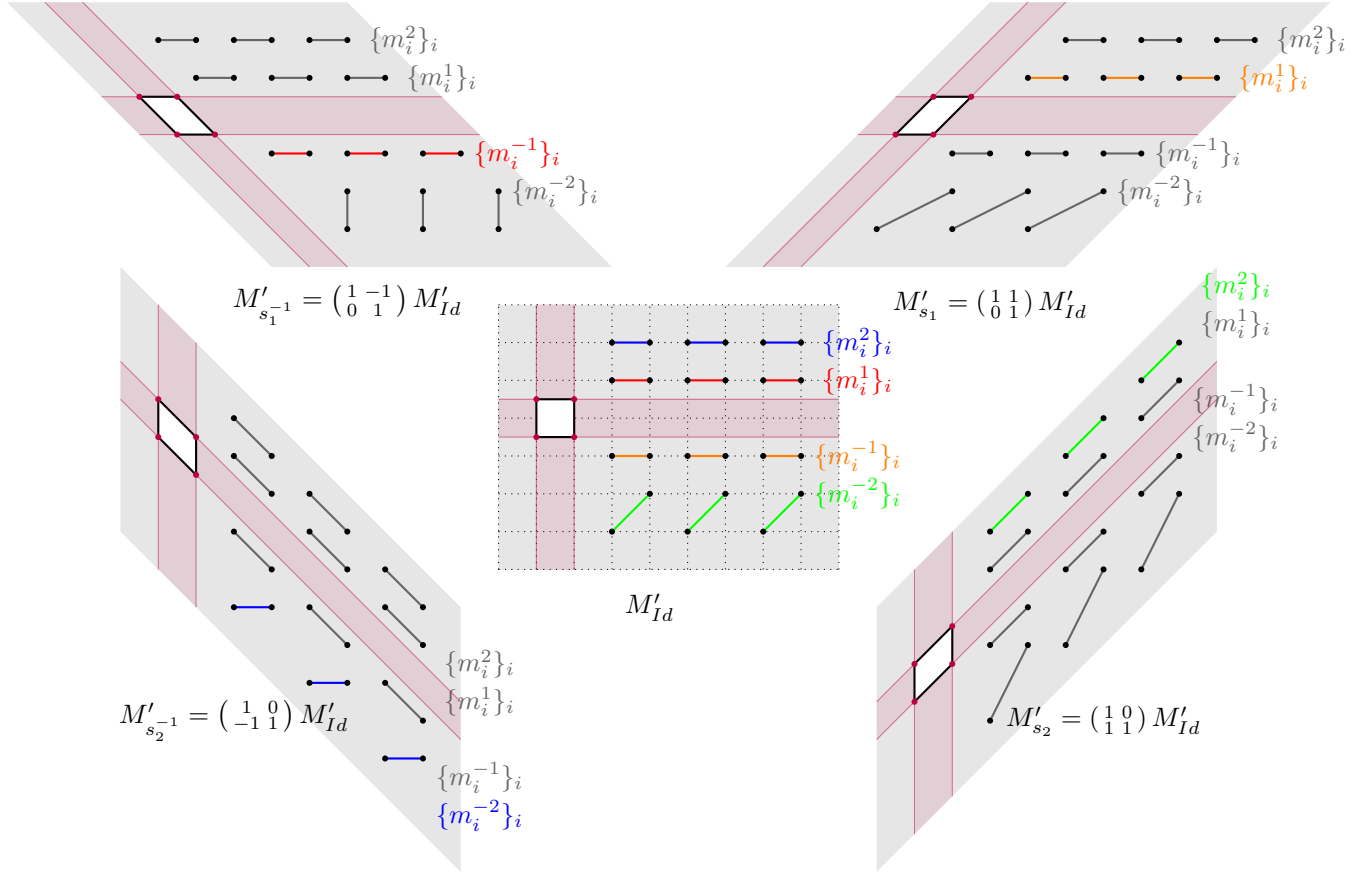


Figure 3.1: An example of the ramified covering $\pi : M \rightarrow M_{Id}$ as in the proof of Theorem 3.2.1. The group is $A = \mathrm{SL}_2(\mathbb{Z})$ and $F : \mathrm{SL}(2, \mathbb{Z}) \rightarrow \mathrm{SL}(2, \mathbb{R})$ is the inclusion.

of a sufficiently small regular neighbourhood of $F(a) \cdot m_i^j$ in M'_a defines a non-separating essential curve in M'_a . Because the slit are disjoint, one can choose these essential curves to be disjoint. Therefore M' has infinite genus. To see that $\mathrm{Ends}(M')$ has only one point we use Lemma 2.1.11 from Section 2.2. If $K \subset M'_a$ is compact then there exists a ball B in $\mathbb{R}^2 \simeq M'_{Id}$ such that K is contained in $\bigcup_{a \in A} s_a(B)$

where $s_a : \mathbb{R}^2 \rightarrow M'_a$ are the sections to the sheets of M' . But since we add infinitely many slits, the complement of this union is still connected. This concludes the fact that M' is a Loch Ness monster.

Step 2. The problem with the surface M' is that it might admit more symmetries than its deck group A . We perform a surgery on M' to avoid this. By doing so, we are careful to preserve the fact that we keep a Loch Ness monster.

Let $[-2, -1] \times [-1/2, 1/2]$ be the square in $M'_{Id} \simeq \mathbb{R}^2$. We remove Z from the surface and glue back the opposite sides left open in the surface M_{Id} . This creates a new surface M_{Id} with a conical singularity p_{Id} of angle 6π with an horizontal and a vertical singularity joining p_{Id} to itself. Moreover, if we draw the eight half-lines starting from the singularity in the direction of these two saddle connections, we separate M'_{Id} into 4 quadrants and 2 infinite bands. Among these quadrants, two of them contain no singularities. See Figure 3.1.

We perform the same surgery in M'_a for $a \in A$ by removing the rectangle $F(a) \cdot Z$. We denote this new surface M_a and by $p_a \in M_a$ the singularity of angle 6π coming from the surgery in the surface M'_a . We denote by M the surface obtained by gluing the slits m_i^j as in step 1. M is homeomorphic to the Loch Ness monster (the same argument we gave in step 1 applies). The deck group A acts by affine transformation and we keep the notation f_a to denote the action of a . Note that M is a ramified covering of the plane \mathbb{R}^2 from which we removed the rectangle $[-2, -1] \times [-1/2, 1/2]$.

Now let $f \in \mathrm{Aff}^+(M)$ be an affine automorphism. Because the only conical singularities of angle 6π in M are the p_a we need to have that $f(p_{Id}) = p_{a_0}$ for some $a_0 \in A$. We claim that $f = f_{a_0}$. Indeed, the horizontal and vertical saddle connections in M_{Id} joining p_{Id} to itself must be mapped to saddle connections joining p_{a_0} to itself. Moreover, the 2 infinite bands parallel to these saddle connections and

the four quadrants are mapped to infinite bands and quadrants. Since two of these quadrants do not contain any singularity, the only possibility is that $f_{a_0} = f$. \square

Recall that the Veech dichotomy for finite-type translation surface states that the translation flows of a Veech surface have very special properties. More precisely if M is of finite-type and its Veech group $\Gamma(M)$ is a lattice, then there are countably many directions stabilized by parabolic elements in $\Gamma(M)$ and for all other directions the translation flows in M are uniquely ergodic (see Theorem C.6.3). The purpose of the exercise below is to show that for infinite-type surface knowing the Veech group is not enough to determine the dynamics of the translation flow.

Exercise 3.2.2

In this exercise we invite the reader to make a modification of the construction in the proof of Theorem 3.2.1 so that we can control the translation flows of the surface M .

Let A be a finitely generated group with generating set $U = \{a_1, \dots, a_k\}$. Let $F : A \rightarrow \mathrm{GL}_+(2, \mathbb{R})$ be a morphism. Show that there exist a translation surface M as in the conclusion of Theorem 3.2.1 that has the following additional property: for every direction θ and for any $x \in M$ then either the orbit $F_\theta^t(x)$ hits a singularity in finite time or it escapes every compact set $K \subset M$ (such flow is in particular *completely dissipative*, see Appendix A).

Hint: in step 1 of the proof of Theorem 3.2.1 instead of considering slits in \mathbb{R}^2 you could consider a cyclic covering of degree $2k$ of \mathbb{R}^2 ramified over the origin and place each family $\{m_i^j\}_i$ of slits in the different sheets of this covering.

Exercise 3.2.2 should warn the reader that the Veech group does not capture interesting properties of translation flows of infinite-type translation surfaces. However, recall that in the proof of Theorem 3.2.1 the surface M is a A -covering of the plane with a square removed. Hence the affine group $\mathrm{Aff}^+(M) \simeq A$ (which is the same thing as the deck group of the covering) acts properly discontinuously (see Definition B.0.1). In many examples in this book such as the infinite staircase, wind-tree models and the Hooper-Thurston-Veech surfaces (that we introduce at the end of this chapter) the affine group does not act properly discontinuously and there is a rich interaction between the affine group and the dynamics of the translation flows.

3.2.2 Veech groups of tame translation surfaces

Recall from Definition 1.1.22 that a translation surface M is tame if $\mathrm{Sing}(M) \subset \widehat{M}$ contains only conical and infinite angle singularities. We now discuss how this geometric condition reflects and imposes constraints on the Veech group.

Definition 3.2.3. A matrix $g \in \mathrm{GL}(2, \mathbb{R})$ is *contracting* if $\lim_{n \rightarrow +\infty} g^n$ exists and is the zero matrix.

Equivalently, a matrix $g \in \mathrm{GL}(2, \mathbb{R})$ is contracting if its eigenvalues are smaller than one in absolute value.

Theorem 3.2.4

Let M be a tame translation surface with at least two singularities. Then its Veech group $\Gamma(M)$ does not contain contracting elements.

Proof. Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^2 . We proceed by contradiction. Let $\phi \in \mathrm{Aff}(M)$ such that $g = D(\phi)$ is contracting. Then, there exists $n_0 > 0$ such that for all $v \in \mathbb{R}^2$ we have $\|g^{n_0}v\| < \|v\|$. Because of this property, the map $\psi := \phi^{n_0}$ is a contraction for the metric on \widehat{M} . Hence it has a unique fixed point $x_0 \in \widehat{M}$ and for all $x \in \widehat{M}$ we have that $\psi^n(x)$ converges to x_0 as $n \rightarrow \infty$ (Banach-fixed point theorem).

By assumption there exists a singularity $x_1 \neq x_0$. Hence $\psi^n(x_1)$ is a sequence of singularities that converges to x_0 . Therefore x_0 must be a wild singularity. This is the desired contradiction. \square

A natural question from Theorem 3.2.4 is whether we can extend the construction of Theorem 3.2.1 to guarantee that the result M is tame. As shown in the following result, the answer is positive.

Theorem 3.2.5: [PSV11]

Let G be a countable subgroup of $\mathrm{GL}^+(2, \mathbb{R})$ without contracting elements. Then there exists a tame translation surface M homeomorphic to the Loch Ness monster such that $\Gamma^+(M) = G$.

Recall that in the proof of Theorem 3.2.1 the surface M is built out of planes with slits M'_a and each of this piece is tame. However, this is not enough to guarantee that the construction produces a tame translation surface (think about the case where $F(A)$ is a group of diagonal matrices). In order to prevent accumulation of singularities the strategy of [PSV11] consists in using what they call “buffer surfaces” to separate M_a from the other pieces M_{a,s_j} it is glued to. We refer the reader to the original article for details about the construction.

From Theorems 3.2.1 and 3.2.5 we conclude that there are no restrictions (other than the obvious ones) to realize countable subgroups of matrices as Veech groups of Loch Ness monsters, even if we impose the translation surface structure to be tame. The following result tells us that there are no restrictions either for the “most complicated” infinite genus surface which is, informally speaking, the Blooming Cantor tree (the surface of infinite genus without planar ends and whose space of ends is homeomorphic to the Cantor set).

Theorem 3.2.6: [RV]

Let $G < \mathrm{GL}_+(2, \mathbb{R})$ be a countable subgroup without contracting elements. Then there exists a tame translation surface M homeomorphic to the Blooming Cantor tree for which $\Gamma^+(M) = G$.

Again, the proof of this result is similar in spirit to the proof of Theorem 3.2.5: given a set of generators U of G one constructs a family of surfaces $\{M_g\}$, each homeomorphic to the Blooming Cantor tree, and glues them following the structure of the Caley graph of G relative to H . Although the proof is simple, it is far from elementary, for the gluings have to be done with care. Results similar to the preceding Theorem hold for an infinite family of topological types of infinite-type translation surfaces. For details we refer the reader to [RV].

Recall from Chapter 2 that to any $g \in \{\mathbb{Z}_{\geq 0} \cup \infty\}$ and a nested couple of closed subsets $C' \subset C$ of the Cantor set corresponds a unique topological type of surface: the one with genus g and such that $C' \subset C$ is homeomorphic to $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$. The following general situation is still an open problem

Question 3.2.7

Let $C' \subseteq C$ be a nested couple of closed subsets of the Cantor set and $g \in \{\mathbb{Z}_{\geq 0} \cup \infty\}$. Is it possible to realise any countable subgroup of $\mathrm{GL}_+(2, \mathbb{R})$ (with or without contracting elements) as the Veech group of a (tame) translation surface M such that $C' \subset C$ is homeomorphic to $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$.

This question remains open even for simple cases such as Jacob’s Ladder, that is when $C' = C$ has only two elements. It is also unknown whether there exist a translation surface homeomorphic to Jacob’s Ladder whose Veech group is $\mathrm{SL}(2, \mathbb{Z})$.

Recall that in Section 3.1 we classified translation surfaces M for which the subgroup $\mathrm{Tr}(M)$ of translations in $\mathrm{Aff}(M)$ is uncountable (see Theorem 3.1.5). Here we classify tame translation surfaces for which the Veech group is uncountable.

Lemma 3.2.8. *Let M be a tame translation surface such that its Veech group $\Gamma^+(M)$ is uncountable. Then $\Gamma^+(M)$ is conjugate to one of $\mathrm{GL}_+(2, \mathbb{R})$,*

$$P := \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} \mid t \in \mathbb{R} \text{ and } s \in \mathbb{R}^+ \right\}$$

or to $P' < \mathrm{GL}_+(2, \mathbb{R})$, the group generated by P and $-Id$.

Proof. Suppose first that M has no saddle connections. If M has no singularities then M is either \mathbb{R}^2 , the cylinder \mathbb{R}^2/\mathbb{Z} or a flat torus. This last case is ruled out for the Veech group of a torus is conjugated to $\mathrm{SL}(2, \mathbb{Z})$. Hence $\Gamma^+(M)$ is $\mathrm{GL}_+(2, \mathbb{R})$ or conjugated to P' . If M has only one singularity, tameness implies that M must be a cyclic covering of the plane ramified over a point, case in which $\Gamma^+(M) = \mathrm{GL}_+(2, \mathbb{R})$.

If M carries saddle connections and the group $\Gamma^+(M)$ is uncountable, then G is conjugate to either P or P' . First remark that tameness implies that all holonomy vectors (and hence saddle connections) must be parallel. Indeed, tameness implies that the set $V_{hol}(M)$ of holonomy vectors of M is at most countable. If there were two linearly independent holonomy vectors v_1, v_2 in $V_{hol}(M)$ then the map $\eta : \Gamma^+(M) \rightarrow V_{hol}(M) \times V_{hol}(M)$ defined by the linear action of matrices of vectors by $\eta(g) = (g(v_1), g(v_2))$ would be an embedding, which is impossible since $V_{hol}(M)$ is countable and $\Gamma(M)$ uncountable. Hence P is a subgroup of $\Gamma^+(M)$. To conclude we claim that $P < \Gamma^+(M) < P'$ and since P is of index 2 in P' we have $G = P$ or $G = P'$. Indeed, the claim follows from the fact that $Spine(M)$, which is the union of all singularities of \widehat{M} , and all horizontal separatrices and saddle connections, is a complete metric space w.r.t. its intrinsic path metric; then if $g \in \Gamma^+(M)$ and e_1 is a unit horizontal vector, then from the preceding discussion we have that $g(e_1) = \pm e_1$, for in any other case g restricted to $Spine(M)$ would be a contraction having a unique fixed point to which singularities must accumulate, contradicting the surface's tameness. \square

Exercise 3.2.9

Prove that if M is a translation surface with Veech group $\Gamma^+(M) = GL_+(2, \mathbb{R})$, then M is either the plane or a covering of the plane ramified over one point.

In the light of the preceding exercise, it is natural to ask if there are topological restrictions for a tame translation surface to have a Veech group conjugated to either P or P' . As the following theorem illustrates, this is not the case for surfaces without planar ends.

Theorem 3.2.10: [RV]

Let C be any closed subset of the Cantor set. Then there exists tame translation surfaces M and M' for which:

- $Ends(M) = Ends_\infty(M)$, $Ends(M') = Ends_\infty(M')$ and both of these spaces are homeomorphic to C , and
- the Veech groups of M and M' are conjugated to P and P' respectively.

Proof. We only provide a sketch of the proof to illustrate the main ideas. Let us first consider the case where C is just one point (*i.e.*, when the surfaces are homeomorphic to a Loch Ness monster) and discuss the key points of the general case. The construction starts from the Euclidean plane \mathbb{R}^2 . We consider the family L^+ of horizontal segments $\ell_i^+ := [((4i - 1), 0), (4i, 0)]$ for $i \in \mathbb{N}$. Let M_P be the tame translation surface obtained by gluing the segment ℓ_{2i-1}^+ to the segment ℓ_{2i}^+ for each $i \in \mathbb{N}$. Given that L is a discrete family of parallel segments contained in the real axis, the surface M_P is a tame Loch Ness monster. Moreover, its Veech group is precisely P because the set of all holonomy vectors of M_P is $\{\pm e_1\}$ and there are no saddle connections on the half plane $x < 0$. To produce a Loch Ness monster with Veech group P' consider on \mathbb{R}^2 the additional families of parallel segments L^- made of $\ell_i^- := [((1 - 4i), 0), (-4i, 0)]$ for $i \in \mathbb{N}$. and define $M_{P'}$ as the surface obtained by gluing ℓ_{2i-1}^+ to ℓ_{2i}^+ and ℓ_{2i-1}^- to ℓ_{2i}^- , for each $i \in \mathbb{N}$. The Veech group of $M_{P'}$ is P' for reasons that are analogous to the ones presented for the surface M_P .

The general case follows from [RV, construction 1.1]. This construction takes as input data a closed subset C of the Cantor set and a tame Loch Ness monster M and produces a tame translation surface M_{elem} (called an *elementary piece*) satisfying $Ends_\infty(M_{elem}) = Ends(M) = C$. When $M = M_P$ (resp. $M_{P'}$) it is possible to assure that P (resp. P') persists through the steps of the construction so that $\Gamma(M_{elem}) = P$ (resp. $\Gamma(M_{elem}) = P'$). The surface M_{elem} is obtained by a cut-and-paste construction similar to the one presented in the proof of theorem 3.2.1, the main difference being that instead of following the Cayley graph of a group as guide for the gluings one emulates the structure of a subtree T_C of the infinite regular tree of degree 3 which satisfies $Ends(T_C) = C$. \square

Question 3.2.11

Let $C' \subseteq C$ be a nested couple of closed subspaces of the Cantor set. Is it possible to realise the group P (resp. P') as Veech group of a tame translation surface M (resp. M') such that $C' \subseteq C$

is homeomorphic to $\text{Ends}_\infty(M) \subset \text{Ends}(M)$?

3.3 Affine groups and Veech groups of coverings

We now study affine groups of translation coverings $p : \widetilde{M} \rightarrow M$ where M is a finite-type translation surface (see Section 2.2, Definition 2.2.1). More precisely, we will study which affine elements of M can be *lifted* to \widetilde{M} ; that is whether there exists $\tilde{f} \in \text{Aff}(\widetilde{M})$ so that $p \circ \tilde{f} = \tilde{f} \circ p$.

In Section 3.3.1 we present the general lifting criterion which follows from standard algebraic topology. Next, in Section 3.3.2 we study the case of Abelian coverings and discuss the importance of the affine group action on homology of the base surface. In Section 3.3.3 we focus on parabolic elements of $\text{Aff}(M, \Sigma)$ that, as we already have seen in the Thurston-Veech construction from Section 1.2.6, are intimately related to cylinder decompositions of M . Then, in Section 3.3.4 we prove a theorem due to Hooper and Weiss that provides a powerful method to show that the Veech group of a covering is non-elementary. Finally, in Section 3.3.5 we apply these results to the wind-tree model.

Most statements concern translation surfaces though they can be applied to half-translation surfaces (see Section 1.1.4) after an appropriate double-covering construction.

3.3.1 The lifting criterion

Let us start with the general lifting criterion. Let $p : \widetilde{M} \rightarrow M$ be a translation covering branched over $\Sigma \subset M$. In order to lift affine elements from M to \widetilde{M} , the branching locus Σ must be preserved. Hence we only deal with elements of the group $\text{Aff}(M, \Sigma)$, which is the subgroup of $\text{Aff}(M)$ preserving the set Σ . The obstruction for a map f in $\text{Aff}(M, \Sigma)$ to be lifted to \widetilde{M} can be measured by its action on the fundamental group of $M \setminus \Sigma$. Recall that if x_0 is any point in M then f induces a group morphism $f_* : \pi_1(M, x_0) \rightarrow \pi_1(M, f(x_0))$. Given that f can change basepoints, f does not necessarily induces an automorphism of $\pi_1(M, x_0)$. However, the action of f on conjugacy classes is well defined. To lighter notations, we will often write $\pi_1(M)$ without reference to the base point when the properties we require are conjugacy invariant.

Theorem 3.3.1: Lifting criterion

Let $p : \widetilde{M} \rightarrow M$ be a translation covering with branching locus $\Sigma \subset M$ and set $\widetilde{\Sigma} := p^{-1}(\Sigma)$. Let $f \in \text{Aff}(M, \Sigma)$ (i.e., $f \in \text{Aff}(M)$ and $f(\Sigma) = \Sigma$). Then there exists a lift $\tilde{f} : \widetilde{M} \rightarrow \widetilde{M}$, (i.e., $f \circ p = p \circ \tilde{f}$ on \widetilde{M}) if and only if the induced outer morphism $f_* \in \text{Out}(\pi_1(M \setminus \Sigma))$ preserves the conjugacy class of $p_*(\pi_1(\widetilde{M} \setminus \widetilde{\Sigma}))$ in $\pi_1(M \setminus \Sigma)$.

Proof. From Proposition 1.33, p. 61 in [Hat02] we have the following topological criterion:

Lemma 3.3.2 (Topological lifting criterion). *Let S be a topological surface, $p : \widetilde{S} \rightarrow S$ a covering map and $f : S \rightarrow S$ a continuous function. Let $x_0 \in S$, $x_1 = f(x_0)$ and $\tilde{x}_0 \in p^{-1}(x_0)$, $\tilde{x}_1 \in p^{-1}(x_1)$. Then there exists a lift $\tilde{f} : \widetilde{S} \rightarrow \widetilde{S}$ satisfying $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$ if and only if the induced morphisms $f_* : \pi_1(S, x_0) \rightarrow \pi_1(S, x_1)$, $p_{0*} : \pi_1(\widetilde{S}, \tilde{x}_0) \rightarrow \pi_1(S, x_0)$ and $p_{1*} : \pi_1(\widetilde{S}, \tilde{x}_1) \rightarrow \pi_1(S, x_1)$ on the corresponding fundamental groups satisfy*

$$f_* p_{0*} \pi_1(\widetilde{S}, \tilde{x}_0) \subset p_{1*} \pi_1(\widetilde{S}, \tilde{x}_1)$$

In order to apply the above Lemma, we fix a basepoint x_0 in $M^0 = M \setminus \Sigma$ and let x_1 be its image under f . Since f is an element of $\text{Aff}(M, \Sigma)$ the image x_1 is not in Σ . Now, a pair of points \tilde{x}_0 and \tilde{x}_1 in \widetilde{M}^0 exist if and only if f preserves the conjugacy class of $(p_0)_*(\pi_1(\widetilde{M}^0, \tilde{x}_0))$. If such a lifting exists, it can be uniquely extended to \widetilde{M} in a continuous way. \square

Exercise 3.3.3

Let $p : \widetilde{M} \rightarrow M$ be a translation covering. We define the subgroups $G_1(p) \subset \text{Aff}(M)$ and $G_2(p) \subset$

$\text{Aff}(\widetilde{M})$ as follows

$$G_1(p) := \{f \in \text{Aff}(M) : \text{there exists } \tilde{f} \in \text{Aff}(\widetilde{M}) \text{ such that } p \circ \tilde{f} = p \circ f\} \quad (3.4)$$

$$G_2(p) := \{\tilde{f} \in \text{Aff}(\widetilde{M}) : \text{there exists } f \in \text{Aff}(M) \text{ such that } p \circ \tilde{f} = p \circ f\} \quad (3.5)$$

1. Prove that the composition with p provides a surjective map

$$\begin{array}{ccc} G_2(p) & \rightarrow & G_1(p) \\ \tilde{f} & \mapsto & p \circ \tilde{f} \end{array}.$$

2. Show that the kernel of the map $G_2(p) \rightarrow G_1(p)$ considered above is $\text{Deck}(p)$.
3. Assume that $p : \widetilde{M} \rightarrow M$ is a normal cover, that is $\text{Deck}(p)$ acts transitively on the fibers of p . Then prove that $G_2(p)$ is the normalizer of $\text{Deck}(p)$ in $\text{Aff}(\widetilde{M})$ (i.e., \tilde{f} in $\text{Aff}(\widetilde{M})$ belongs to $G_2(p)$ if and only if $\tilde{f} \text{Deck}(p) \tilde{f}^{-1} = \text{Deck}(p)$).
4. Prove that if $\tilde{f} \in G_2(p)$ maps to $f \in G_1(p)$ then the derivatives $D(\tilde{f})$ and $D(f)$ are equal.

Definition 3.3.4. Let $p : \widetilde{M} \rightarrow M$ be a translation covering. The *relative Veech group* $\Gamma(p)$ or $\Gamma(\widetilde{M} \rightarrow M)$ of the covering p is the group $D(G_1(p)) = D(G_2(p))$, that is, the image by the derivative homomorphism of the groups $G_1(p)$ and $G_2(p)$ defined in (3.4) and (3.5).

The following direct consequence of Proposition 3.3.1 is already a rich source of examples.

Corollary 3.3.5. Let M be a translation surface, $\Sigma \subset M$ a discrete subset and $M^0 = M \setminus \Sigma$. Let $p : \widetilde{M} \rightarrow M$ be a translation covering defined by a characteristic subgroup¹ of $\pi_1(M^0)$. Then every element of $\text{Aff}(M, \Sigma)$ lifts to $\text{Aff}(\widetilde{M})$. In particular, $\Gamma(M, \Sigma) \subset \Gamma(\widetilde{M})$.

Example 3.3.6. Let G be a group. We consider the following families of subgroups of G :

1. The lower central series:

$$\gamma_0(G) \supseteq \gamma_1(G) \supseteq \dots \supseteq \gamma_n(G) \supseteq \dots \quad (3.6)$$

where $\gamma_0(G) := G$ and $\gamma_{n+1}(G) := [\gamma_n(G), G]$ for each $n > 0$.

2. The derived series:

$$\delta_0(G) \supseteq \delta_1(G) \supseteq \dots \supseteq \delta_n(G) \supseteq \dots \quad (3.7)$$

where $\delta_0(G) = G$ and $\delta_{n+1}(G) := [\delta_n(G), \delta_n(G)]$ for $n > 0$.

3. The subgroup of n -th powers:

$$G^n := \langle g^n : g \in G \rangle \quad (3.8)$$

It is not difficult to check that all subgroups of G listed above are characteristic. In particular Corollary 3.3.5 applies. Let us consider the case of square-tiled surfaces where everything can be made more explicit.

Corollary 3.3.7. Let $p : \widetilde{M} \rightarrow \mathbb{T}^2$ be a square-tiled surface determined by a subgroup of $F_2 \simeq \pi_1(\mathbb{T}^2 \setminus \{0\})$ in one of the families (3.6), (3.7) or (3.8) defined above. Then every element of $\text{Aff}(\mathbb{T}^2, \{0\}) \simeq \text{SL}(2, \mathbb{Z})$ lifts to \widetilde{M} , in particular $\Gamma(\widetilde{M})$ contains $\text{SL}(2, \mathbb{Z})$. Moreover, if p is ramified then $\Gamma(\widetilde{M}) = \text{SL}(2, \mathbb{Z})$.

Proof. The first part of the statement directly follows from Corollary 3.3.5. The Veech group of the torus with a marked point is $\text{SL}(2, \mathbb{Z})$. In particular the Veech group of \widetilde{M} contains $\text{SL}(2, \mathbb{Z})$. However, it might not be equal to it: the Euclidean plane admits affine automorphisms that permute the fibers of the covering map $p : \mathbb{C} \rightarrow \mathbb{T}^2$. When the covering is ramified, the fact that the covering subgroup is characteristic ensures that the corners of each square in \widetilde{M} are conical singularities (when the ramification order is finite) or punctures that gives rise to infinite angle singularity in the metric completion (when the order is infinite). All other points of \widetilde{M} are regular. Hence, any element of $\text{Aff}(\widetilde{M})$ preserves $\widehat{p}^{-1}(\{0\})$, where \widehat{p} is the unique extension of p to the metric completion (see Definition 2.2.1 and the paragraph that follows). As a consequence every element of $\text{Aff}(\widetilde{M})$ descends to M . \square

¹A subgroup H of a group G is *characteristic* if for any automorphism $\phi : G \rightarrow G$ we have $\phi(H) = H$.

To illustrate let us consider the subgroup $\gamma_1(F_2) = \delta_1(F_2) = [F_2, F_2]$ of the fundamental group $F_2 = \pi_1(\mathbb{T}^2 \setminus \{0\})$. The associated (non-ramified) covering space $p: \mathbb{C} \setminus \mathbb{Z}^2 \rightarrow \mathbb{T}^2 \setminus \{0\}$ is a \mathbb{Z}^2 -covering, because $\mathbb{Z}^2 = F_2/\delta_1(F_2)$. Now, recall from Definition 2.2.1 in Chapter 2 that the corresponding translation covering is obtained by adding the conical singularities and regular points in the metric completion of $\mathbb{C} \setminus \mathbb{Z}^2$ to which p can be extended continuously. In the case of $\mathbb{C} \setminus \mathbb{Z}^2$, the metric completion is obtained by adding \mathbb{Z}^2 which are all regular points and p is the (unramified) universal cover $\hat{p}: \mathbb{C} \rightarrow \mathbb{T}^2$. An other explicit example is given by the second group in the central series: $F_2/\gamma_2(F_2)$ is isomorphic to the *Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\},$$

More generally, subgroups of the central series gives rise to a nilpotent coverings.

The second group in the derived series $\delta_2(F_2)$ is however less explicit. The quotient $F_2/\delta_2(F_2)$ is called the *free metabelian group of rank 2*.

Exercise 3.3.8

1. Show that for $n \geq 0$, $[\gamma_{n+1}(F_2) : \gamma_n(F_2)] = +\infty$ and $[\delta_{n+1}(F_2) : \gamma_n(F_2)] = +\infty$.
2. Show that the subgroups $\gamma_n(F_2)$ and $\delta_n(F_2)$ are pairwise distinct except: $\gamma_0(F_2) = \delta_0(F_2) = F_2$ and $\gamma_1(F_2) = \delta_1(F_2) = [F_2, F_2]$.
(*hint*: use the fact that subgroups of F_2 are free groups)

The study of quotients by the n -th power subgroup is much more delicate and has a long history. The quotient $B(2, n) := F_2/(F_2)^n$ is called the *free Burnside group* (each element in $B(2, n)$ has order at most n). For small values of n , one can check that $B(2, n)$ is actually finite. However, it is a deep result that there exists n_0 such that $B(2, n)$ is infinite for every $n \geq n_0$ (Adian and Novikov (1968), Ol'shanskii (1982), Ivanov (1994) and Lysenok (1996)). Let us also mention that it is still unknown whether $B(2, 5)$ is infinite.

We now present two examples of coverings that are not translation coverings but for which the deck group acts by affine transformations. The first example are irrational billiards (for which the deck group acts by isometries) and the second one dilation surfaces (for which the Veech group acts by dilations).

Example 3.3.9. In Section 1.2.1 we discussed how to translate the dynamics of a billiard ball in a polygon P into the dynamics of the translation flow on a translation surface $M(P)$. Moreover, Theorem 1.2.3 proven in Example 2.2.15 shows that if P is an irrational polygon (*i.e.*, none of its interior angles is commensurable with π) then $M(P)$ is homeomorphic to the Loch Ness monster.

Theorem 3.3.10: [Val12]

Let P be a simply connected Euclidean polygon with interior angles $\{\lambda_j \pi\}_{j=1}^N$. If there exists $\lambda_j \in \mathbb{R} \setminus \mathbb{Q}$ then the Veech group $\Gamma^+(M(P))$ is a subgroup of $\text{SO}(2, \mathbb{R})$. Moreover, the subgroup of $\text{SO}(2, \mathbb{R})$ generated by the rotations $z \rightarrow e^{2\lambda_j \pi} z$, $j = 1, \dots, N$ is a subgroup of $\Gamma^+(M(P))$ of finite index.

Proof. We discuss the case of triangles P , for the case of a general polygon is analogous. From Section 1.2.1 and the proof of Theorem ?? presented in Example 2.2.15 we can deduce that there exists a non-ramified covering $p: M(P) \rightarrow \mathbb{S}^2(P)$ whose base is a three-punctured sphere made of two copies of P (with the vertices removed) and whose deck transformation group is isometric to $\mathbb{Z} \times A$ or $\mathbb{Z}^2 \times A$ (where A is a finite abelian group) depending on whether the rank of P is equal to 1 or 2, see (1.8) in Section 1.2.1. Let us emphasize that here $\mathbb{S}^2(P)$ is not a translation surfaces and p is not a translation covering. The surface $\mathbb{S}^2(P)$ has a $\text{Isom}^+(\mathbb{R}^2)$ -structure (see Appendix E). Let $\text{hol}: \pi_1(M(P)) \rightarrow \text{Isom}^+(\mathbb{R}^2)$ be its holonomy. Then the covering p is defined by the normal subgroup $\text{hol}^{-1}(\text{Trans}(\mathbb{R}^2))$.

By construction, the image of hol is contained in the group generated by the reflections with respect to the sides of P . Let $R(P)$ be the group generated by the linear part of the reflections with respect to the sides of P and $R^+(P) = R(P) \cap \text{SO}(2)$. The deck group of p acts on the translation surface $M(P)$ by affine homeomorphisms whose linear parts generate the group $R^+(P)$. We claim that actually $\Gamma^+(M(P))$ is formed only by rotations. Indeed, in $M(P)$ there are saddle connections of minimal length, say $L > 0$, corresponding to geodesics of minimal length in the Euclidean surface $\mathbb{S}^2(P)$. Since one of

the λ_j is irrational, $R^+(P)$ contains an irrational rotation. In particular, the holonomy vectors of length L are dense in the circle $\{z \in \mathbb{C} : |z| = L\}$. Now, if we had an element $A \in \Gamma^+(M(P))$ different from a rotation, then the image of $\{z \in \mathbb{C} : |z| = L\}$ under A would be an ellipse containing infinitely many points corresponding to holonomy vectors of norm less than L , which is a contradiction to the fact that L is the minimal length of a saddle connection. To finish the proof remark that the set of holonomy vectors of minimal length of $M(P)$ is of the form $\sqcup_{i=1}^k R^+(P)v_i$, where v_i are minimal length holonomy vectors and the group $\Gamma^+(M(P))/R^+(P)$ acts freely on the set of $R^+(P)$ -orbits of holonomy vectors of minimal length. \square

Veech groups of translation coverings of dilation surfaces. At the end of Section 2.2 we explained that each dilation surface S has a natural normal covering $\pi : \tilde{S} \rightarrow S$ defined by the inverse image of $\text{Trans}(\mathbb{R}^2)$ under the holonomy $\text{hol} : \pi_1(S) \rightarrow \text{Dil}^+(\mathbb{R}^2)$ of the $\text{Dil}^+(\mathbb{R}^2)$ -structure of S . The pullback of the dilation surface structure of S defines thus a translation surface structure on \tilde{S} and the deck transformation group of π acts on this translation surface by dilations. In particular, if Δ denotes the image of $\pi_1(S) \rightarrow \text{Dil}^+(\mathbb{R}^2) \rightarrow \mathbb{R}^*$, where the second arrow is just the derivative map, then the Veech group of \tilde{S} contains an element of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, for each $a \in \Delta$.

3.3.2 Abelian coverings and $\text{Aff}(M, \Sigma)$ -action on $H_1(M, \Sigma; \mathbb{Z})$

The lifting criterion in Proposition 3.3.1 is defined in terms of the action of $\text{Aff}(M)$ on the fundamental group. In the case of Abelian coverings this can naturally be reformulated in terms of the action of $\text{Aff}(M, \Sigma)$ on the relative homology space $H_1(M, \Sigma; \mathbb{Q})$. The central theme of this section and the next is to study this action.

Let us first mention that the properties of the action of the affine group on $H_1(M; \mathbb{Z})$ is tightly linked to dynamical properties of translation flows or foliations in \tilde{M} (see for example Theorem 1.2.18 from the introduction). Dynamical properties of Abelian coverings will be investigated in Chapters 5 and 6.

Recall from Lemma 2.2.5 that an Abelian covering of a translation surface M at most branched over Σ is determined by a subspace $V \subset H_1(M, \Sigma; \mathbb{Q})$. On the other hand, the group $\text{Aff}(M, \Sigma)$ has an induced action on $H_1(M, \Sigma; \mathbb{Q})$. As a consequence of the lifting criterion (Proposition 3.3.1) we have the following.

Theorem 3.3.11: Abelian lifting criterion

Let M be a compact translation and $\Sigma \subset M$ a finite set. Let V be a vector subspace of $H_1(M, \Sigma; \mathbb{Q})$ and $p : \tilde{M} \rightarrow M$ the Abelian translation covering it determines. Then an element $f \in \text{Aff}(M, \Sigma)$ lifts to \tilde{M} if and only if $f_*V = V$.

In Example 3.3.6, we already encountered an Abelian covering. Namely, the covering determined by the vector subspace $V = H_1(M, \Sigma; \mathbb{Q})$ which corresponds to the derived subgroup $\delta_1(\pi_1(M \setminus \Sigma))$ of the fundamental group $\pi_1(M \setminus \Sigma)$. In this situation, all elements of $\text{Aff}(M, \Sigma)$ lift to \tilde{M} (see Corollary 3.3.5).

In view of Theorem 3.3.11, studying the affine groups of Abelian translation covers of a compact surface M amounts to study the stabilizers of vector subspaces of $H_1(M, \Sigma; \mathbb{Q})$ under the action of the affine group $\text{Aff}(M, \Sigma)$. An important subspace that is stabilized by $\text{Aff}(M, \Sigma)$ is the kernel of the holonomy map, which we define as follows:

Definition 3.3.12. Let $M = (X, \omega)$ be a finite-type translation surface and Σ be a finite subset in M (possibly empty). We define $\text{kerhol}(M, \Sigma)$ as the kernel of the holonomy map

$$\begin{aligned} H_1(M, \Sigma; \mathbb{Q}) &\rightarrow \mathbb{C} \\ c &\mapsto \int_c \omega \end{aligned}$$

When $\Sigma = \emptyset$, we will simply use the simpler notation $\text{kerhol}(M)$ for $\text{kerhol}(M, \emptyset)$.

Let us emphasize that in the definition of kerhol we used the homology with rational coefficients $H_1(M, \Sigma; \mathbb{Q})$ and not $H_1(M, \Sigma; \mathbb{R})$. The latter group always has codimension 2 while, as we will see in 3.3.18, the codimension of $\text{kerhol}(M)$ is intimately related to the arithmetic properties of the holomy vectors of M .

Exercise 3.3.13

Let M be a finite-type translation surface and $\Sigma \subset M$ a finite subset.

1. Prove that $\ker\text{hol}(M, \Sigma)$ is invariant under the action of $\text{Aff}(M, \Sigma)$.
2. Prove that all elements of $\text{Aff}(M, \Sigma)$ lift to the Abelian covering defined by $V = \ker\text{hol}(M, \Sigma)$ (*hint*: use Theorem 3.3.11).

Most Abelian coverings that we study in this book are defined by subspaces of $\ker\text{hol}(M, \Sigma)$. This is the case of the infinite staircase: as it can be seen in Figure 2.3 in Example 2.2.3 this \mathbb{Z} -covering is defined by the cycle $B - A \in H_1(M, \Sigma; \mathbb{Z})$ and $\text{hol}(A) = \text{hol}(B) = 1$. It is also the case, for the Wind-tree models $W_{a,b} \rightarrow X_{a,b}$ for any $(a, b) \in (0, 1) \times (0, 1)$. Let us consider Figure 1.14b in which the cycles Z_{ij} and Y_{ij} appear. Let us define the two absolute cycles

$$Y = Y_{00} + Y_{10} - Y_{01} - Y_{11} \quad \text{and} \quad Z = Z_{00} + Z_{01} - Z_{10} - Z_{11} \quad (3.9)$$

in $H_1(X_{a,b}; \mathbb{Z})$. The subspace of $H_1(X_{a,b}; \mathbb{Q})$ generated by Y and Z defines the unramified covering $W_{a,b} \rightarrow X_{a,b}$. To see that it belongs to $\ker\text{hol}(X_{a,b})$ it is enough to notice that

$$\forall i, j \in \{0, 1\}, \quad \text{hol}(Z_{i,j}) = 1 \quad \text{and} \quad \text{hol}(Y_{i,j}) = \sqrt{-1}.$$

As shown in the following exercise, half-translation surfaces (or quadratic differentials) provides examples of \mathbb{Z}^d -coverings defined by subspaces in $\ker\text{hol}$.

Exercise 3.3.14

Let M be a half-translation surface and $p : \widetilde{M} \rightarrow M$ a \mathbb{Z}^d -half-translation covering branched over Σ such that \widetilde{M} is not a translation surface (Panov planes from 1.2.5 provide examples of such situation).

1. Show that we have a commutative square

$$\begin{array}{ccc} \widetilde{M}' & \longrightarrow & \widetilde{M} \\ p' \downarrow & & \downarrow p \\ M' & \longrightarrow & M \end{array}$$

where M' and \widetilde{M}' are respectively the double orientation coverings of M and \widetilde{M} and p' is a \mathbb{Z}^d -covering.

2. Show that the \mathbb{Z}^d -covering p' is defined by a subspace of $\ker\text{hol}(M', \Sigma')$ where Σ' is the union of the preimages of Σ , the preimages of the zeros of odd order in M , and the preimages of poles in M .

In the following exercise we show that for a square-tiled surface M , the space $\ker\text{hol}(M, \Sigma)$ admits a natural complement in $H_1(M, \Sigma; \mathbb{Q})$. This result is generalized in Theorem 3.3.18 and Lemma 3.3.20 to any finite-type translation surface with non-elementary Veech group.

Exercise 3.3.15

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the square torus and $\Sigma \subset \mathbb{T}^2$ a finite set that contains the origin 0.

1. Show that $\dim \ker\text{hol}(\mathbb{T}^2, \Sigma) \leq |\Sigma| - 1$ with equality if and only if Σ are rational points, that is $\Sigma \subset \mathbb{Q}^2/\mathbb{Z}^2$.
2. Prove that if $\Sigma \subset \mathbb{Q}^2/\mathbb{Z}^2$ then there is a decomposition

$$H_1(\mathbb{T}^2, \Sigma; \mathbb{Q}) = H_1(\mathbb{T}^2; \mathbb{Q}) \oplus \ker\text{hol}(\mathbb{T}^2, \Sigma).$$

3. Let $p : M \rightarrow \mathbb{T}^2$ be a finite-type square-tiled surface. Let $T_M \subset H_1(M; \mathbb{Q})$ be the subspace

generated by the preimages under p of closed curves in \mathbb{T}^2 . Show that $p_* : T_M \rightarrow H_1(\mathbb{T}^2; \mathbb{Q})$ is an isomorphism and that we have a decomposition

$$H_1(M; \mathbb{Q}) = T_M \oplus \ker \text{hol}(M) \quad (3.10)$$

that is orthogonal with respect to the algebraic intersection form on $H_1(M; \mathbb{Q})$.

4. Show that T_M defined in the previous question is $\text{Aff}(M)$ -invariant.
5. Let M be a finite-type square-tiled surface as before. Let $\Sigma' \subset M$ be a finite set contained in the preimage of the rational points of the torus. Show that

$$H_1(M, \Sigma'; \mathbb{Q}) = T_M \oplus \ker \text{hol}(M, \Sigma'). \quad (3.11)$$

We now study in full generality a finite-type surface M with non-elementary Veech group. It will clarify the special role that $\ker \text{hol}(M, \Sigma)$ plays when considering the Veech groups of a \mathbb{Z}^d -coverings of M . The discussion requires some technical concepts from compact translation surfaces. A reader that is not familiar with these notions should not be afraid of not understanding everything as most of the results developed here will not be used intensively in the following sections.

Recall that a subgroup of $\text{SL}(2, \mathbb{R})$ is called non-elementary if it contains two hyperbolic elements with disjoint axes (see other equivalent definitions in Lemma B.0.2 from Appendix B). By extension, we say that the affine group $\text{Aff}(M, \Sigma)$ of a finite-type translation surface is non-elementary if its image $\Gamma(M, \Sigma)$ by the derivative homomorphism $D : \text{Aff}(M, \Sigma) \rightarrow \text{SL}(2, \mathbb{R})$ is non-elementary (since we assume that M is finite-type, the affine elements necessarily preserve the area and hence image is necessarily in $\text{SL}(2, \mathbb{R})$).

An important source of examples of finite-type translation surface M with non-elementary Veech groups are the ones obtained from the Thurston-Veech construction (see Section 1.2.6). In this case, the Veech group $\Gamma(M)$ contains two parabolic elements (with distinct fixed point on $\partial\mathbb{H}^2$). Among these Thurston-Veech surfaces, there are some very exceptional ones for which the Veech group $\Gamma(M)$ is of finite covolume: these are the so-called Veech surfaces. Finite-type square-tiled surfaces (see Example 2.2.9) are Veech surfaces and their Veech groups are commensurable with $\text{SL}(2, \mathbb{Z})$. An other example of surface with non-elementary Veech group but which does not come from a Thurston-Veech construction is the Arnoux-Yoccoz surface [HLM09]. Let us terminate our list with a general construction from [HS04]. We need two definitions.

Definition 3.3.16. Let M be a compact translation surface and Σ be its set of conical singularities. Let $\Lambda = \{\int_\gamma \omega : \gamma \in H_1(M, \Sigma; \mathbb{Q})\}$. A point in M is called *rational* if the holonomy of any path from x to a point in Σ belongs to Λ

Definition 3.3.17. Let M be a surface with non-elementary Veech group. A point x in M is called *periodic* if its stabilizer in $\text{Aff}(M)$ has finite index (or equivalently, if its orbit is finite).

For square-tiled surfaces, the rational points are the elements of $\mathbb{Q}^2/\mathbb{Z}^2$ which are all periodic. This justifies our terminology. For a surface with non-elementary Veech group the periodic points are rational points, however the converse is not true: in a Veech surface the set of periodic points is finite [GHS03, MÖ6]. The fact that there exist rational points which are not periodic can be used to construct finite-type translation surfaces whose Veech group do not admit a finite set of generators.

Let M be a translation surface whose Veech group is non-elementary. To M one can associate its *holonomy field* $k \subset \mathbb{R}$ that admits three equivalent definitions

1. k is the subfield of \mathbb{R} generated by the cross-ratios $(s_1, s_2; s_3, s_4)$ of slopes s_1, s_2, s_3, s_4 of holonomies of saddle connections in M .
2. Let $T_M^{id} := \mathbb{R} \text{Re}(\omega) + \mathbb{R} \text{Im}(\omega) \subset H^1(M; \mathbb{R})$ be the *tautological plane*. Then k is the field of definition of T_M^{id} , that is the smallest field such that one can write T_M^{id} as a kernel of a matrix with coefficients in k (in a \mathbb{Q} -basis of $H^1(M; \mathbb{Q})$),
3. For any hyperbolic element f in $\text{Aff}(M)$ we have $k = \mathbb{Q}[\text{tr}(D(f))]$,

We refer to [KS00], [GJ00], [HL06a] and [McM03] for more details. A square-tiled surface has holonomy field $k = \mathbb{Q}$.

For each embedding $\sigma : k \rightarrow \mathbb{R}$ or a pair of complex conjugate embeddings $\sigma, \bar{\sigma} : k \rightarrow \mathbb{C}$ we can define a Galois conjugates $T_M^\sigma \subset H^1(M; \mathbb{R})$ of the tautological plane by applying σ on the defining equations of T_M^{id} . For real embedding T_M^σ has dimension 2 while for pair of complex it has dimension 4.

Theorem 3.3.18

Let M be a compact translation surface whose affine group is non-elementary and k be its holonomy field. Then each subspace T_M^σ is symplectic and $\text{Aff}(M)$ -invariant. There is direct sum decomposition orthogonal with respect to the intersection form

$$T_M := \bigoplus_{\sigma \in \text{Emb}(k)} T_M^\sigma \quad (3.12)$$

where $\text{Emb}(k)$ is the set of real embeddings and pairs of complex conjugate embeddings of k .

The space T_M is a subspace of $H_1(M; \mathbb{R})$ defined over \mathbb{Q} and its annihilator is exactly $\ker \text{hol}(M)$. As a consequence $\ker \text{hol}(M)$ is symplectic and we have a direct sum decomposition orthogonal with respect to the algebraic intersection form

$$H_1(M; \mathbb{R}) = T_M \oplus (\ker \text{hol}(M) \otimes \mathbb{R}) \quad (3.13)$$

where T_M is identified to its the Poincaré dual in $H_1(M; \mathbb{Q})$.

Any subspace $V \subset H_1(M; \mathbb{R})$ that has a non-elementary stabilizer in $\text{Aff}(M)$ decomposes as

$$V = \bigoplus_{\sigma \in I} T_M^\sigma \oplus (\ker \text{hol}(M) \otimes \mathbb{R} \cap V) \quad (3.14)$$

where $I \subset \text{Emb}(k)$.

Proof. We only provide a proof when k is totally real. The general case introduce some technical details that would obfuscate the proof.

The action of $\text{Aff}(M)$ on the tautological plane $T_M^{\text{id}} = \mathbb{R} \text{Re}(\omega) \oplus \mathbb{R} \text{Im}(\omega)$ identifies with the derivative map. Indeed, by definition of the affine action

$$\begin{pmatrix} f^* \text{Re}(\omega) \\ f^* \text{Im}(\omega) \end{pmatrix} = D(f) \begin{pmatrix} \text{Re}(\omega) \\ \text{Im}(\omega) \end{pmatrix}.$$

In particular, T_M^{id} is invariant under $\text{Aff}(M)$.

Each Galois conjugate T_M^σ satisfies the same property: the action of $\text{Aff}(M)$ on this subspace identifies with another embedding of the Veech group in $\text{SL}(2, \mathbb{R})$, T_M^σ is $\text{Aff}(M)$ -invariant and the action of $\text{Aff}(M)$ on T_M^σ identifies with another embedding of the Veech group $\Gamma(M)$ in $\text{SL}_2(\mathbb{R})$. Since the Galois action of \mathbb{Q}/\mathbb{Q} is \mathbb{Q} -linear, each plane T_M^σ is symplectic.

To prove that the sum is direct we used the third definition of the holonomy field (as the field generated by the traces of the matrices $D(f)$ for $f \in \text{Aff}(M)$). Because $\text{Aff}(M)$ is non-elementary, it admits a hyperbolic element. Its trace $t = \text{tr}(D(f))$ is a generator of k and hence the embeddings $\sigma(t)$ are pairwise distinct. The eigenvalues of $D(f)$ is a pair $\lambda, 1/\lambda$ such that $t = \lambda + 1/\lambda$. Moreover λ is simple as an eigenvalue of the action f_* of f on $H^1(M; \mathbb{R})$ (see Theorem C.6.1 in Appendix C). Hence $T_M^{\text{id}} = E_\lambda \oplus E_{1/\lambda}$ where E_λ and $E_{1/\lambda}$ are the associated eigenspaces. This is also true for each Galois conjugate. But by the uniqueness of the decomposition into eigenspaces, the sum is direct.

Since $\ker \text{hol}(M)$ is a \mathbb{Q} -subspace of $H_1(M; \mathbb{Q})$ it is also annihilated by the T_M^σ . The direct sum in equation (3.13) follows.

Now we want to prove the last part of the Theorem. Let V be a subspace of $H^1(M; \mathbb{R})$ with a non-elementary stabilizer under the action of $\text{Aff}(M)$. We use again the eigenspace decomposition of hyperbolic elements in $\text{Aff}(M)$. Let f_1 and f_2 be two hyperbolic elements in the stabilizer of V in $\text{Aff}(M)$ such that the fixed points of $D(f_1)$ and $D(f_2)$ in $\partial \mathbb{H}^2$ are pairwise distinct (in other words, there is no common eigenvector for $D(f_1)$ and $D(f_2)$). This is possible because we assumed that this stabilizer is non-elementary. Galois conjugation preserves the property of transversality of the matrices. Hence for each Galois conjugate T_M^σ the two decompositions into eigenspaces $T_M^\sigma = E_{\lambda_1^\sigma} \oplus E_{1/\lambda_1^\sigma}$ for f_1 and $T_M^\sigma = E_{\lambda_2^\sigma} \oplus E_{2/\lambda_2^\sigma}$ for f_2 satisfy

$$E_{\lambda_1^\sigma} \cap E_{\lambda_2^\sigma} = E_{\lambda_1^\sigma} \cap E_{1/\lambda_2^\sigma} = E_{1/\lambda_1^\sigma} \cap E_{\lambda_2^\sigma} = E_{1/\lambda_1^\sigma} \cap E_{1/\lambda_2^\sigma} = \{0\}$$

Now the invariant subspace V must decompose along the eigenspaces of any element in its stabilizer. Since the decompositions in eigenspaces of f_1 and f_2 lead to distinct decompositions of T_M^σ , we obtain the decomposition (3.14). \square

Corollary 3.3.19. *Let M be finite type translation surface and $\Sigma \subset M$ a finite set such that $\text{Aff}(M, \Sigma)$ is non-elementary. Let $T_M = \bigoplus_{\sigma \in \text{Emb}(k)} T_M^\sigma$ as in Theorem 3.3.18.*

Let V be a vector subspace of $H_1(M, \Sigma; \mathbb{Q})$. If the stabilizer of V in $\text{Aff}(M, \Sigma)$ is non-elementary then either V contains T_M or V is contained in $\ker \text{hol}(M, \Sigma)$.

In the particular case of \mathbb{Z} -coverings, that is $\dim_{\mathbb{Q}} V = 1$, only the second possibility occurs.

Proof. The decomposition (3.13) from Theorem 3.3.18 is also valid for relative homology

$$H_1(M, \Sigma; \mathbb{Q}) = \bigoplus_{\sigma \in \text{Emb}(k)} T_M^\sigma \oplus \ker \text{hol}(M, \Sigma). \quad (3.15)$$

The reason is that the class of $\text{Re}(\omega)$ and $\text{Im}(\omega)$ are well defined in $H^1(M \setminus \Sigma; \mathbb{Q})$ which by Poincaré duality can be identified to subspaces of $H_1(M, \Sigma; \mathbb{Q})$.

Now, the second part of Theorem 3.3.18 shows that an invariant subspace must respect the decomposition (3.15). Since V is assumed to be defined over \mathbb{Q} it must either contain the whole sum $\bigoplus_{\sigma \in \text{Emb}(k)} T_M^\sigma$ (that is the smallest subspace defined over \mathbb{Q} that contains T_M^{id}) or be contained in $\ker \text{hol}(M, \Sigma)$

With the additional hypothesis that $\dim_{\mathbb{Q}} V = 1$, it can not contain T_M which has non-zero even dimension. \square

Let us mention two important elementary results about $\ker \text{hol}(M, \Sigma)$. The first one concerns the behavior under finite coverings.

Lemma 3.3.20. *Let M be a compact translation surface with non-elementary Veech group, k its holonomy field and $\Sigma \subset M$ a finite set of rational points. Let $p: \widetilde{M} \rightarrow M$ a finite covering branched over Σ and $\widetilde{\Sigma} := p^{-1}(\Sigma)$. Then $p_*: H_1(\widetilde{M}, \widetilde{\Sigma}; \mathbb{Q}) \rightarrow H_1(M, \Sigma; \mathbb{Q})$ induces an isomorphism between $T_{\widetilde{M}}^\sigma$ and T_M^σ defined in Theorem an isomorphism bijection in $\text{restr } \text{codim } \ker \text{hol}(M, \Sigma) = \text{codim } \ker \text{hol}(\widetilde{M}, \widetilde{\Sigma})$ (where the codimensions are respectively measured in $H_1(M, \Sigma; \mathbb{Q})$ and $H_1(\widetilde{M}, \widetilde{\Sigma}; \mathbb{Q})$).*

The second lemma states that the surfaces for which $\ker \text{hol}(M)$ is maximal are exactly the square-tiled surfaces.

Lemma 3.3.21. *Let M be a compact translation surface and $\Sigma \subset M$ a (possibly empty) finite set of points. Then $\text{codim } \ker \text{hol}(M, \Sigma) = 2$ (where the codimension is taken inside $H_1(M, \Sigma; \mathbb{Q})$) if and only if $p: M \rightarrow \mathbb{T}^2$ is a square-tiled surface and $\Sigma \subset p^{-1}(0)$.*

Proof of Lemma 3.3.20. Let

$$\Lambda := \left\{ \int_{\gamma} \omega : \gamma \in H_1(M, \Sigma; \mathbb{Z}) \right\} \quad \text{and} \quad \Lambda' := \left\{ \int_{\gamma'} \omega' : \gamma' \in H_1(M', \Sigma'; \mathbb{Z}) \right\}.$$

By lifting paths it is easy to show that $\Lambda = \Lambda'$. Now, by definition of $\ker \text{hol}(M, \Sigma)$ and $\ker \text{hol}(M', \Sigma')$ we have $\text{codim } \ker \text{hol}(M, \Sigma) = \text{rk}_{\mathbb{Z}} \Lambda_M$ and $\text{codim } \ker \text{hol}(M', \Sigma') = \text{rk}_{\mathbb{Z}} \Lambda_{M'}$ from which the lemma follows. \square

Proof of Lemma 3.3.21. Let us define

$$\Lambda := \left\{ \int_{\gamma} \omega : \gamma \in H_1(M, \Sigma; \mathbb{Z}) \right\} \subset \mathbb{C}.$$

Fix a reference point $x_0 \in \Sigma$ if Σ is non-empty and any point in M otherwise. Then consider the period mapping

$$P: \begin{array}{ccc} M & \rightarrow & \mathbb{C}/\Lambda \\ x & \mapsto & \int_{x_0}^x \omega. \end{array}$$

The map is well defined since two paths from x_0 to x differ by an element in absolute homology.

Now $\text{codim } \ker \text{hol}(M, \Sigma) = 2$ if and only if Λ is a lattice in \mathbb{C} . Hence if $\text{codim } \ker \text{hol}(M, \Sigma) = 2$, the map P provides a translation covering with a torus target. And by construction, all points of Σ are mapped to 0 in the quotient.

The proof of the converse is a particular case of Lemma 3.3.20. \square

A “generic” surface M in a given stratum $\mathcal{H}(\kappa)$ of Abelian differentials satisfies $\dim \ker \text{hol}(M) = 0$. However, Lemmas 3.3.20 and 3.3.21 show that when M comes from a covering then $\ker \text{hol}(M)$ is non-trivial.

Exercise 3.3.22

Let L be the L-shaped surface made of 3 squares as in Figure 3.4 and let $\alpha = \gamma_2 - \gamma_1$ and $\beta = \gamma_3 - \gamma_4$ (as elements of the absolute homology $H_1(L; \mathbb{Z})$).

1. Show that $\ker \text{hol}(L) = \mathbb{Q}\alpha \oplus \mathbb{Q}\beta$,
2. Show that L admits no translation automorphism (so that we can identify the Veech group and the affine group).
3. Show that the following three matrices belong to the Veech group of L

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

(actually these 3 matrices generate the Veech group)

4. Let $\rho : \text{Aff}(M) \rightarrow \text{GL}(\ker \text{hol}(M))$. Show that in the basis (α, β) of $\ker \text{hol}(M)$ one has

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(h_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(v_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

5. Conclude that the subgroup of $\text{Aff}(L)$ that consists of elements that act trivially on $\ker \text{hol}(L)$ is a non-trivial normal subgroup of $\text{Aff}(L)$ of infinite index. Does it contain a parabolic element?

Affine action on absolute and relative homologies

In this section we study the differences between the action of affine groups on absolute and relative homology groups.

Let M be a finite-type translation surface and $\Sigma \subset M$ be a finite set so that $\text{Aff}(M, \Sigma)$ is non-elementary. Recall that both $\ker \text{hol}(M)$ and $\ker \text{hol}(M, \Sigma)$ are $\text{Aff}(M, \Sigma)$ -invariant subgroups of $H_1(M, \Sigma; \mathbb{Q})$ (see Exercise 3.3.13).

Theorem 3.3.23: [HW12]

Let M be a finite-type translation surface and let $\psi : \text{Aff}(M, \Sigma) \rightarrow \text{GL}(\ker \text{hol}(M, \Sigma))$ and $\psi_0 : \text{Aff}(M, \Sigma) \rightarrow \text{GL}(\ker \text{hol}(M))$ be the induced actions. Then, $\ker(\psi)$ is a normal subgroup of $\ker(\psi_0)$ and the quotient is virtually Abelian of rank at most $r_0(r - r_0)$ where $r_0 = \dim(\ker \text{hol}(M))$ and $r = \dim(\ker \text{hol}(M, \Sigma))$.

In particular, if $\Gamma(M)$ is non-elementary and $\ker(\psi)$ is non-trivial then $\Lambda \ker \psi = \Lambda \ker \psi_0 = \Lambda \Gamma(M)$.

Proof. Because $\ker \text{hol}(M) \subset \ker \text{hol}(M, \Sigma)$, in an appropriate basis of $\ker \text{hol}(M, \Sigma)$, ψ has the form

$$\psi(f) = \begin{pmatrix} \psi_0(f) & \alpha(f) \\ 0 & \sigma(f) \end{pmatrix} \quad (3.16)$$

where $\sigma : \text{Aff}(M, \Sigma) \rightarrow \text{GL}(V)$ is the action induced by permutations on $V = \{v \in \mathbb{Q}^\Sigma : \sum v_i = 0\}$ and $\alpha(f)$ is some $r_0 \times (r - r_0)$ matrix with integer coefficients.

Hence, $\ker(\sigma)$ is of finite index (the image of σ is a permutation group) and the image of α is an Abelian group whose rank is given by the dimensions of the matrix: $r_0(r - r_0)$.

The last part of the proof concerning the limit sets is a direct application of Lemma B.0.4 from the appendix. \square

Exercise 3.3.24

This exercise is a continuation of Exercise 3.3.15. Recall that $\text{Aff}(\mathbb{T}^2, \{0\}) \simeq \text{SL}(2, \mathbb{Z})$. Instead of considering $\text{Aff}(\mathbb{T}^2, \Sigma)$ (that can admit translations) we consider subgroups of $\text{Aff}(\mathbb{T}^2, \{0\})$ that we

identify to subgroups of $SL(2, \mathbb{Z})$.

The n -torsion points in \mathbb{T}^2 are the n^2 points $(\frac{1}{n}(\mathbb{Z}^2))/\mathbb{Z}^2$. The *principal congruence subgroup* of degree n is

$$\Gamma(n) := \{A \in SL(2, \mathbb{Z}) : A \equiv Id \pmod{n}\}$$

1. Show that $\Gamma(n)$ is the pointwise stabilizer of the n -torsion points (*i.e.*, $As = s$ for each n -torsion point s).
2. Assume furthermore that Σ is contained in the n -torsion points. Prove that $\Gamma(n)$ preserves $\ker\text{hol}(M, \Sigma)$ and that for each element in $\Gamma(n)$ the restriction of its action on $H_1(M, \Sigma; \mathbb{Q})$ to $\ker\text{hol}(M, \Sigma)$ is trivial.
3. Conclude that if $V \subset \ker\text{hol}(\mathbb{T}^2, \Sigma)$ is a subvector space and \widetilde{M} the Abelian covering associated to V then the affine group of \widetilde{M} contains $\Gamma(n)$.

In the above exercise, we have seen a family of examples where the affine group $\text{Aff}(M, \Sigma)$ acts via a finite group on $\ker\text{hol}(M, \Sigma)$. However, as shown in the following example this is not the case in general.

Example 3.3.25. We now present an example of a finite-type square-tiled surface $Z_{3,1}$ made of 4 squares in the stratum $\mathcal{H}(1, 1)$ of Abelian differentials. We compute explicitly the action of $\text{Aff}(O) = \text{Aff}(O, \Sigma)$ on $H_1(M; \mathbb{Z})$ and $H_1(M, \Sigma; \mathbb{Z})$ where Σ is the set made of the two conical singularities of O . The Veech

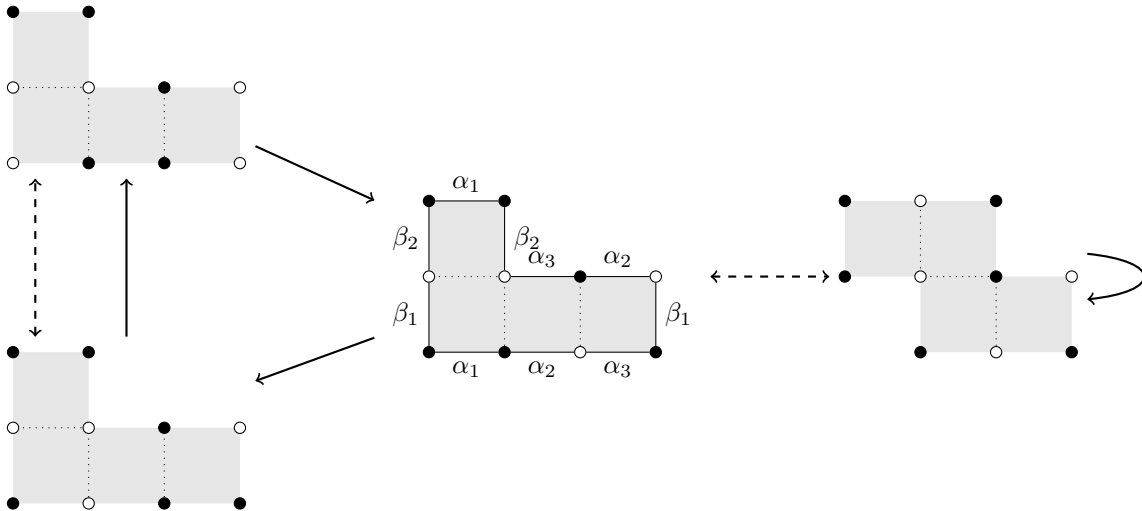


Figure 3.2: The $SL(2, \mathbb{Z})$ -orbit of the square-tiled surface $Z_{3,1}$. The plain arrows correspond to the action of the horizontal parabolic element in $SL_2(\mathbb{Z})$ while the dotted ones to the rotation by $\pi/2$.

group (which in that case identifies to the affine group) is an index 4 subgroup of $SL(2, \mathbb{Z})$ generated by the matrices

$$g_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad g_3 = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}.$$

Then in the basis $\{2\alpha_1 - \alpha_2 - \alpha_3, 2\alpha_2 - 2\alpha_1 + \beta_2 - \beta_1, \alpha_2 - \alpha_1\}$ of $\ker\text{hol}(Z_{3,1}, \Sigma)$ depicted in Figure 3.2 the map ψ_0 is given by

$$\psi_0(g_1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \psi_0(g_2) = \begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \psi_0(g_3) = \begin{pmatrix} -2 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can show then show that for $g = \begin{pmatrix} 37 & 60 \\ 8 & 13 \end{pmatrix} = g_1 g_3 (g_2)^2 g_3 g_2$ then $\alpha(g) = (-1, 0)$. Which shows that $\ker(\psi)$ and $\ker(\psi_0)$ are different. Moreover, taking its conjugate $g' = \begin{pmatrix} 13 & 24 \\ 20 & 37 \end{pmatrix} = g_2 g_3 g_2 g_1 g_3 g_2$ we have $\alpha(g') = (-1, -1)$. Hence, the Abelian group of Lemma 3.3.23 has rank 2 in this example (the maximal possible).

Let us also mention that this example shows that the $\text{Aff}(M, \Sigma)$ -invariant subspace $H_1(M; \mathbb{R})$ inside $H_1(M, \Sigma; \mathbb{R})$ does not necessarily admits a complement that is also invariant.

3.3.3 Lifting parabolic elements in G -coverings

In this section we study parabolic elements in more detail. Because a parabolic element in the affine group of a finite-type translation surface is associated to a cylinder decomposition their properties are often much easier to understand than for hyperbolic elements. As we will see in the next chapters, cylinders are extremely useful to study recurrence and ergodicity of coverings.

Cylinders and strips in translation surfaces were defined in Definition 1.1.30. Let us first consider the general question of preimages of cylinders. Let $p : \widetilde{M} \rightarrow M$ be a translation covering. Then, the preimages in \widetilde{M} of cylinders in M are either cylinders or strips. The following statement gives a precise description.

Lemma 3.3.26. *Let $p : \widetilde{M} \rightarrow M$ be a G -covering branched over Σ and determined by the kernel of the morphism $\rho : \pi_1(M \setminus \Sigma) \rightarrow G$. Let C be a maximal cylinder in $M \setminus \Sigma$ with core curve γ seen as a conjugacy class in $\pi_1(M \setminus \Sigma)$. Let $k \in \{1, 2, \dots\} \cup \{\infty\}$ be the order of $\rho(\gamma)$ in G . Then, if k is finite, the preimage of C in \widetilde{M} consists of a disjoint collection of copies of cylinders with same height and whose circumference is k -times longer than the circumference of C . If k is infinite, the preimage of C consists of a disjoint union of strips of the same height as C .*

Proof. The argumentation is based on the path-lifting property from the theory of covering spaces. Namely, let us fix a basepoint x on the curve γ . Then this curve can be lifted from any point \tilde{x} in the preimage of x . The endpoint of the lifting is then $\tilde{x} \cdot \gamma$ where \cdot represents the right action of the fundamental group $\pi_1(M \setminus \Sigma, x)$ on the fiber of x . Repeating this lifting from $\tilde{x} \cdot \gamma$ we end up in $\tilde{x} \cdot (\gamma)^2$, etc. This procedure comes back for the first time to \tilde{x} in k steps if and only if the order of $\rho(\gamma)$ is k . This proves the part of the Lemma concerning the circumferences.

Since the boundary of the cylinders consists of saddle connections that also lifts to saddle connection the boundary of the cylinders in \widetilde{M} are unchanged. The height of C_i and its preimages are the same. \square

Definition 3.3.27. Let M be a translation or half-translation surface. We say that $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is:

1. *completely periodic* if there exists a collection of maximal cylinders $\{C_i\}_{i \in I}$ parallel to θ such that $\cup_{i \in I} C_i$ is dense in M . Such a collection of cylinders is called a *cylinder decomposition* in direction θ .
2. an *affine multitwist direction* if it is completely periodic and there exists an affine multitwist f in $\text{Aff}(M)$ that acts as a product of Dehn twists in these cylinders
3. A *parabolic direction* if there exists $f \in \text{Aff}(M)$ parabolic such that $D(f)$ fixes the direction θ .

Remark 3.3.28. For finite-type translation surface M , Lemma 1.2.24 shows that parabolic and affine multitwist directions are the same thing. Moreover, the existence of an affine multitwist for a given cylinder is equivalent to the commensurability of the cylinders.

As shown in Example 3.3.32 below, a parabolic direction in an affine translation surface is not necessarily an affine multitwist direction. However, as we will see in Section 3.4, the condition for a completely periodic direction to be stabilized by an affine multitwist can still be formulated in terms of moduli of the cylinders.

As a consequence of Lemma 3.3.26 we have the following general statement for lifting parabolic affine automorphisms.

Corollary 3.3.29. *Let M be a finite-type translation surface and let $p : \widetilde{M} \rightarrow M$ be a G -translation covering branched over Σ defined by a morphism $\rho : \pi_1(M \setminus \Sigma) \rightarrow G$. Let $f \in \text{Aff}(M, \Sigma)$ be a parabolic element for which C_1, \dots, C_k are the maximal cylinders in the corresponding decomposition of M . Let γ_i be the core curve of C_i considered as a conjugacy class in $\pi_1(M \setminus \Sigma)$. If all the elements $\rho(\gamma_1), \dots, \rho(\gamma_k)$ have finite order m_1, \dots, m_k in G , then $f^{\text{lcm}(m_1, \dots, m_k)}$ lifts to \widetilde{M} ,*

Proof of Corollary 3.3.29. Let us assume that $\rho(\gamma_1), \dots, \rho(\gamma_k)$ have finite order m_1, \dots, m_k . Then, by Lemma 3.3.26, the preimage of the cylinder C_i in \widetilde{M} is a disjoint union of cylinders with same height and whose circumference is m_i times the one of C_i . In particular, the moduli of any of these cylinder in \widetilde{M} is $\frac{\mu(C_i)}{m_i}$. Let $f \in \text{Aff}(M)$ be the multitwist whose derivative stabilizes the parabolic direction. Let $m = \text{lcm}(m_1, \dots, m_k)$. Then f^m lifts as a multitwist in \widetilde{M} . \square

We now turn to the case of Abelian coverings. As we already saw in Theorem 3.3.11 from Section 3.3.2 the lifting of elements of $\text{Aff}(M, \Sigma)$ to the covering \widetilde{M} can be expressed in terms of the action of this element on the relative homology group $H_1(M, \Sigma; \mathbb{Q})$.

Definition 3.3.30. Let M be a finite-type translation surface and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a completely periodic direction. Let $\gamma_1, \dots, \gamma_k$ be the core curves of the maximal cylinders in the corresponding decomposition of M . The *homological dimension* of the completely periodic direction θ in M is the dimension of the subspace spanned by $\gamma_1, \dots, \gamma_k$ in absolute homology $H_1(M; \mathbb{Q})$.

Note that the homological dimension is always smaller than or equal to the number of cylinders.

Theorem 3.3.31: one-cylinder trick

Let M be a finite-type translation surface and $p : \widetilde{M} \rightarrow M$ be a \mathbb{Z}^d -translation covering with branching points Σ defined by a subspace of $\text{kerhol}(M, \Sigma)$. Let $f \in \text{Aff}(M)$ be a multitwist whose derivative stabilizes a completely periodic direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

1. If there is *only one* cylinder C in the cylinder decomposition of M in direction θ , then the preimages of this cylinder in \widetilde{M} are disjoint isometric copies of C .
2. The direction θ has homological dimension 1 in M if and only if f_* acts as the identity on $\text{kerhol}(M, \Sigma)$.

In both situations above f lifts to an element $\widetilde{f} \in \text{Aff}(\widetilde{M})$.

Example 3.3.32. Let us first consider the situation of the infinite staircase from Section 1.2.3 to which Theorem 3.3.31 applies. Recall that the infinite staircase is a \mathbb{Z} -translation covering $p : \widetilde{M} \rightarrow M$, where the base M is a torus and the covering map p is branched over two points (see Example 2.2.3). In the horizontal and vertical directions, the cylinder decompositions consist of a single cylinder and hence the first item of Theorem 3.3.31 applies. In the diagonal direction $\theta = \pi/4$ there are two cylinders but their core curves are homologous and the second item of Theorem 3.3.31 applies. Note that in this diagonal direction, the staircase decomposes into two strips. The parabolic element \widetilde{f} that stabilizes this strip decomposition is not a multitwist in the sense of Definition 1.2.25.

We will actually prove results stronger than Theorem 3.3.31. In particular, we will describe explicitly the action of multitwists on homology. The algebraic intersection form

$$\langle \cdot, \cdot \rangle : H_1(M \setminus \Sigma; \mathbb{R}) \times H_1(M, \Sigma; \mathbb{R}) \rightarrow \mathbb{R}$$

will play an important role.

Lemma 3.3.33. Let M be a finite-type translation surface and V a subspace of $\text{kerhol}(M, \Sigma)$ defining a \mathbb{Z}^d -covering $p : \widetilde{M} \rightarrow M$ branched over $\Sigma \subset M$. Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an affine multitwist direction in $M \setminus \Sigma$ with corresponding multitwist $f \in \text{Aff}(M, \Sigma)$. Let $\gamma_1, \dots, \gamma_k$ in $H_1(M \setminus \Sigma; \mathbb{Q})$ be the core curves of the cylinders C_1, \dots, C_k in the and μ_1, \dots, μ_k their moduli. Let us define

$$\phi : \begin{array}{ccc} H_1(M, \Sigma; \mathbb{Q}) & \rightarrow & H_1(M, \Sigma; \mathbb{Q}) \\ v & \mapsto & \sum_{i=1}^k \mu_i \langle \gamma_i, v \rangle \overline{\gamma}_i \end{array}$$

where $\overline{\gamma}_i$ denotes the image of γ_i in $H_1(M, \Sigma; \mathbb{Q})$. Then

1. $\langle \gamma_i, V \rangle = 0$ for each $i = 1, \dots, k$ if and only if the preimage of each C_i in \widetilde{M} is a disjoint union of cylinders isometric to C_i .
2. $\phi(V) = 0$ if and only if f_* acts trivially on V ,
3. $\phi(V) \subset V$ if and only if f lifts to the cover defined by V .

The last ingredient in the proof of Theorem 3.3.31 is the explicit expression of $\text{Im}(\omega)$ for horizontal completely periodic directions.

Lemma 3.3.34. *Let M be a finite-type translation surface whose horizontal direction is completely periodic. Let $\Sigma \subset M$ be a finite set of points and let $\gamma_1, \dots, \gamma_k$ in $H_1(M \setminus \Sigma; \mathbb{R})$ be the core curves of maximal cylinders in $M \setminus \Sigma$ and h_1, h_2, \dots, h_k their heights. Then the cycle $\gamma := h_1\gamma_1 + h_2\gamma_2 + \dots + h_k\gamma_k$ is the Poincaré dual of $\text{Im}(\omega)$. That is to say*

$$\forall \eta \in H_1(M, \Sigma; \mathbb{R}), \quad \langle \gamma, \eta \rangle = \int_{\eta} \text{Im}(\omega).$$

We first prove Theorem 3.3.31 as a consequence of Lemma 3.3.33 and Lemma 3.3.34.

Proof of Theorem 3.3.31. Let us assume there the surface M decomposes as a single cylinder C with core curve γ in direction θ . By Lemma 3.3.34 the core curve is the Poincaré dual of an element in the tautological plane $\mathbb{R} \text{Re}(\omega) \oplus \mathbb{R} \text{Im}(\omega)$. Hence $\langle \gamma, \ker \text{hol}(M, \Sigma) \rangle = 0$ and apply the first item in Lemma 3.3.33 to conclude that the preimages of C in \widetilde{M} are disjoint isometric copies of C .

Now let us prove the second item. We assume that the horizontal direction has homological dimension one. That is to say for each $i = 1, \dots, k$ there exists $\alpha_i \in \mathbb{Q}_{>0}$ so that $\overline{\gamma}_i = \alpha_i \overline{\gamma}_1$. Since the core curves are primitive elements of $H_1(M; \mathbb{Z})$ we have $\alpha_i = 1$ for all $i = 1, \dots, k$. Now recall that the modulus $\mu_i = h_i/w_i$ where h_i and w_i are the heights and circumferences of cylinders. But since the γ_i are homologous, the w_i are equal and $\mu_i = h_i/w_1$. In particular, we have for any $v \in H_1(M, \Sigma; \mathbb{Q})$

$$\phi(v) = \sum_{i=1}^k \mu_i \langle \gamma_i, v \rangle \overline{\gamma}_i = \frac{1}{w_1} \left\langle \sum_{i=1}^k h_i \gamma_i, v \right\rangle \overline{\gamma}_1.$$

Now, similarly to the one-cylinder case we just apply Lemma 3.3.34 to conclude that $\left\langle \sum_{i=1}^k h_i \gamma_i, v \right\rangle$ vanishes for vectors $v \in \ker \text{hol}(M, \Sigma)$. \square

Proof of Lemma 3.3.34. Let us consider the following generating system of $H_1(M, \Sigma; \mathbb{R})$. Let $\eta_1, \eta_2, \dots, \eta_n$ be the saddle connections on the boundary of the cylinders appearing in the horizontal direction. We orient them in such way that $\int_{\eta_i} \omega > 0$, for all $i = 1, \dots, n$. Next, for each cylinder choose a saddle connection that goes from the top to the bottom. We denote ξ_i the transverse saddle connection in the i -th cylinder. The set $\{\eta_1, \dots, \eta_n, \xi_1, \dots, \xi_k\}$ generates $H_1(M, \Sigma; \mathbb{R})$ and

- for all $i = 1, \dots, n$ and $j = 1, \dots, k$ we have $\langle \eta_i, \gamma_j \rangle = 0$
- for all $i = 1, \dots, k$ and $j = 1, \dots, k$ we have $\langle \xi_i, \gamma_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker symbol.

As $\int_{\eta_i} \text{Im}(\omega) = 0$ and $\int_{\xi_i} \text{Im}(\omega) = h_i$ we obtain the result. \square

Proof of Lemma 3.3.33. Suppose that the \mathbb{Z}^d -covering $p: \widetilde{M} \rightarrow M$ is defined by the kernel of $\rho: \pi_1(M \setminus \Sigma) \rightarrow G \simeq \mathbb{Z}^d$. Then $\langle \gamma_i, V \rangle = 0$ if and only if $\rho(\gamma_i) = 0$ for every $i = 1, \dots, k$. From Lemma 3.3.26 we conclude that this last condition happens if and only if for every $i = 1, \dots, k$ the preimage of each C_i in \widetilde{M} is a disjoint union of cylinders isometric to C_i .

The multitwist f acts as a power of a Dehn twist in each cylinder. Let m_i be the multiplicity of the twist in the cylinder C_i . Then the action of f on $H_1(M, \Sigma; \mathbb{Z})$ is given by:

$$f_*(v) = v + \sum_{i=1}^k m_i \langle \gamma_i, v \rangle \overline{\gamma}_i$$

Now, recall that this multiplicity m_i is proportional to the moduli μ_i of the cylinders. Namely if the parabolic stabilizes the horizontal direction and is hence written as $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ then $\lambda \mu_i = m_i$. This implies that $f_*(v) = v + \frac{1}{\lambda} \phi(v)$ where ϕ was defined in the statement of Lemma 3.3.33. Hence $f_*(V) = V$ if and only if $\phi(V) = V$. From here the second item follows trivially and the last item is deduced from the Abelian lifting criterion (Theorem 3.3.11). \square

Recall from Definition 3.3.17 and the discussion after that a periodic point in a square tiled surface $p: M \rightarrow \mathbb{T}^2$ is a point that projects to $\mathbb{Q}^2/\mathbb{Z}^2$ under p .

Corollary 3.3.35 (one-cylinder trick for square-tiled surfaces). *Let M be a square-tiled surface and let $\Sigma \subset M$ be a finite set of periodic points. Assume that M admits a completely periodic direction of homological dimension one. Let $\widetilde{M} \rightarrow M$ be a \mathbb{Z}^d -covering determined by a subspace V of $\ker \text{hol}(M, \Sigma)$. Then the Veech group $\Gamma(\widetilde{M} \rightarrow M)$ contains a non-trivial normal subgroup of $\Gamma(M)$ generated by parabolic elements.*

Let us mention that a non-trivial normal subgroup of the lattice $\Gamma(M)$ is somehow "big": its limit set is the whole circle (see Lemma B.0.4). A particular case of Corollary 3.3.35 appeared in [HS10].

Proof. Let G be the subgroup of $\text{Aff}(M)$ generated by the matrices Df where f is a multitwist that stabilizes a completely periodic direction with homological dimension one in M . Then by assumption G is non-trivial. It also follows from the one-cylinder trick (Theorem 3.3.31) that any element in G lifts to \widetilde{M} . Let us prove that G is a normal subgroup. If f is a multitwist in G stabilizing a direction θ and $h \in \text{Aff}(M)$, then hfh^{-1} is the multitwist stabilizing the direction $h\theta$. Moreover, the direction θ and $h\theta$ are affinely equivalent. In particular the completely periodic direction θ and $h\theta$ in M have the same homological dimensions and hfh^{-1} lifts to \widetilde{M} . \square

Example 3.3.36. The eierlegende Wollmilchsau $p : W \rightarrow \mathbb{T}^2$ is the square-tiled surface depicted in figure 3.3. It was discovered independently by Herrlich and Weitze-Schmithüsen [HS08] and Forni [For06]. W is a genus 3 translation surface with 4 conical singularities of total angle 4π (i.e., it belongs to the stratum $\mathcal{H}(1, 1, 1, 1)$).

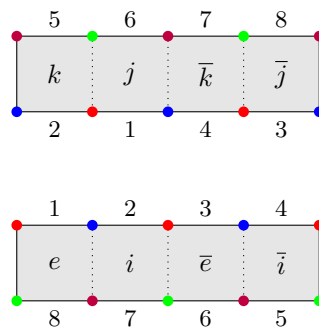


Figure 3.3: The eierlegende Wollmilchsau square-tiled surface W . The integers on the outside of the polygons correspond to the edge pairing. The bold labels in the squares show that W has Deck group transformation the quaternion group.

Exercise 3.3.37

Let W be the eierlegende Wollmilchsau square-tiled surface and Σ be its set of four conical singularities.

1. Show that the Deck group of $p : W \rightarrow \mathbb{T}^2$ can be identified with the quaternion group

$$Q := \{e, i, j, k, \bar{e}, \bar{i}, \bar{j}, \bar{k}\}$$

with neutral element e and multiplication rules

- $i^2 = j^2 = k^2 = \bar{e}$
- $ij = k, ki = j$ and $jk = i$
- $\forall g \in Q, g^{-1} = \bar{g}$ where $\bar{\bar{g}} = g$.

2. Show that the subgroup of $\pi_1(\mathbb{T}^2 \setminus \{0\})$ that defines W is characteristic. Conclude that the Veech group of W is $\text{SL}(2, \mathbb{Z})$ (see Corollary 3.3.5).
3. Use the one-cylinder trick (Theorem 3.3.31) to prove that the horizontal multitwist in W acts trivially on $\ker\text{hol}(W, \Sigma)$.
4. Using the fact that the Veech group of W is $\text{SL}(2, \mathbb{Z})$, prove that any multitwist in $\text{Aff}(W, \Sigma)$ acts trivially on $\ker\text{hol}(W, \Sigma)$. *Hint:* there is only one conjugacy class of parabolic elements in $\Gamma(W) = \text{SL}(2, \mathbb{Z})$.

5. Show that the derivative map induces an isomorphism between the subgroup of $\text{Aff}(W, \Sigma)$ generated by the multitwists of the previous question and $\Gamma(4)$ (*hint*: for surjectivity, use the fact that $\Gamma(4)$ is generated by its parabolic elements)
6. Conclude that the Veech group of any \mathbb{Z}^d -covering \widetilde{W} of W at most branched over Σ contains the congruence group $\Gamma(4)$ (*hint*: use Theorem 3.3.11).

In the above exercise we explicitated the action of a finite index subgroup of $\text{Aff}(M, \Sigma)$ on $\ker\text{hol}(W, \Sigma)$. The complete description of the action of $\text{Aff}(W, \Sigma)$ on $\ker\text{hol}(W, \Sigma)$ is given in [MY10]. Other properties of the eierlegende Wollmilchsau are discussed in [For06] and [HS08].

Let us also mention that the eierlegende Wollmilchsau W is not the only know example of finite-type square-tiled surface such that a finite index subgroup of its affine group acts trivially in homology. There is a cousin of the eierlegende Wollmilchsau: a genus 4 square-tiled surface called the *ornithorynque* (the french for platypus). Similarly to what is proposed in Exercise 3.3.37, one can prove that a subgroup of its affine group isomorphic to $\Gamma(3)$ acts trivially on homology. We refer to [MY10] and [FMZ11], [FM14, Chapter 7.2] for more details on this example.

3.3.4 The Hooper-Weiss theorem

Let us recall from Definition 3.3.17 and the remark that follows that a periodic point in a square-tiled surface $p : M \rightarrow \mathbb{Z}^2$ is a point that projects under p to a point in $\mathbb{Q}^2/\mathbb{Z}^2$. The aim of this section is to prove the following result.

Theorem 3.3.38: [HW12]

Let M be a square-tiled surface of genus 2 and let $\Sigma \subset M$ be a finite set of periodic points. Let \widetilde{M} be the translation covering of M determined by a subspace V of $\ker\text{hol}(M, \Sigma)$. Then the Veech group $\Gamma(\widetilde{M} \rightarrow M)$ contains a non-trivial normal subgroup^a of $\Gamma(M, \Sigma)$.

^aSee Defition 1.1.27

The reader should notice the similarity with Corollary 3.3.35 of the one-cylinder trick. In both situations, the kernel of the action $\text{Aff}(M) \rightarrow \text{GL}(\ker\text{hol}(M, \Sigma))$ is infinite (or equivalently the image of the kernel in $\Gamma(M)$ is non-trivial).

Proof of Theorem 3.3.38. Since Σ are periodic points, the relative Veech group $\Gamma(M, \Sigma)$ is a lattice of finite index in $\Gamma(M)$. Let $\psi : \text{Aff}(M, \Sigma) \rightarrow \text{GL}(\ker\text{hol}(M, \Sigma))$ and $\psi_0 : \text{Aff}(M, \Sigma) \rightarrow \text{GL}(\ker\text{hol}(M))$ be the morphisms induced by the action of the affine group on $H_1(M, \Sigma; \mathbb{Z})$. By Lemma 3.3.23, it suffices to show that the image of $\ker(\psi_0)$ in $\Gamma(M)$ is non-trivial. We suppose that $D \ker(\psi_0)$ is trivial and derive a contradiction.

Because M has genus 2, $\ker\text{hol}(M)$ has dimension 2. We also know from Theorem 3.3.18 that $\ker\text{hol}(M)$ is symplectic. Hence the image of ψ_0 is commensurable with $\text{SL}(2, \mathbb{Z})$. Up to passing to a finite index subgroup A of $\text{Aff}(M, \Sigma)$ we can further suppose that:

1. the image $\psi_0(A)$ is torsion-free (*e.g.*, consider $\psi_0^{-1}(\Gamma(2))$ where $\Gamma(2)$ denote the principal congruence subgroup)
2. that both D and ψ_0 are faithful representations of A into $\text{SL}(2, \mathbb{Z})$ (because $\text{Tr}(M) = \ker(\text{Aff}(M) \rightarrow \Gamma(M))$ is finite).

From now on we fix such finite index subgroup $A \subset \text{Aff}(M, \Sigma)$.

Now, let $f \in A$. Then

1. if $D(f)$ is parabolic then $\psi_0(f)$ is parabolic (indeed it can not be identity or elliptic since $\psi|_A$ is injective)
2. if $D(f)$ is hyperbolic, then $2 \leq |\text{tr}(\psi_0(f))| < |\text{tr}(D(f))|$ (the dominant eigenvalue of $D(f)$ is the stretch-factor of the pseudo-Anosov and dominates any other eigenvalue of the action of f on $H_1(M; \mathbb{R})$, see Theorem C.6.1).

Now consider the hyperbolic surfaces $S = \mathbb{H}^2/D(A)$ and $S_0 = \mathbb{H}^2/\psi_0(A)$. Because both ψ_0 and D are faithful, there exists an homotopy equivalence $\phi : S \rightarrow S_0$ induced by the isomorphism at the level of fundamental groups. Because parabolic elements in A are mapped to parabolic elements by ψ_0 we know that the number of cusps n of S is smaller than or equal to the one n_0 of S_0 . We claim that there are actually equal. Indeed, let us denote by $\gamma_1, \dots, \gamma_n$ in $\pi_1(S)$ the generators of peripheral curves in S . If we had $n_0 > n$ then these linearly dependent elements would be mapped to linearly independent elements of $H_1(S_0; \mathbb{Z})$. Hence, S and S_0 have the same number of cusps and, because of the homotopy equivalence, the same genus. In other words, the underlying topological surface of S and S_0 is the same topological surface.

Now for each non-peripheral $\gamma \in \pi_1(S)$ let $\ell(\gamma)$ denote the length of the geodesic representative on S , and let $\ell_0(\gamma)$ denote the length of the geodesic representative of $\phi_*(\gamma)$. It is a deep result by Thurston (see [Thu85, Theorem 3.1]) that

$$\sup_{\gamma \in \pi_1(S)} \frac{\ell_0(\phi_*(\gamma))}{\ell(\gamma)} \geq 1, \quad (3.17)$$

with equality only if ϕ is homotopic to an isometry of the hyperbolic surfaces S and S_0 .

Now a closed geodesic γ in S corresponds to the conjugacy class $D(f)$ of an hyperbolic affine element f in $\text{Aff}(M)$. Its image $\gamma_0 = \phi(\gamma)$ corresponds to the conjugacy class of $\psi_0(f)$. The hyperbolic lengths of the loops γ and γ_0 are respectively the logarithms of the dominant eigenvalues of $D(f)$ and $\psi_0(f)$. Thus, equality in 3.17 is a direct contradiction to the fact that $|\text{tr}(\psi_0(f))| < |\text{tr}(D(f))|$. \square

Remark 3.3.39. The proof of Theorem 3.3.38 is actually more general. Along the same line one can prove the following generalization.

Theorem 3.3.40

Let M be a Veech surface. Assume that there exists a rank 2 subspace $E \subset \ker \text{hol}(M)$ defined over \mathbb{Q} and preserved by a subgroup A of finite index of $\text{Aff}(M)$. Then the kernel of the induced action $A \rightarrow \text{GL}(E)$ is infinite or, equivalently, the image of this kernel in $\Gamma(M)$ is non-trivial.

We will even see a more general version in Lemma 3.3.49 and Remark 3.3.50.

Exercise 3.3.41

Let $M = (X, q)$ be a meromorphic quadratic differential with at most simple poles on a torus such that its orientation cover is a Veech surface. Let $\tilde{M} \rightarrow (X, q)$ be the \mathbb{Z}^2 -half-translation covering obtained by taking the universal cover of the torus (*i.e.*, a Poincaré plane as in Section 1.2.5). Show that Theorem 3.3.40 can be applied to $\text{Aff}(\tilde{M}) \rightarrow \text{GL}(H_1(\tilde{M}, \mathbb{Z}))$. *Hint:* use Exercise 3.3.14.

Example 3.3.42. We now come back to the square-tiled surface $Z_{3,1}$ of Example 3.3.25. This example also appears as [HW12, Example 7.2]. As can be seen on Figure 3.2 any cylinder decomposition of the square-tiled surface $Z_{3,1}$ has homological dimension 2. As a consequence, the kernel of the action $\text{Aff}(Z_{3,1}) \rightarrow \text{GL}(\ker \text{hol}(Z_{3,1}))$ does not contain any parabolic element (see Theorem 3.3.31). However, the Hooper-Weiss theorem does apply: this kernel is infinite.

The square-tiled surface $Z_{3,1}$ belongs to the stratum $\mathcal{H}(1, 1)$. Let us mention that in the other stratum of genus 2, $\mathcal{H}(2)$ any square-tiled surface admits a completely periodic direction made of a single maximal cylinder [HL06b].

The following is a delicate open question.

Question 3.3.43

Does there exist a square-tiled surface $p : M \rightarrow \mathbb{T}^2$ of genus 3 such that the kernel of the induced action $\text{Aff}(M) \rightarrow \text{GL}(\ker \text{hol}(M))$ is finite?

Note that Theorem 3.3.31 discards any square-tiled surface that admits a completely periodic of homological dimension one. The reader interested by this question should consult [HM].

3.3.5 Affine symmetries of wind-tree models

In this section we study the affine symmetries of the wind-tree models $W_{a,b}$ that were defined in Section 1.2.4. We recall that $W_{a,b}$ is a \mathbb{Z}^2 -covering of a finite-type translation surface $X_{a,b}$ that belongs to the stratum $\mathcal{H}(2^4)$. Moreover, $X_{a,b}$ is a K -covering of $L_{a,b}$ in $\mathcal{H}(2)$ where $K = (\mathbb{Z}/(2\mathbb{Z})) \times (\mathbb{Z}/(2\mathbb{Z}))$ is the Klein group (see Figure 3.4).

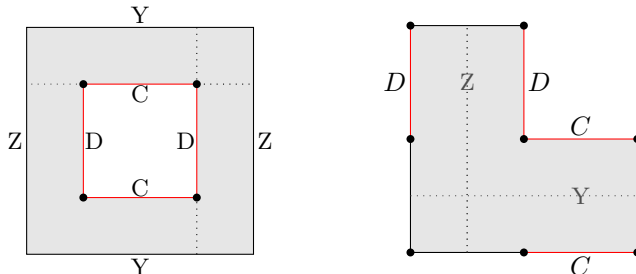


Figure 3.4: Two views of $L_{1/2,1/2}$. The left picture is close to Figure 1.14b for $X_{a,b}$. The right picture is obtained from the left one by cutting along the dotted lines and gluing along the sides Y and Z .

The affine symmetries of the surface $L_{a,b}$ ore generally of translation surfaces of genus 2 have been studied in important works of Calta and McMullen (a complete statement is given in Theorem C.6.6 from Appendix C.6.3). First of all $L_{a,b}$ is a Veech surface if and only if the parameters (a, b) satisfy one of the two following conditions

- (a, b) is rational; in this case, $L_{a,b}$ is a square tiled surface up to multiplication by a diagonal matrix,
- there exists a square-free positive integer D and rational numbers x, y such that $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x) + y\sqrt{D}$.

The following subset of parameters plays an important role

$$\mathcal{E} := \left\{ \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathbb{Q}^2 : 0 < p < q, 0 < r < s, p, q \text{ odd and } r, s \text{ even} \right\}. \quad (3.18)$$

See Lemma C.6.8 for alternative definitions of the set \mathcal{E} .

Theorem 3.3.44

Let (a, b) be parameters for the wind-tree model so that $L_{a,b}$ is a Veech surface. Then

1. $\Gamma(W_{a,b} \rightarrow L_{a,b})$ has infinite index in $\Gamma(L_{a,b})$,
2. $\Gamma(W_{a,b} \rightarrow L_{a,b})$ contains a non-trivial normal subgroup, in particular its limit set is the whole circle
3. If furthermore $(a, b) \in \mathcal{E}$, where \mathcal{E} is defined in (3.18), then $\Gamma(W_{a,b} \rightarrow L_{a,b})$ contains a parabolic element.

The rest of the section is dedicated to the proof of Theorem 3.3.44. We will prove the first item about the index of $\Gamma(W_{a,b} \rightarrow L_{a,b})$ in $\Gamma(L_{a,b})$. Then, we consider parameters (a, b) in \mathcal{E} and use the one-cylinder trick (Theorem 3.3.31) to prove the third item. Finally, we apply a variant of the Hooper-Weiss theorem (Theorem 3.3.38) for all parameters (a, b) which provides a proof of the second item.

$\Gamma(W_{a,b} \rightarrow L_{a,b})$ has infinite index in $\Gamma(L_{a,b})$

We begin the proof of Theorem 3.3.44 by proving that $\Gamma(W_{a,b} \rightarrow L_{a,b})$ is not a lattice. We use an argument that appears in [Cab12].

Let a, b be parameters so that $L_{a,b}$ is a Veech surface. Let $f \in \text{Aff}(X_{a,b})$ be the multitwist stabilizing the horizontal direction. We claim that no power of f lifts to $W_{a,b}$. Let V be the subspace of $\ker \text{hol}(X_{a,b})$ generated by the elements Y and Z as in (3.9). We want to show that no power of f_* preserves V . By a direct computation, one can see that f_* fixes Y but for any $n \geq 1$, we have $(f_*)^n Z \notin V$. As a consequence no power of f_* lifts to $W_{a,b}$ and the index of $\Gamma(W_{a,b} \rightarrow L_{a,b})$ in $\Gamma(L_{a,b})$ is infinite.

Parabolic elements for parameters in \mathcal{E}

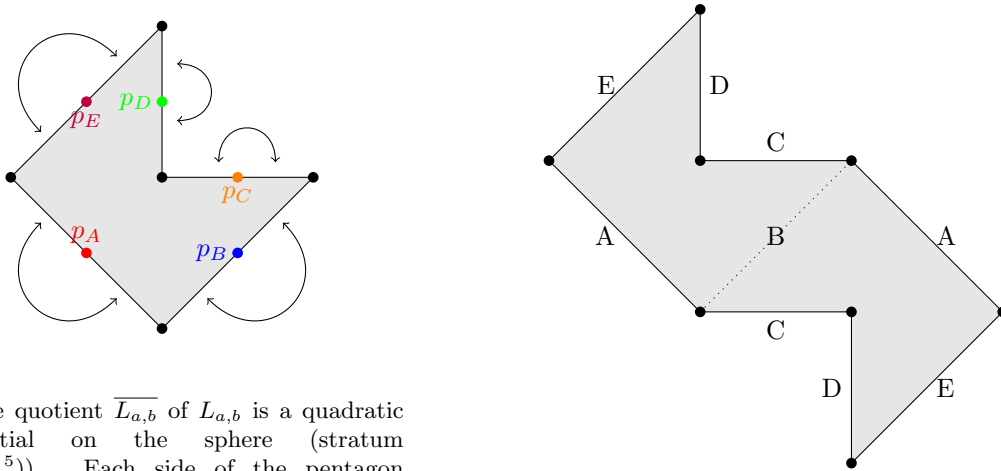
Let us recall that in (3.18) we defined a subset of rational parameters \mathcal{E} for the windtree-model. For (a, b) in \mathcal{E} we consider parabolic elements in $L_{a,b}$ and show that some of them can be lifted to $W_{a,b}$.

Lemma 3.3.45. *Let $(a, b) \in \mathcal{E}$. Then $L_{a,b}$ admits a completely periodic direction made of a single cylinder. For any such direction, if the surface $L_{a,b}$ decomposes into a cylinder of modulus μ then the surface $X_{a,b}$ decomposes into two cylinders of modulus $\mu/2$ and whose core curves are homologous.*

As a consequence of the above Lemma and Theorem 3.3.31 we have.

Corollary 3.3.46. *Let $(a, b) \in \mathcal{E}$. The square of any multitwist stabilizing a one cylinder direction in $L_{a,b}$ lifts to $X_{a,b}$ and $W_{a,b}$. In particular, $\Gamma(W_{a,b} \rightarrow X_{a,b})$ contains a parabolic element.*

Proof of Lemma 3.3.45. We consider the quotient $\overline{L_{a,b}}$ of $L_{a,b}$ by the hyperelliptic involution (the unique affine element in $\text{Aff}(L_{a,b})$ whose derivative is minus identity). The surface $\overline{L_{a,b}}$ is depicted in Figure 3.5. The advantage of this surface is that it has genus 0, and describing a covering of a genus 0 surface is very convenient (the fundamental group is generated by loops around the punctures). The first step,



(a) The quotient $\overline{L_{a,b}}$ of $L_{a,b}$ is a quadratic differential on the sphere (stratum $\mathcal{Q}(1, -1^5)$). Each side of the pentagon is glued to itself via a 180° rotation and contains a pole in its center. The five poles are labeled p_A, p_B, p_C, p_D, p_E in counterclockwise order.

(b) The surface $L_{a,b}$ as two copies of a pentagon. The sides C and D are the same as in Figure 3.4.

Figure 3.5: $\overline{L_{a,b}}$ and $L_{a,b}$.

is to describe the coverings $L_{a,b} \rightarrow \overline{L_{a,b}}$ and $X_{a,b} \rightarrow \overline{L_{a,b}}$. Both involve homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ relative to the set of singularities $\overline{\Sigma}$ of $\overline{L_{a,b}}$ that consists of the five poles p_A, \dots, p_E and a conical singularity of angle 3π . We denote by $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ and \overline{E} the elements in $H_1(\overline{L_{a,b}}, \overline{\Sigma}; \mathbb{Z}/2\mathbb{Z})$ that corresponds to the quotients of the sides A, B, C, D and E of the figure $L_{a,b}$ (see Figure 3.5). Since we consider $\mathbb{Z}/2\mathbb{Z}$ coefficients, the orientation of these curves does not matter. These 5 elements form a basis of $H_1(\overline{L_{a,b}}, \overline{\Sigma}; \mathbb{Z}/2\mathbb{Z})$.

In this basis the cycle $c_L \in H_1(\overline{L_{a,b}}, \overline{\Sigma}; \mathbb{Z}/2\mathbb{Z})$ that corresponds to the degree 2 covering $L_{a,b} \rightarrow \overline{L_{a,b}}$ and $c_X \in H_1(\overline{L_{a,b}}, \overline{\Sigma}; (\mathbb{Z}/2\mathbb{Z})^3)$ that corresponds to the degree 8 covering $X_{a,b} \rightarrow \overline{L_{a,b}}$ are

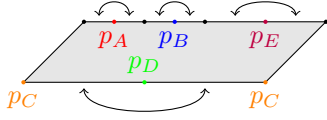
$$c_L = \overline{A} + \overline{B} + \overline{C} + \overline{D} + \overline{E} \quad c_X = (1, 1, 0)\overline{D} + (1, 0, 1)\overline{C} + (1, 0, 0)(\overline{A} + \overline{B} + \overline{E}). \quad (3.19)$$

The formula for c_L and c_X can be deduced by analyzing the action of the Deck group of $X_{a,b} \rightarrow L_{a,b}$ on $H_1(X_{a,b}; \mathbb{Z})$.

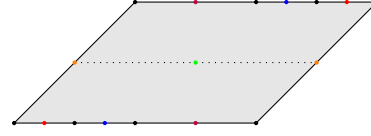
A direction is completely periodic in $L_{a,b}$ if and only if it is completely periodic in $\overline{L_{a,b}}$. Furthermore, if the direction in $L_{a,b}$ is made of a single cylinder then it is also the case in $\overline{L_{a,b}}$ which is a quotient.

Now, by [HLT11] for any completely periodic direction in $\overline{L_{a,b}}$ made of a single cylinder, one side of this cylinder contains the two poles p_E and p_F while the three other poles and the conical singularity of angle 3π are on the other side (for a proof, see also Section C.6.3 of Appendix C). This fact holds exactly for the parameters (a, b) in \mathcal{E} and this is where this hypothesis is crucial.

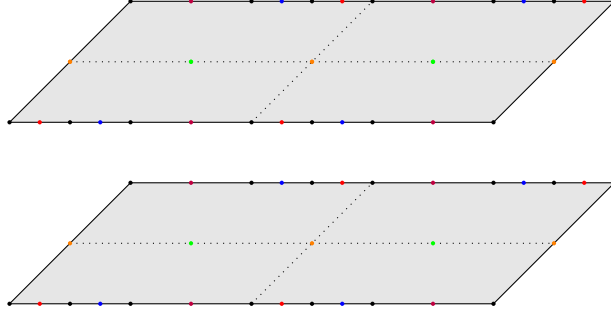
Now, a completely periodic direction in $\overline{L_{a,b}}$ made of a single cylinder of modulus $\overline{\mu}$ is of the form shown in Figure 3.6a (up to permutation of the poles p_A, p_B, p_E located on the top of the cylinder). The



(a) A one cylinder direction in $\overline{L_{a,b}}$ when $(a,b) \in \mathcal{E}$ has one of its sides with the poles $\{p_C, p_D\}$ while the other three are on the other side.



(b) The preimage of the one cylinder direction of Figure 3.6a in $L_{a,b}$.



(c) The preimage of the one cylinder direction of Figure 3.6a in $X_{a,b}$.

Figure 3.6: A one cylinder direction in $\overline{L_{a,b}}$ and its preimages in $L_{a,b}$ and $X_{a,b}$.

preimage of this cylinder in $L_{a,b}$ and $X_{a,b}$ can be computed from the formulas (3.19). The preimage in $L_{a,b}$ consists of a single cylinder of modulus $2\bar{\mu}$ (see Figure 3.6b) while in $X_{a,b}$ it consists of two cylinders of modulus $\bar{\mu}$ with homologous core curves (see Figure 3.6c). \square

Remark 3.3.47. For rational parameters (a,b) that do not belong to \mathcal{E} it can be shown that no parabolic element of $\text{Aff}(X_{a,b})$ lifts to $W_{a,b}$.

Homology action of $\text{Aff}(X_{a,b})$

We now prove the second item in Theorem 3.3.44 by using a variant of Hooper-Weiss Theorem.

First of all the surface $X_{a,b}$ admits a finer decomposition of $H_1(X_{a,b}; \mathbb{Q})$ than what we have discussed for a general surface in Theorem 3.3.18. This is a very general phenomenon for normal covers: the deck group of the cover acts on homology and the affine groups preserves the isotypical components of this action ²

Let τ_h be the translation of $X_{a,b}$ that exchanges the polygon 00 with 10 and 01 with 11 of Figure 1.14b. In other words, τ_h exchanges the two copies drawn on the same horizontal. Similarly, let τ_v be the one exchanging the copies 00 with 01 and 10 with 11.

Lemma 3.3.48 ([DHL14]). *Let (a,b) be any parameter for the surface $L_{a,b}$. Let τ_h and τ_v be the generators of the Deck transformations of the covering $X_{a,b} \mapsto L_{a,b}$. Let us define for each $s, t \in \{+1, -1\}$ the following subspaces:*

$$E^{st} := \{v \in H_1(X_{a,b}; \mathbb{R}) : \tau_h v = sv \quad \tau_v v = tv\}.$$

Then

1. $\dim E^{++} = 4$ and $\dim E^{+-} = \dim E^{-+} = \dim E^{--} = 2$,
2. there is a symplectic orthogonal splitting $H_1(X_{a,b}; \mathbb{R}) = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}$ and $E^{+-} \oplus E^{-+} \oplus E^{--} \subset \ker \text{hol}(X_{a,b})$,
3. each element of $\text{Aff}(X_{a,b})$ preserves E^{++} and E^{--} ,
4. each element of $\text{Aff}(X_{a,b})$ either preserves or exchanges E^{+-} and E^{-+} .

²The isotypical components of a finite group are defined via the representation theory of finite groups. For a definition in the context of translation surface see [MYZ14].

Proof. This is straightforward computation that can be done in at least two ways. First, one can fix a basis of $H_1(X_{a,b}; \mathbb{Z})$ and compute the action of $\langle \tau_h, \tau_v \rangle$. An alternative approach consists in computing the topology of the quotients. First of all $L_{a,b} = X_{a,b}/\langle \tau_h, \tau_v \rangle$ hence $E^{++} \simeq H_1(L_{a,b}; \mathbb{Q})$ which has rank 4. Next $X_{a,b}/\langle \tau_h \rangle$, $X_{a,b}/\langle \tau_v \rangle$ and $X_{a,b}/\langle \tau_h, \tau_v \rangle$ have all genus 3. \square

Lemma 3.3.49 ([Parb]). *Let (a, b) be parameters so that $L_{a,b}$ is a Veech surface. Let $H_1(X_{a,b}; \mathbb{R}) = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}$ be as in Theorem 3.3.48. Then the kernel of the group action $\text{Aff}(X_{a,b}) \rightarrow \text{GL}(E^{+-} \oplus E^{-+} \oplus E^{--})$ is non-trivial.*

Proof. Up to passing to a subgroup A of index 2 of $\text{Aff}(X_{a,b})$, we can assume that each individual component E^{st} for $(s, t) \in \{+-, -+, --\}$ is preserved. Since each component E^{st} has rank 2 and is contained in $\ker \text{hol}(X_{a,b})$ the general Hooper-Weiss theorem (Theorem 3.3.40) implies that the kernels of $A \rightarrow \text{GL}(E^{st})$ are infinite. Their images in $\Gamma(X_{a,b})$ are non-trivial normal subgroups and hence non-elementary (see Lemma B.0.4). In particular there exist ϕ_{+-} , ϕ_{-+} and ϕ_{--} in A so that ϕ_{st} acts trivially on E^{st} .

Assume that $\phi_{+-} \neq \phi_{-+}$. Then, by conjugating by an element in A , we can ensure that the fixed points of $g_1 := D(\phi_{-+})$ and $g_2 := D(\phi_{+-})$ on $\partial\mathbb{H}^2$ are disjoint. For such conjugates, there exist integers m_1 and m_2 so that $(g_1)^{m_1}$ and $(g_2)^{m_2}$ play ping-pong, that is generate a free-group that consists only of hyperbolic elements. In particular the commutator $[(\phi_{+-})^{m_1}, (\phi_{-+})^{m_2}]$ is an hyperbolic affine element. Moreover it acts trivially on both E^{+-} and E^{-+} .

Applying the same argument to $[(\phi_{+-})^{m_1}, (\phi_{-+})^{m_2}]$ and ϕ_{--} we construct an element that acts trivially on the sum $E^{-+} \oplus E^{+-} \oplus E^{--}$. \square

Remark 3.3.50. The proof of Lemma 3.3.49 does actually provide an extension of the general Hooper-Weiss theorem. If M is a Veech surface and there exist subspaces E_1, \dots, E_k of $\ker \text{hol}$ in $H_1(M; \mathbb{Z})$

1. each E_i has rank 2 and they are in direct sum
2. a subgroup of finite index A of $\text{Aff}(M)$ preserves each E_i

Then the kernel of the induced action $A \rightarrow \text{GL}(E_1) \times \dots \times \text{GL}(E_k)$ is infinite and its image in $\Gamma(A)$ is a non-trivial normal subgroup.

3.4 Hooper-Thurston-Veech construction

In Section 1.2.6 we introduced the Thurston-Veech construction, which given two multicurves filling a finite-type topological surface S produces a half-translation surface M on S admitting a horizontal and a vertical cylinder decomposition where all cylinders involved have the same moduli. As a consequence the surface M has a lot of affine pseudo-Anosov homeomorphisms. In the following paragraphs we detail an extension of the Thurston-Veech construction based on the ideas of P. Hooper [Hoo15] and illustrate it with several examples. Infinite-type quadratic differentials constructed this way have non-elementary Veech groups and their translation flows exhibit rich dynamics, which we study in detail in Chapters 5 and 6.

Remark 3.4.1. In this chapter we construct surfaces where changes of coordinates are translations or half-translations. For the sake of a simple nomenclature, we will henceforth use the equivalent term quadratic differential (see Chapter 1).

The whole construction is based in the following statement, which is the extension of Lemma 1.2.24 to infinite-type surfaces.

Lemma 3.4.2. *Let M be a half-translation surface for which the horizontal direction is completely periodic and suppose that every maximal cylinder in the corresponding cylinder decomposition has moduli equal to $\frac{1}{\lambda}$ for some $\lambda \in \mathbb{R}^*$. Then there exists a unique affine automorphism φ_h which fixes the boundaries of the cylinders and has derivative $D\varphi_h = h_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. Moreover, the automorphism φ_h acts as a Dehn twist along the core curve of each cylinder.*

Remark 3.4.3. If M is a half-translation surface and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a completely periodic direction for which every cylinder has moduli equal to $\frac{1}{\lambda}$ for some $\lambda \in \mathbb{R}^*$, then one can apply to M the rotation $R_\theta \in \text{SO}(2, \mathbb{R})$ that takes θ to the horizontal direction and apply the preceding lemma. For example, if $\theta = \frac{\pi}{2}$, then there exists a unique affine automorphism φ_v which fixes the boundaries of the vertical cylinders, acting on each cylinder as a Dehn twist and with derivative $D\varphi_v = v_{-\lambda} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$.

Exercise 3.4.4

Let M be a half-translation surface and suppose that $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is a direction in which M into maximal cylinders $\{C_i\}_{i \in I}$ of moduli $\{\mu_i\}$. Moreover, suppose that there exists $s \in \mathbb{R}^*$ such that for every $i \in I$ we have $m_i := s\mu_i \in \mathbb{Z}$. Show that there exist a unique $f \in \text{Aff}(M)$ fixing the boundary of C_i for all $i \in I$ which has derivative $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ up to conjugation in $\text{SO}(2, \mathbb{R})$.

The proof of Lemma 3.4.2 can be deduced from the preceding exercise and we leave it to the reader.

Let us remark that Lemma 1.2.24 is an equivalence between parabolic directions and affine multitwist directions for finite-type surfaces. Though for infinite-type surfaces, only the implication stated in Lemma 3.4.2 is valid. Indeed, we already encountered many counterexamples to the converse. First of all, in Example 3.3.32 we showed that the Veech group of the infinite staircase has a parabolic element fixing the direction $\theta = \frac{\pi}{4}$ and in this direction the staircase decomposes into two strips. Secondly, iff one considers the plane or any cyclic cover ramified over the origin, its Veech group is $\text{SL}(2, \mathbb{R})$. In this case, any direction θ is parabolic but none of them contains a cylinder. Finally, more exotic examples can be constructed from Theorem 3.2.1 and Exercise 3.2.2. Note that all these examples have infinite area.

Question 3.4.5

Let M be an infinite-type translation surface of finite area. Are parabolic directions in M affine multitwist directions?

Definition 3.4.6. A quadratic differential M is called a *Hooper-Thurston-Veech* surface if it admits horizontal and vertical cylinder decompositions where all cylinders have the same moduli. The common moduli of the cylinders is called the moduli of the Hooper-Thurston-Veech surface.

Remark 3.4.7. If $M = M(\alpha, \beta, \mathbf{h})$ is a Hooper-Thurston-Veech surface of moduli $\frac{1}{\lambda}$ then $\text{Aff}(M)$ has two affine multitwists ϕ_h and ϕ_v with $D\phi_h = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and $D\phi_v = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$. Moreover, if $\lambda \geq 2$ then

$$G_\lambda = \langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \rangle \quad (3.20)$$

is a free subgroup of the Veech group $\Gamma(M)$ (see lemma B.2.1 in Appendix B).

Note that finite-type Hooper-Thurston-Veech surfaces are those given by the classical Thurston-Veech construction discussed in Section 1.2.6. Translation coverings provides an elementary way of obtaining Hooper-Thurston-Veech surfaces. More precisely the lifting criterion for cylinders stated in Lemma 3.3.26 implies the following.

Lemma 3.4.8. *Let M be a finite-type translation surface obtained by the Thurston-Veech construction for the multicurves $\alpha = \cup_{i \in I} \alpha_i$ and $\beta = \cup_{j \in J} \beta_j$. Let $p: \widetilde{M} \rightarrow M$ be the G -covering branched at most over the singularities Σ of M defined by a morphism $\rho: \pi_1(M \setminus \Sigma) \rightarrow G$ such that for all $i \in I$ we have $\rho(\alpha_i) = 1$ and for all $j \in J$ we have $\rho(\beta_j) = 1$. Then \widetilde{M} is a Hooper-Thurston-Veech surface.*

The rest of the section is dedicated to the general construction of Hooper-Thurston-Veech surfaces starting from a pair of transverse multicurves in an infinite-type surface.

3.4.1 From cylinder decompositions to bipartite graphs

Let S be a topological surface. Recall from Definition 1.2.25 that a multicurve in S is a locally finite family of essential simple closed curves in S , pairwise non-intersecting and pairwise non-homotopic. A pair of simple closed curves c and c' are in *minimal position* if $|c \cap c'| = \iota(c, c')$, where $\iota(\cdot, \cdot)$ denotes the geometric intersection between isotopy classes of curves.

Definition 3.4.9. We say that a pair of multicurves $\alpha = \cup \alpha_i, \beta = \cup \beta_j$

- is in *minimal position* if every pair α_i, β_j of components of α and β are in minimal position,
- *fills* S if every connected component of the complement of the union $\alpha \cup \beta$ is a topological disc with at most one puncture.

Exercise 3.4.10

Let α and β two multicurves in a topological surface S . Show that each component α_i of α intersects finitely many components of β .

hint: use the fact that a multicurve is by definition locally finite.

Definition 3.4.11. Let $\alpha = \cup_{i \in I} \alpha_i$ and $\beta = \cup_{j \in J} \beta_j$ be two multicurves in a topological surface S in minimal position (not necessarily filling). The *configuration graph* of the pair (α, β) is the bipartite graph $\mathcal{G}(\alpha \cup \beta)$ whose vertex set is $I \sqcup J$ and where there is an edge between two vertices $i \in I$ and $j \in J$ for every intersection between the corresponding curves α_i and β_j .

Before elaborating the construction for infinite-type surfaces, we explain how the configuration graph already appeared in disguised form in the Thurston-Veech construction from Section 1.2.6 (more precisely Exercises 1.2.26 and 1.2.27). Let S be a finite-type surface and $\alpha = \cup \alpha_i, \beta = \cup \beta_j$ a pair of filling multicurves in minimal position. As we have seen there is a half-translation structure on S which admits an horizontal and a vertical affine multitwist along the multicurves α and β respectively. To construct such half-translation structure, we considered the intersection matrix $E = (\iota(\alpha_i, \beta_j))_{i \in I, j \in J}$ and shown that the heights of the horizontal cylinders \mathbf{h}_h and the heights of the vertical cylinders \mathbf{h}_v must satisfy $E\mathbf{h}_v = \lambda\mathbf{h}_h$ and $E^t\mathbf{h}_h = \lambda\mathbf{h}_v$. Let us consider the following matrix

$$A = \begin{pmatrix} 0 & E \\ E^t & 0 \end{pmatrix}.$$

Then A is the adjacency matrix of the configuration graph $\mathcal{G}(\alpha \cup \beta)$ and the vector $\mathbf{h} = (\mathbf{h}_h, \mathbf{h}_v)$ is an eigenvector of A .

Now, let us consider a Hooper-Thurston-Veech surface with moduli $1/\lambda$. Let $H = \{H_i\}_{i \in I}$ and $V = \{V_j\}_{j \in J}$ be respectively the horizontal and vertical cylinders of M . For every $i \in I$ let α_i be the core curve of H_i and for every $j \in J$ let β_j be the core curve of V_j . Then $\alpha = \cup \alpha_i$ and $\beta = \cup \beta_j$ are multicurves in M . Let $\mathbf{h} : I \cup J \rightarrow \mathbb{R}_{>0}$ be the function which to an index associates the width of the corresponding cylinder. Then it satisfies the equation $A\mathbf{h} = \lambda\mathbf{h}$ where A is the adjacency operator of the graph $\mathcal{G}(\alpha \cup \beta)$ defined by

$$(A\mathbf{h})(v) := \sum_{w \sim v} \mathbf{h}(w) \tag{3.21}$$

where the sum above is taken over the vertices w adjacent to v in $\mathcal{G}(\alpha \cup \beta)$ (with the multiplicity given by the number of edges between v and w).

Definition 3.4.12. Let $\mathcal{G} = (V, E)$ be a graph with vertices of finite degree. Let $A : V^{\mathbb{R}} \rightarrow \mathbb{R}$ its adjacency operator. A function $\mathbf{h} : V \rightarrow \mathbb{R}$ that satisfies $A\mathbf{h} = \lambda\mathbf{h}$ is called a λ -harmonic function.

Exercise 3.4.13

Let $H = \{H_i\}$ and $V = \{V_j\}$ be horizontal and vertical cylinder decompositions of a quadratic differential M , and let $\mathcal{G}(\alpha \cup \beta)$ be the configuration graph of the corresponding core curves. Suppose that each horizontal cylinder intersects finitely many vertical cylinders and *vice versa*. Show that the union of the core curves of the cylinders in $H \cup V$ fills M , *i.e.*, their complement in M is a union of discs with at most one puncture. On the other hand, show that the core curves of the horizontal and vertical cylinder decompositions of the infinite-type translation surface M in Example 2.3.5, Chapter 2, do not fill.

We now present the main construction that extends to infinite-type surfaces the Thurston-Veech construction of Exercise 1.2.27.

Theorem 3.4.14

Let S be topological surface, $\alpha = \cup_{i \in I} \alpha_i$ and $\beta = \cup_{j \in J} \beta_j$ two multicurves in minimal position whose union fills S . Suppose that:

- **C1.** The configuration graph $\mathcal{G}(\alpha \cup \beta)$ has finite degree (*i.e.*, there is an upper bound on the degree of the vertices in $\mathcal{G}(\alpha \cup \beta)$),

- **C2.** For every component D of the complement of $\alpha \cup \beta$ in S , its boundary ∂D in S is connected.
- **C3.** For every component D of the complement of $\alpha \cup \beta$ for which ∂D intersects infinitely many curves in $\alpha \cup \beta$, then D is a disk *without* punctures.

Then there exists a positive λ -harmonic function $\mathbf{h} : \mathcal{G}(\alpha \cup \beta) \rightarrow \mathbb{R}_{>0}$ of the configuration graph. Moreover, for any positive λ -harmonic function \mathbf{h} , there exist a half-translation structure $M = M(\alpha, \beta, \mathbf{h})$ on S which is a Hooper-Thurston-Veech surface of moduli $1/\lambda$ whose horizontal cylinders (respectively vertical cylinders) have core curves $\{\alpha_i\}_{i \in I}$ (resp. $\{\beta_j\}_{j \in J}$).

Note that if S is of finite-type then the conditions **C1**, **C2** and **C3** above are satisfied automatically and the result follows from the Thurston-Veech construction as explained in Section 1.2.6.

Proof. The union $\alpha \cup \beta$ of the multicurves α and β define a graph embedded in S : the vertices are points in $\bigcup_{(i,j) \in I \times J} \alpha_i \cap \beta_j$ and edges are the segments forming the connected components of $\alpha \cup \beta \setminus \bigcup_{(i,j) \in I \times J} \alpha_i \cap \beta_j$. It is important not to confuse the (geometric) graph $\alpha \cup \beta$ with the (abstract) configuration graph $\mathcal{G}(\alpha \cup \beta)$.

To define the quadratic differential we consider a dual embedded graph $(\alpha \cup \beta)^*$ in S . When all complementary components of $S \setminus \alpha \cup \beta$ have a compact boundary $(\alpha \cup \beta)^*$ is just the dual graph of $\alpha \cup \beta$. In the case when there are complementary components of $S \setminus \alpha \cup \beta$ whose boundary is formed by infinitely many vertices and edges we need to define $(\alpha \cup \beta)^*$ more carefully. The vertices of $(\alpha \cup \beta)^*$ are defined as follows:

- **A1.** For every connected component D of $S \setminus \alpha \cup \beta$ such that ∂D is formed by finitely many edges of $\alpha \cup \beta$ choose a unique point v_D inside the connected component. If D is a punctured disc, then choose v_D to be the puncture.
- **A2.** If D is a connected component of $S \setminus \alpha \cup \beta$ such that ∂D is formed by infinitely many edges of $\alpha \cup \beta$, then by hypothesis **C3** above D is homeomorphic to a disc. Moreover, $D \cup \partial D$ has only one end to which we associate an ideal point v_D . We think of v_D as a vertex “at infinity” of $(\alpha \cup \beta)^*$, see Figure 3.7.

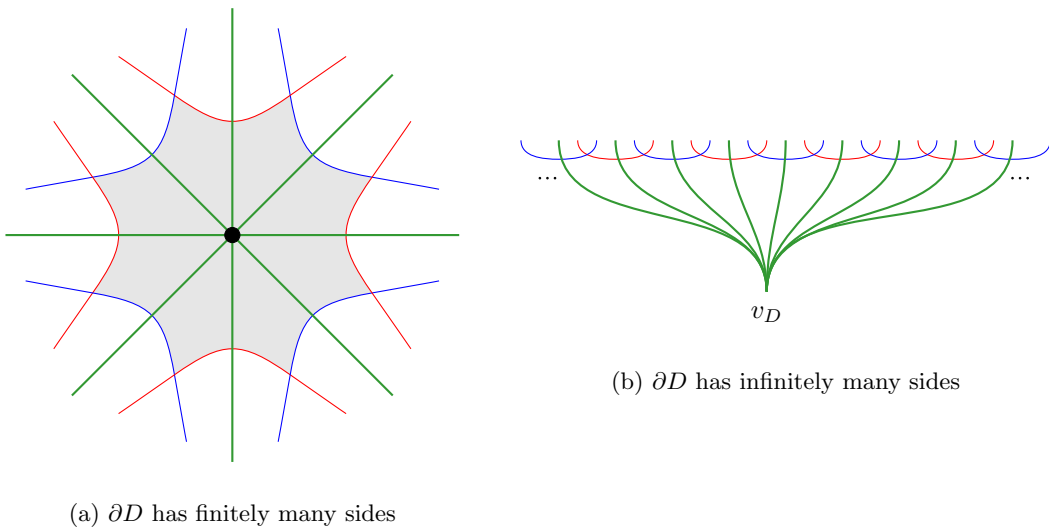


Figure 3.7: Edges of the graph $(\alpha \cup \beta)^*$.

The points v_D chosen above are the vertices of $(\alpha \cup \beta)^*$. Two vertices in this graph are joined by an edge in S if the closures of the corresponding connected components of $S \setminus \alpha \cup \beta$ share an edge of $\alpha \cup \beta$. Edges are chosen to be pairwise disjoint. For edges with vertices at infinity edges are chosen to be proper rays defining the end of $D \cup \partial D$ associated to v_D , see Figure 3.7.

Given that $\alpha \cup \beta$ fills and the boundary ∂D of each complementary component D of $\alpha \cup \beta$ is connected (hypothesis **C2**), every connected component $S \setminus (\alpha \cup \beta)^*$ is a topological quadrilateral which contains a unique vertex of $\alpha \cup \beta$. Hence there is a well defined bijection between edges in the abstract graph $\mathcal{G}(\alpha \cup \beta)$ and the set of these quadrilaterals. This way, for every edge $e \in E(\mathcal{G}(\alpha \cup \beta))$ we denote by R_e

the closure in S of the corresponding topological quadrilateral with the convention to add to R_e vertices v_D corresponding to punctures in S and points at infinity (as chosen above) so that R_e has four vertices.

Note that there are only two sides of R_e intersecting the multicurve α , which henceforth are called *vertical sides*. The other two sides are in consequence called *horizontal*, see Figure 3.8.

We now build a half-translation atlas on S by mapping the topological quadrilaterals R_e of the dual graph $(\alpha \cup \beta)^*$ into Euclidean rectangles. As in the finite-type case, the λ -harmonic function $\mathbf{h} : \mathcal{G}(\alpha \cup \beta) \rightarrow \mathbb{R}_{>0}$ gives us compatible heights of horizontal and vertical cylinders. More precisely, let us define the maps

$$p_\alpha : E(\mathcal{G}(\alpha \cup \beta)) \rightarrow V(\mathcal{G}(\alpha \cup \beta)) \quad \text{and} \quad p_\beta : E(\mathcal{G}(\alpha \cup \beta)) \rightarrow V(\mathcal{G}(\alpha \cup \beta))$$

which to an edge e of the configuration graph $\mathcal{G}(\alpha \cup \beta)$ associate its endpoints $p_\alpha(e)$ in I and $p_\beta(e)$ in J . Our aim is to construct a homeomorphism

$$R_e \longrightarrow [0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)] \quad (3.22)$$

that will determine a unique half-translation atlas for S with the required properties (see Figure 3.8).

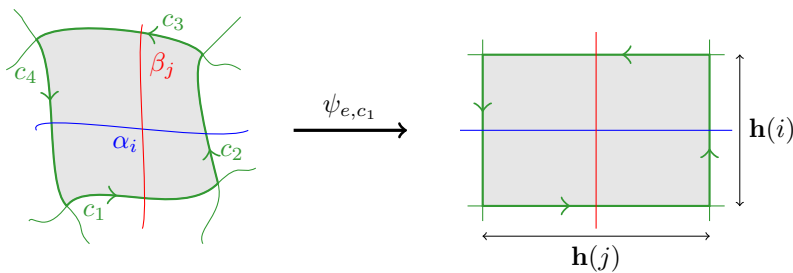


Figure 3.8: Transforming topological Rectangles into Euclidean rectangles.

In order to ensure compatibility of the atlas we first consider mapping of the edges. For each oriented edge c of $(\alpha \cup \beta)^*$ crossing a component α_i of α we choose an homeomorphism $\theta_c : c \rightarrow [0, \mathbf{h}(i)]$ such that

- the start point of c is mapped to 0 and its endpoint to $\mathbf{h}(i)$,
- $c^{-1}(\mathbf{h}(i)/2)$ is the intersection of c with α_i ,
- if c' is the same edge with opposite orientation $\theta_{c'}(x) = \mathbf{h}(i) - \theta_c(x)$.

Then for each rectangle R_e (corresponding to an edge e of $\mathcal{G}(\alpha \cup \beta)$) and one of its horizontal sides c_1 we construct a map ψ_{e,c_1} as follows. Choose a cyclic ordering of the sides c_1, c_2, c_3, c_4 following the orientation of the surface. Then define on the boundary ∂R_e

$$\psi_{e,c_1} : x \mapsto \begin{cases} (\theta_{c_1}(x), 0) & \text{if } x \in c_1 \\ (\mathbf{h}(j), \theta_{c_2}(x)) & \text{if } x \in c_2 \\ (\mathbf{h}(j) - \theta_{c_3}(x), \mathbf{h}(i)) & \text{if } x \in c_3 \\ (0, \mathbf{h}(i) - \theta_{c_4}(x)) & \text{if } x \in c_4 \end{cases}$$

Note that by our choice of compatibilities of the maps θ_c we have that $\psi_{e,c_1}|_{\partial R_e} = -\psi_{e,c'_3}|_{\partial R_e}$ where c'_3 is the edge c_3 with reversed orientation.

Now, we extend each ψ_{e,c_1} into a homeomorphism such that

- $\alpha_i \cap R_e$ is mapped to $[0, \mathbf{h}(j)] \times \{\mathbf{h}(i)/2\}$,
- $\beta_j \cap R_e$ is mapped to $\{\mathbf{h}(j)/2\} \times [0, \mathbf{h}(i)]$,
- $\psi_{e,c_1} = -\psi_{e,c'_3}$.

We claim that the family $\{\psi_{e,c}\}$ we constructed above extends to a unique half-translation atlas on S . First of all, the coordinate change between ψ_{e,c_1} and ψ_{e,c'_3} is a point reflection. Notice also that each point in the interior of a rectangle is covered by two charts. Now, for each edge c in the intersection

of two neighbouring rectangles R_e and $R_{e'}$, the compatibility of the maps ψ_e on the boundaries via the maps θ_c can be used to provide charts compatible with the atlas.

Note that the argument is similar to the one we used in the introduction to show that a constructive translation surface (obtained from gluing polygons) is also a geometric translation surface (endowed with a translation atlas). See item 1 of Theorem 1.1.11.

We denote the resulting half-translation surface $M(\alpha, \beta, \mathbf{h})$.

Now, for every $i \in I$, the curve α_i is the core curve of the horizontal cylinder $H_i := \cup_{e \in p_\alpha^{-1}(i)} R_e$. Because \mathbf{h} is harmonic we have

$$\sum_{e \in p_\alpha^{-1}(i)} \mathbf{h}(p_\beta(e)) = \sum_{j \sim i} \mathbf{h}(j) = \lambda \mathbf{h}(i).$$

In other words, the circumference $\sum_{e \in p_\alpha^{-1}(i)} \mathbf{h}(p_\beta(e))$ of H_i is λ times its height $\mathbf{h}(i)$. That is, the moduli of $\mu(H_i)$ of H_i is equal to $\frac{1}{\lambda}$. The same computation with β_j shows that the vertical cylinders $V_j := \cup_{e \in p_\beta^{-1}(j)} R_e$ have core curve β_j and moduli $\frac{1}{\lambda}$. □

Remark 3.4.15. Let $M = M(\alpha, \beta, \mathbf{h})$ be a Hooper-Thurston-Veech surface as in Theorem 3.4.14. For every connected component D of $S \setminus \alpha \cup \beta$ we have that:

1. If ∂D is formed by $2k$ edges of $\alpha \cup \beta$ then the vertex v_D of $(\alpha \cup \beta)^*$ is a conical singularity of total angle $k\pi$ in³ $\text{Sing}(M)$ and all other points of D are regular.
2. If ∂D is formed by infinitely many edges of $\alpha \cup \beta$ then the vertex v_D of $(\alpha \cup \beta)^*$ is either an infinite angle singularity or a wild singularity.

This follows from the fact that in the proof of Theorem 3.4.14 we are glueing Euclidean rectangles with a common vertex at v_D . Moreover, when v_D is a wild singularity, it has a bi-infinite rotational component⁴ $\text{Rot}(\gamma)$. However, not all rotational components of linear approaches to v_D are bi-infinite: as seen in the next section baker's surfaces are examples of Hooper-Thurston-Veech surfaces.

In particular, the number and type of singularities in $M(\alpha, \beta, \mathbf{h})$ does not depend on the λ -harmonic function \mathbf{h} .

In the rest of this chapter we illustrate with several examples the quadratic differential how to determine explicitly the quadratic differential $M(\alpha, \beta, \mathbf{h})$.

Example 3.4.16. In the following paragraphs we complete the discussion started in Example 1.2.30. In particular, we show that the infinite staircase and all λ -staircases are Hooper-Thurston-Veech surfaces homeomorphic to the Loch Ness monster. Figure 3.9 shows a Loch Ness monster with two multicurves α and β in minimal position satisfyin contidions **C1–C4** of Theorem 3.4.14 and the corresponding configuration graph $\mathcal{G}(\alpha \cup \beta)$, to which we refer hereafter as the \mathbb{Z} -graph because its vertices are indexed by \mathbb{Z} .

A λ -harmonic function on the \mathbb{Z} -graph satisfies

$$\mathbf{h}(n+1) = \lambda \mathbf{h}(n) - \mathbf{h}(n-1) \tag{3.23}$$

for every $n \in \mathbb{Z}$. This is a linear recurrence relation whose characteristic polynomial is $x^2 - \lambda x + 1$. Let $r_+ := \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}$ and $r_- := \frac{1}{r_+}$ be the corresponding roots. There are two cases to consider:

- $\lambda = 2$. Since $r_+ = r_- = 1$ the general solution of (3.23) is of the form $\mathbf{h}(n) = C + Dn$. Positive solutions only exist if $D = 0$ and hence the only positive λ -harmonic function in this case (up to scaling) is $\mathbf{h}(n) = 1$ for all $n \in \mathbb{Z}$.
- $\lambda > 2$. For each fixed value of λ , the general solution of (3.23) is a linear combination of the positive eigenfunctions $\mathbf{h}_1(n) = r_+^n$ and $\mathbf{h}_2(n) = r_+^{-n}$.

We now explain a systematic way to picture $M(\alpha, \beta, \mathbf{h})$. In general it can be difficult to visualize the graph $\alpha \cup \beta$ and how the rectangles lie in S . The following steps are intended to overcome this difficulty.

³For $k = 2$, the vertex v_D is a regular point of M .

⁴*i.e.* the map dir restrited to each rotational component defines a bijection with \mathbb{R} , see (2.7, in Section 2.3.2).

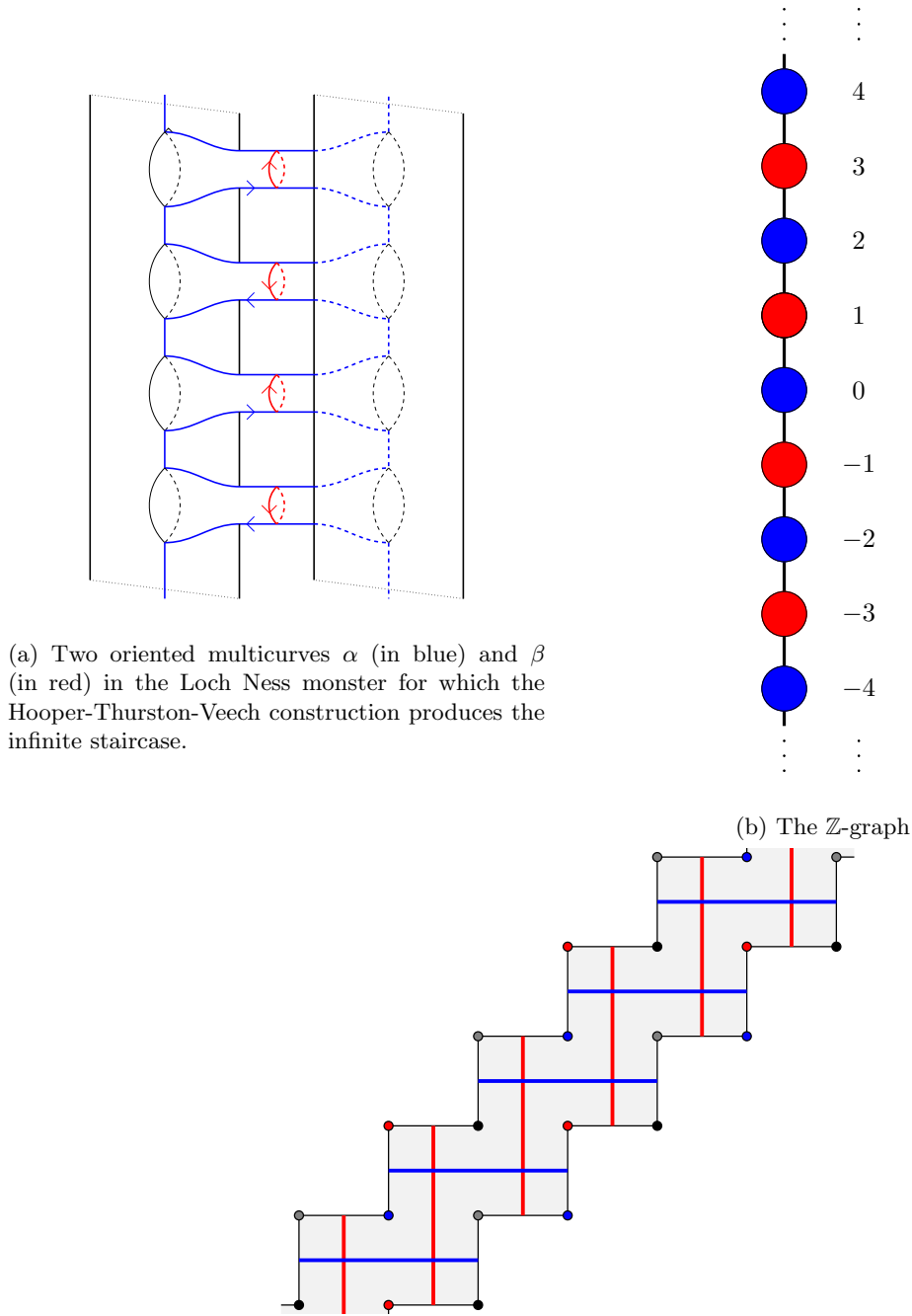


Figure 3.9: The infinite staircase as a Hooper-Thurston-Veech surface.

1. *Orientations.* Fix an orientation on each of the curves in $\alpha \cup \beta$ and the standard orientation in \mathbb{R}^2 . Let ψ_{e,c_1} be an orientation preserving homeomorphism as defined in the proof of Theorem 3.4.14. We say that R_e is *positive* (respectively *negative*) if the algebraic intersection of $\psi_{e,c_1}(\alpha \cap R_e)$ with $\psi_{e,c_1}(\beta \cap R_e)$ is positive (respect. negative). In Figure 3.10 we illustrate positive and negative rectangles.
2. *Gluing rectangles.* The orientation of elements in $\alpha \cup \beta$ defines two permutations

$$r : E(\mathcal{G}(\alpha \cup \beta)) \rightarrow E(\mathcal{G}(\alpha \cup \beta)) \quad \text{and} \quad u : E(\mathcal{G}(\alpha \cup \beta)) \rightarrow E(\mathcal{G}(\alpha \cup \beta)) \quad (3.24)$$

as follows. Let $e \in E(\mathcal{G}(\alpha \cup \beta))$ be determined by a point $p_{ij} \in \alpha_i \cap \beta_j$ and let $p_{ij'} \in \alpha_i \cap \beta_j$ be the point of $\alpha_i \cap \beta$ that follows from p_{ij} when considering the orientation of α_i . If $e' \in E(\mathcal{G}(\alpha \cup \beta))$ denotes the edge determined by $p_{ij'}$ then $r(e) = e'$. The permutation u is defined in an analogous

way by considering points in $\alpha \cap \beta_j$. Note that for every vertex $i \in I$ (respectively $j \in J$) of $\mathcal{G}(\alpha \cup \beta)$, r defines a cyclic permutation of edges adjacent to i (respect. u for edges adjacent to j).

As discussed in the proof of Theorem 3.4.14, the rectangles $[0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)]$ are glued along parallel edges following the configuration of the multicurves $\alpha \cup \beta$. More precisely: for each $e \in E$ is glued to the right side of the Euclidean rectangle $[0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)]$ to the left side of $R_{r(e)}$ using a translation if both rectangles R_e and $R_{r(e)}$ are positive or using a half-translation if both rectangles are negative. On the other the upper side of $[0, \mathbf{h} \circ p_\beta(e)] \times [0, \mathbf{h} \circ p_\alpha(e)]$ is glued to the lower side of $R_{u(e)}$ always using a translation.

3. *Singularities.* Remove all corners of rectangles glued as above which correspond to marked points in S or which produce singularities which are not conical.

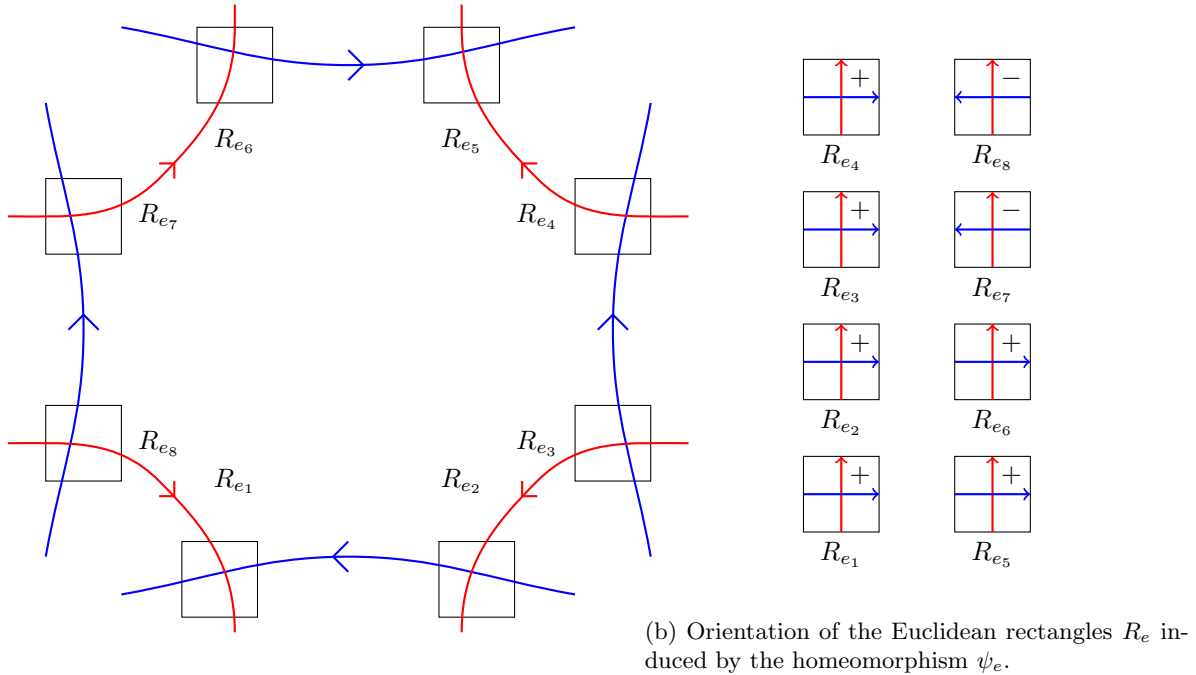


Figure 3.10: Positive and negative rectangles R_e

To visualize concretely the steps explained above consider the oriented multicurves depicted in Figure 3.9 and the λ -harmonic function $\mathbf{h}(n) = 1$ for all $n \in \mathbb{Z}$ of the adjacency operator of the \mathbb{Z} -graph (*i.e.* case $\lambda = 2$). For each edge e in the \mathbb{Z} -graph the rectangle R_e is positive and identified with a unit square, therefore all gluings of these are done using translations. Therefore $M(\alpha, \beta, \mathbf{h})$ is a translation surface. Moreover, it is one of our favorites examples in this book: the infinite staircase. See Figure 3.9.

Exercise 3.4.17

Let $M(\alpha, \beta, \mathbf{h})$ be a Hooper-Thurston-Veech surface of infinite type.

1. Let $C : E(\mathcal{G}(\alpha \cup \beta)) \rightarrow \{-1, +1\}$ be the function that assigns to each rectangle its sign (*i.e.* $+1$ to positives) as described in step 1 in Example 3.4.16. Show that $M(\alpha, \beta, \mathbf{h})$ is a translation surface if and only if for any closed walk $\{e_0, e_1, \dots, e_n\}$, $e_i \in E(\mathcal{G}(\alpha \cup \beta))$, we have that $\prod_{i=0}^n C(e_i) = 1$. Conclude that the λ -harmonic function \mathbf{h} plays no role on determining whether change of coordinates in $M(\alpha, \beta, \mathbf{h})$ are translations or half-translations.

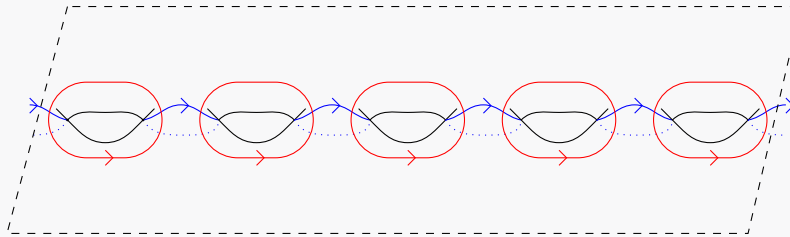
Compare with Exercise 1.2.29 in Chapter 1.

Exercise 3.4.18

For $\lambda > 2$, consider any positive linear combination \mathbf{h} of the λ -harmonic functions of the \mathbb{Z} -graph $\mathbf{h}_1(n) = r_+^n$ and $\mathbf{h}_2(n) = r_+^{-n}$. Show that $M(\alpha, \beta, \mathbf{h})$ is a translation surface. Draw the rectangles corresponding rectangles and identifications between them, and compare your results with Figure 1.21 in Chapter 1.

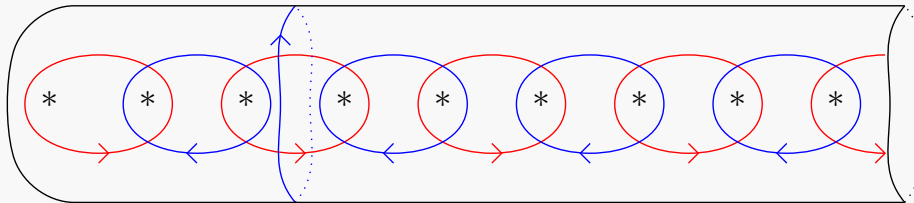
Exercise 3.4.19

Consider the oriented multicurves α and β on the Loch Ness monster shown in the Figure below. Show that the configuration graph $\mathcal{G}(\alpha \cup \beta)$ is isomorphic to the \mathbb{Z} -graph. Construct the quadratic differential $M(\alpha, \beta, \mathbf{h})$ following the steps explained in Example 3.4.16 for the 2-harmonic function $\mathbf{h}(n) = 1$, for all $n \in \mathbb{Z}$. Show that $M(\alpha, \beta, \mathbf{h})$ is not isomorphic to the infinite-staircase and explain what went wrong. Is $M(\alpha, \beta, \mathbf{h})$ homeomorphic to the Loch Ness monster?



Exercise 3.4.20

Consider the oriented multicurves $\alpha = \{\alpha_i\}_{i \geq 0}$ and $\beta = \{\beta_j\}_{j \geq 0}$ on the surface S depicted in the figure below. Let \mathbf{h} be any positive λ -harmonic function of $\mathcal{G}(\alpha \cup \beta)$. Show that the quadratic differential $M(\alpha, \beta, \mathbf{h})$ is not the square of an Abelian differential, *i.e.* there are changes of coordinates in the corresponding atlas that are half-translations.



Constructing Hooper-Thurston-Veech surfaces from bipartite graphs

In the preceding section, we have shown that a Hooper-Thurston-Veech surface is encoded by a pair of multicurves α and β (the core curves of the horizontal and vertical cylinders) and a λ -harmonic function $\mathbf{h} : \mathcal{G}(\alpha \cup \beta) \rightarrow \mathbb{R}_{>0}$ of the configuration graph (which encodes the heights of cylinders). As explained in Example 3.4.16, an orientation of the components of α and β defines a pair of permutations r, u on the edges of $\mathcal{G}(\alpha \cup \beta)$ (see (3.24)) which encode how cylinders intersect. In this section we explain how to construct a Hooper-Thurston-Veech from an abstract bipartite graph of finite degree, a pair of permutations and a λ -harmonic function. This is the way the construction is presented in [Hoo15].

Definition 3.4.21. Let \mathcal{G} be a bipartite graph with vertices $I \sqcup J$. A *ribbon structure* on \mathcal{G} is a choice for each vertex v of a cyclic ordering to the edges adjacent to v . In other words, it is a pair of permutations $r : E(\mathcal{G}) \rightarrow E(\mathcal{G})$ and $u : E(\mathcal{G}) \rightarrow E(\mathcal{G})$ such that for each edge adjacent to $i \in I$ (respectively $j \in J$) the orbit of i by r (resp. of j by u) are the edges adjacent to i (resp. j).

Theorem 3.4.22: [Hoo15]

Let \mathcal{G} be a connected bipartite graph of finite degree endowed with a ribbon structure. Then for every positive λ -harmonic function \mathbf{h} of \mathcal{G} there is an Abelian differential $M = M(\mathcal{G}, \mathbf{h})$ which

is a Hooper-Thurston-Veech surface of moduli $\frac{1}{\lambda}$. Let $H = \{H_i\}_{i \in I}$ and $V = \{V_j\}_{j \in J}$ be the corresponding horizontal and vertical cylinder decompositions; and α and β be the multicurves given by their core curves. Then \mathcal{G} is the configuration graph $\mathcal{G}(\alpha \cup \beta)$.

Proof. We sketch the main ingredients of the proof. Let $E(\mathcal{G}) = I \sqcup J$ be the set of vertices of the bipartite graph \mathcal{G} . We have natural projections:

$$p_I : E(\mathcal{G}) \rightarrow I \quad \text{and} \quad p_J : E(\mathcal{G}) \rightarrow J$$

which assign to an edge $e = ij$ the vertices $p_I(e) = i$ and $p_J(e) = j$. Consider a λ -harmonic function \mathbf{h} of \mathcal{G} and for each $e \in E(\mathcal{G})$ define $R_e = [0, \mathbf{h} \circ p_J(e)] \times [0, \mathbf{h} \circ p_I(e)]$. The ribbon structure of \mathcal{G} defines a pair of permutations

$$r : E(\mathcal{G}) \rightarrow E(\mathcal{G}) \quad \text{and} \quad u : E(\mathcal{G}) \rightarrow E(\mathcal{G})$$

as follows: $r(e) = S_{I(e)}(e)$ and $u(e) = S_{J(e)}(e)$. The r -orbit (respectively u -orbit) of e is formed by the all edges adjacent to $I(e)$ (respect. $J(e)$). To obtained the desired translation surface glue using a translation the right side of the Euclidean rectangle R_e to the left side of $R_{r(e)}$, and the upper side of R_e to the bottom side of $R_{u(e)}$. The rest of the proof follows from arguments analogous to the ones used in the proof of Theorem 3.4.14. \square

Remark 3.4.23. Note that the steps of the construction of $M(\mathcal{G}, \mathbf{h})$ are very similar to those explained in Example 3.4.16, except that we do not have to care about orientations for there is no surface involved in the input data. On the other hand, if we consider α and β the multicurves formed by the core curves of the cylinder decompositions H and V of $M(\mathcal{G}, \mathbf{h})$ respectively, then $M(\mathcal{G}, \mathbf{h})$ is equal to $M(\alpha, \beta, \mathbf{h})$, the Hooper-Thurston-Veech surface given by Theorem 3.4.14.

Exercise 3.4.24

Let \mathbf{h} be a λ -harmonic function used to construct a Hooper-Thurston-Veech surface M of the form $M(\alpha, \beta, \mathbf{h})$ or $M(\mathcal{G}, \mathbf{h})$. Show that the area of M is given by

$$\text{Area}(M) = \frac{\lambda}{2} \sum_{v \in V(\mathcal{G})} \mathbf{h}_v^2.$$

Deduce that \mathbf{h} is a sequence in $\ell^2(V)$ if and only the surface $M(\mathcal{G}, \mathbf{h})$ has finite area.

Example 3.4.25. *The \mathbb{N} -graph.* Let \mathcal{G} be the graph depicted in Figure 3.11. Henceforth we call \mathcal{G} the \mathbb{N} -graph and as indicated in the Figure vertices are indexed by the non-negative integers. Given that every vertex except one has degree 2, there is a unique ribbon structure on this graph. As in Example 3.4.16, a λ -harmonic function \mathbf{h} on \mathcal{G} satisfies the (linear recurrence) relation $\mathbf{h}(n) = \lambda \mathbf{h}(n-1) - \mathbf{h}(n-2)$, for $n \geq 2$ with initial condition $\lambda \mathbf{h}(0) = \mathbf{h}(1)$. The corresponding characteristic polynomial is $x^2 - \lambda x + 1$. For $\lambda = 2$ there is only one root and $\mathbf{h}(n) = A + Bn$, $A, B \in \mathbb{R}$ is a general solution.

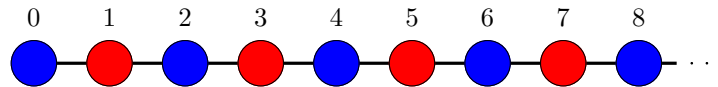


Figure 3.11: The \mathbb{N} -graph.

Let's now consider the case of simple roots $r_+ > r_-$ corresponding to $\lambda > 2$. The general solution in this case is of the form $\mathbf{h}(n) = Ar_+^n + Br_-^n$.

Exercise 3.4.26

Draw the infinite area surface $M(\mathcal{G}, \mathbf{h})$ for \mathcal{G} equal to \mathbb{N} -graph and \mathbf{h} is the 2-harmonic function $\mathbf{h}(n) = n + 1$. For $\lambda > 2$ and $\mathbf{h}(0) = r_+ - r_-$ show that $\mathbf{h}(n) = (r_+^{n+1} - r_-^{n+1})$ is a λ -harmonic function and draw the corresponding Hooper-Thurston-Veech surface $M(\mathcal{G}, \mathbf{h})$. You should get in both cases something that resembles a staircase going upwards. One can turn the \mathbb{N} -graph into a metric graph by declaring that each edge has length 1. Consider then the subset of each surface corresponding to the ball of radius n in the \mathbb{N} -graph centered at the origin. Denote this subset

$Y_n(\lambda)$. Compare the growth of the area of $Y_n(\lambda)$ for parameters $\lambda = 2$ and $\lambda > 2$.

Example 3.4.27. *Modified \mathbb{N} -graph.* Let \mathcal{G}_k be the graph obtained by adding $k \geq 1$ vertices of valence 1 to the vertex 0 in the \mathbb{N} -graph, see Figure 3.12. Henceforth we call \mathcal{G}_k the modified \mathbb{N} -graph. Label by $\{(-1, 1), \dots, (-1, k)\}$ the k vertices of valence 1 adjacent to 0 the only vertex of valence $n + 1$. Any λ -harmonic function \mathbf{h} satisfies $\lambda \mathbf{h}((-1, j)) = \mathbf{h}(0)$ for all $j = 1, \dots, k$ and $\mathbf{h}(1) = \mathbf{h}(0)(\lambda - \frac{k}{\lambda})$. For every $n \geq 1$ we have the same linear recurrence relations as in Example 3.4.25, namely $\mathbf{h}(n) = \lambda \mathbf{h}(n - 1) - \mathbf{h}(n - 2)$. Its characteristic polynomial is $x^2 - \lambda x + 1$. There are hence two cases to consider:

1. $\lambda = 2$. Here the general solution of the linear recurrence relation that a 2-harmonic function must satisfy is of the form $\mathbf{h}(n) = A + Bn$. By evaluating in $n = 0$ and $n = 1$, and substituting the value for $\mathbf{h}(1)$ calculated above we obtain that $\mathbf{h}(n) = \mathbf{h}(0) + (\lambda \mathbf{h}(0) - \frac{k}{\lambda})n$. The only possible values that yield a positive 2-harmonic function are $k = 1$ (which is just the case of the \mathbb{N} -graph treated before) and $k = 2$.

Exercise 3.4.28

Show that for $k = 2$ the corresponding Hooper-Thurston-Veech surface is tiled by infinitely many unit squares but is not isometric to the infinite staircase.

2. $\lambda > 2$. In this case the general solution of the linear recurrence relation that \mathbf{h} satisfies must satisfy of the form

$$\mathbf{h}(n) = Ar_+^n + Br_-^n \tag{3.25}$$

(where $r_+ > r_-$ are the roots of the characteristic polynomial). Therefore:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{\mathbf{h}(0)}{r_+ - r_-} \begin{pmatrix} -r_- & 1 \\ r_+ & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda - \frac{k}{\lambda} \end{pmatrix}$$

Hence:

$$\mathbf{h}(n) = \frac{\mathbf{h}(0)}{r_+ - r_-} \left((\lambda - \frac{k}{\lambda} - r_-)r_+^n - (r_+ - \lambda + \frac{k}{\lambda})r_-^n \right) \tag{3.26}$$

Exercise 3.4.29

Show that for $k \geq 2$ fixed, the condition $(\lambda - \frac{k}{\lambda})r_+ > 1$ is sufficient to guarantee that we have a positive eigenvector for the adjacency operator. *hint: the sequence $\mu_n := \frac{r_+^n - r_-^n}{r_+^{n-1} - r_-^{n-1}}$, $n \geq 2$ is decreasing and $\lim_{n \rightarrow \infty} \mu_n = r_+$.*

For fixed k find λ such that $A = 0$. Use (3.25) to obtain $\mathbf{h}(n)$ and use Exercise 3.4.24 to prove that in this case the corresponding Hooper-Thurston-Veech surface has finite area.

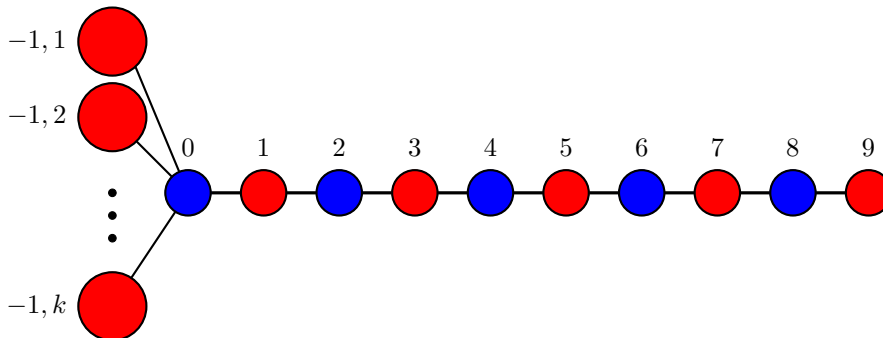


Figure 3.12: Modified \mathbb{N} -graph \mathcal{G}_k .

3.5 The Veech group of baker's surfaces $B_{\frac{1}{q}}$.

In this section we revisit baker's surfaces B_α , with $\alpha > 0$. These finite area infinite-type surfaces were introduced in Example 1.2.2 from Chapter 1. For parameters $\alpha = \frac{1}{q}$, $q \geq 2$ we show that B_α is a Hooper-Thurston-Veech surface and we describe its Veech group explicitly.

Theorem 3.5.1

For $\alpha = \frac{1}{q}$, $q \geq 2$, the baker's surface $B_{\frac{1}{q}}$ is affinely equivalent a Hooper-Thurston-Veech surface of moduli $\frac{\sqrt{q}}{q+1}$. That is, there exists a matrix $N \in \text{SL}(2, \mathbb{R})$ such that ${}^a N \cdot B_{\frac{1}{q}}$ is a Hooper-Thurston-Veech surface.

^aHere $N \cdot$ denote the natural action of $\text{GL}(2, \mathbb{R})$ on translation surfaces.

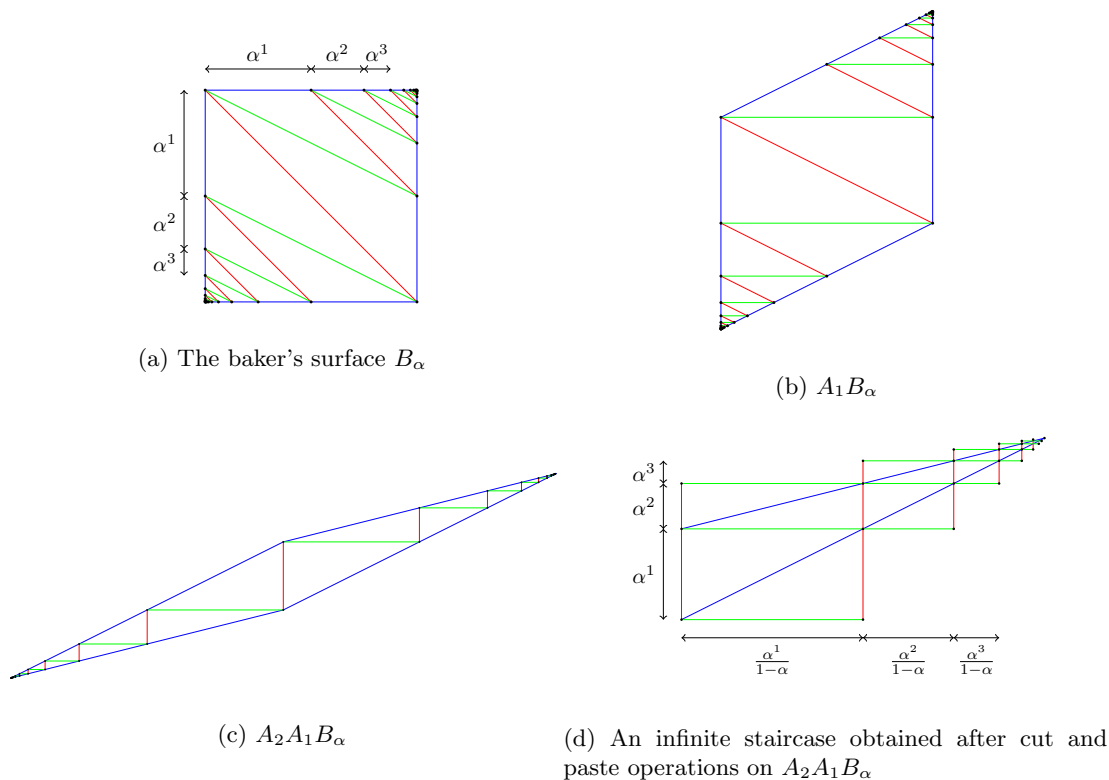


Figure 3.13: The baker's surface as an infinite staircase: an example of the Hooper-Thurston-Veech construction.

Proof. On the top left part of Figure 3.13 we depict two families of parallel saddle connections that bound two transverse cylinder decompositions of baker's surface B_α . The first step to see that B_α can be obtained by a Hooper-Thurston-Veech construction is to rend cylinders in these two decompositions vertical and horizontal respectively. This is achieved by applying first an affine map with derivative $A_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and then another with derivative $A_2 = \begin{pmatrix} 1 & \frac{1}{1-\alpha} \\ 0 & 1 \end{pmatrix}$. The result is the image depicted at the bottom left part of figure 3.13. By cutting and pasting we can present the obtained surface as an infinite (half) staircase going upwards, as shown in the bottom right part of figure 3.13. Let us denote this (half) infinite staircase B'_α . From the figure it is clear that the horizontal and vertical translation flow decompose B'_α into two infinite families of cylinders that we denote $H = \{H_0, H_1, \dots\}$ and $V = \{V_0, V_1, \dots\}$ respectively. Here the enumeration of horizontal cylinders goes from bottom to top and for vertical cylinders from left to right. Let us denote by $\mu(-) = \frac{\text{height}(-)}{\text{circumference}(-)}$ the function associating to a cylinder its moduli. A direct

calculation shows that:

$$\mu(H_0) = 1 - \alpha, \quad \mu(H_j) = \frac{\alpha(1 - \alpha)}{1 + \alpha} \text{ for } j \geq 1 \quad \text{and} \quad \mu(V_k) = \frac{1}{(1 + \alpha)(1 - \alpha)} \text{ for } k \geq 0.$$

In other words, $A_2A_1B_\alpha$ is not a Thurston-Veech surface for the moduli of the horizontal and vertical cylinders is not the same. To fix this issue one can apply an affine map with hyperbolic derivative of the form $A_3(X) = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix}$. This hyperbolic affine map changes moduli of horizontal and vertical cylinders by a factor of X^{-2} and X^2 respectively. A direct calculation shows that if we pick $A_3(\sqrt[4]{\alpha(1 - \alpha)^2})$, then $\mu(A_3H_j) = \mu(A_3V_k) = \frac{\sqrt{\alpha}}{1 + \alpha}$ for all $j \geq 1, k \geq 0$ but $\mu(A_3H_0) = \frac{1}{\sqrt{\alpha}}$. To fix this last issue we can artificially subdivide the horizontal cylinder H_0 into q horizontal cylinders $H_{0,i}, i = 1, \dots, q$ of equal height and moduli $\mu(H_{0,i}) = \frac{1}{q\sqrt{\alpha}}$. Hence, $A_3A_2A_1B_\alpha$ after this subdivision of A_3H_0 is a Hooper-Thurston-Veech surface if and only if $\alpha = \frac{1}{q-1}$. \square

Exercise 3.5.2

From figure 3.13 deduce that baker's surface B_α for $\alpha = \frac{1}{q}$ is a Hooper-Thurston-Veech surface obtained by a construction on the modified \mathbb{N} -graph \mathcal{G}_{q+1} as explained in example 3.4.27. Moreover, show that in this case the eigenvalue of the adjacency operator is $\lambda = \frac{\sqrt{q}}{q+1}$.

Exercise 3.5.3

Prove that the surface B_α for $\alpha \in \mathbb{Q}$ is given by a Hooper-Thurston-Veech construction, up to application of an affine transformation and a subdivision of its horizontal and vertical cylinders.

We now determine the Veech groups of baker's surfaces for parameters $\alpha = \frac{1}{q}, q \geq 2$. Let us recall that from (3.20) that for every $\lambda \in \mathbb{R}, G_\lambda$ denotes the subgroup of $SL(2, \mathbb{R})$ generated by:

$$h_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad v_\lambda = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

Theorem 3.5.4

The Veech group of baker's surface B_α , for $\alpha = \frac{1}{q}, q \geq 2$, is conjugated in $SL(2, \mathbb{R})$ to the group generated by $-\text{Id}$ and G_λ , where $\lambda = \frac{q+1}{\sqrt{q}}$.

Remark 3.5.5. Theorem 3.5.4 was originally stated by Chamanara in [Cha04]. However there is an issue with the proof Chamanara presents: he claimed that for any translation surface of finite area the translation flow on the invariant direction of a parabolic affine automorphism always defines a cylinder decomposition of the surface. The proof of this (Proposition 2 in [Cha04]) assumes that a generalized version of Keane's theorem holds (Proposition 1 in [Cha04]). We will provide a counterexample to this assumption in Section 6.3.2. Herrlich and Randecker have also computed the Veech group of $B_{\frac{1}{2}}$, see [HR].

proof of Theorem 3.5.4. From Theorem 3.5.1 we know that Baker's surface $B_{\frac{1}{q}}$ is, up to composition with an affine map, a Hooper-Thurston-Veech surface. Indeed, if we apply to $B_{\frac{1}{q}}$ the matrix

$$A = \begin{pmatrix} \sqrt[4]{\alpha(1 - \alpha)^2} & 0 \\ 0 & \frac{1}{\sqrt[4]{\alpha(1 - \alpha)^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{1 - \alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

where $\alpha = \frac{1}{q}$ using the $SL(2, \mathbb{R})$ -action on translation surfaces, the result is a Hooper-Thurston-Veech surface $M(\mathcal{G}_{q+1}, h)$, where \mathcal{G}_{q+1} is a modified \mathbb{N} -graph and $\lambda = \frac{q+1}{\sqrt{q}}$. This surface can be depicted as a (half) staircase and we denote it $M_{\frac{1}{q}}$, see figure 3.13. Given that $B_{\frac{1}{q}}$ and $M_{\frac{1}{q}}$ are affinely equivalent, their Veech groups are conjugated. More precisely $\Gamma(M_{\frac{1}{q}}) = A\Gamma(B_{\frac{1}{q}})A^{-1}$.

Rotational components were introduced in Section 2.3.2 in Chapter 2 in order to describe wild singularities of translation surfaces. Moreover, we started a discussion on rotational components of baker's surfaces in Example 2.3.22 in Chapter 2. In the following lemma we continue this discussion by describing the set of rotational components of $B_{\frac{1}{q}}$.

We define $G'_\lambda := A^{-1}G_\lambda A < \Gamma(B_{\frac{1}{q}})$, $h' = \begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix}$ and H as the quotient of the group generated by $\{G'_\lambda, -Id\}$ modulo the subgroup generated by the hyperbolic element h' .

Lemma 3.5.6. *The rotational components of $B_{\frac{1}{q}}$ are:*

- two bi-infinite rotational⁵ components isometric to \mathbb{R} .
- a countable collection $\{R_i\}$ of open rotational components⁶ of finite total angle given by the H -orbit of a rotational component of length $\frac{\pi}{2}$.

Moreover, for every direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ there is at most one rotational component of finite total angle containing a linear approach parallel to θ .

Proof. The projection $\mathbb{R}^2 \setminus \{y = 0\} \rightarrow \mathbb{P}_{\mathbb{R}}^1$ that associates to each non-horizontal line through the origin its co-slope is given by $(x, y) \rightarrow x/y$. The linear action of $SL(2, \mathbb{R})$ on co-slopes corresponding to lines through the origin intersecting $\{y = 1\}$ is given by its action on \mathbb{R} by fractional linear transformations $x \rightarrow \frac{ax+b}{cx+d}$.

Two bi-infinite rotational components are depicted in red and blue in figure 1.11, section 1.2.2. These are the only rotational components of infinite total angle and that they are isometric to \mathbb{R} . In $B_{\frac{1}{q}}$ there exist two rotational components R'_0 and $-R'_0$ of total angle $\frac{\pi}{2}$: they are defined by segments approaching the vertices **b** and **d** in the figure respectively. Moreover, the rotational components R'_0 and $-R'_0$ are fixed by a (hyperbolic) affine automorphism with derivative $h' = \begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix}$. Let us denote by R_0 the finite angle rotational component in $M_{\frac{1}{q}}$ corresponding to R'_0 . The projection of the set of directions of linear approaches in R_0 via $\mathbb{R}^2 \setminus \{y = 0\} \rightarrow \mathbb{P}_{\mathbb{R}}^1$ is the interval $I_0 = (r_2, r_1)$, where $r_2 = \frac{1}{\sqrt{q}}$ and $r_1 = \sqrt{q}$. Moreover, the open interval I_0 is the $h = Ah'A^{-1}$ orbit of the interval $]\frac{2}{\lambda}, \frac{2}{\lambda} + \frac{\lambda}{2}]$, where $\lambda = \frac{q+1}{\sqrt{q}}$, see figure 3.14. From Appendix B.2 on the limit set of G_λ , we know that the G_λ -orbit of the interval I_0 is equal to $\mathbb{R} \setminus \Lambda(G_\lambda)$, where $\Lambda(G_\lambda)$ denotes the limit set of G_λ . In particular, the complement in \mathbb{R} of the G_λ -orbit of I_0 has empty interior.

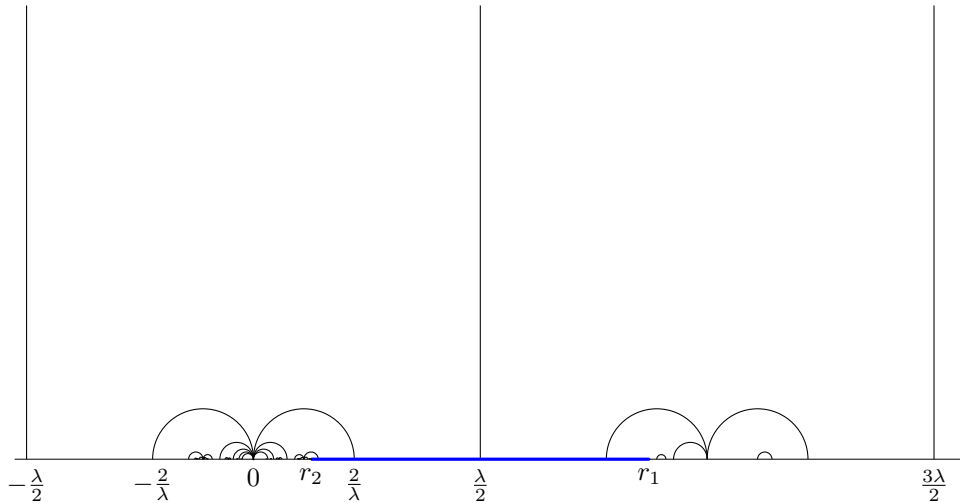


Figure 3.14: Fundamental domain of G_λ .

Now we use the flat geometry of Baker's surface illustrated in Figure 1.11, section 1.2.2. There are no rotational components of total angle equal to zero. This implies that the set of rotational components with finite total angle of $B_{\frac{1}{q}}$ is given by the H -orbit of R'_0 and $-R'_0$. Moreover, from this Figure we

⁵i.e. the map $\widetilde{\text{dir}}$ restricted to each rotational component defines a bijection with \mathbb{R} , see (2.7).

⁶i.e. the map $\widetilde{\text{dir}}$ restricted to each rotational component R_i defines a bijection with an open subinterval of \mathbb{R} , see (2.7).

deduce that linear approaches parallel to a direction in I_0 encounter a unique rotational component of finite total angle. This implies that for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ there is at most one rotational component of finite total angle containing a linear approach parallel to θ . \square

We now finish the proof of Theorem 3.5.4. Let g be an element of the Veech group of $M_{\frac{1}{q}}$ different from $-Id$. By the preceding lemma, G_λ acts transitively on the set of rotational components of finite total angle and hence there exists $g' \in G_\lambda$ such that $g'g$ fixes the interval I_0 , which is the set of directions defining the rotational component of finite total angle fixed by the hyperbolic map $A \begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix} A^{-1}$. Therefore $A^{-1}g'gA$ must be a hyperbolic matrix that fixes the horizontal and vertical direction. Since in $B_{\frac{1}{q}}$ all saddle connections in the vertical and horizontal direction have lengths in the set $\{q^n\}_{n \in \mathbb{Z}}$ we conclude that $A^{-1}g'gA$ must be a power of the hyperbolic matrix $\begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix}$ and hence $g \in G_\lambda$. \square

In general there is no recipe to calculate the Veech group a given translation surface, not even for Hooper-Thurston-Veech surfaces. So far, all Veech groups of infinite-type finite area translation surfaces that have been calculated are not lattices in $SL(2, \mathbb{R})$. We conclude this chapter with the following open problem.

Question 3.5.7

Does there exist a finite area translation surface whose Veech group is a lattice in $SL(2, \mathbb{R})$?

