

The logistic map

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Some general definitions

Let X be a set and f be a function from X to X . Since the domain and the codomain of f are equal, we can *iterate* the function f i.e. the functions $f, f^2 := f \circ f, f^3 := f \circ f \circ f, \dots$ are well defined. For every x_0 in X , we can define the *orbit* of x_0 under f as the sequence x_0, x_1, x_2, \dots where $x_{i+1} := f(x_i) = f^{i+1}(x_0)$.

A point $p \in X$ is said to be a *fixed point* if $f(p) = p$. Such points p have a constant orbit.

A point p in X is said to be *periodic* if there exists a positive integer n such that $f^n(p) = p$. The smallest such n is called the *period* of f .

A family of functions

For every fixed $r \in [0, 4]$, we define the function $g_r : [0, 1] \rightarrow [0, 1]$ by $g_r(x) := rx(1-x)$. Such map is called a *logistic map*.

Prove that for any parameter $r \in [0, 4]$, the function g_r preserves the interval $[0, 1]$ (hence it is well defined, and we can iterate it).

Draw the graph of the map g_r for various values of the parameter r

Hint: Look at the function `plot` (to draw the graph of a function) and `Graphics()` (that creates an empty graphics). To superpose images you need to use `+`.

Drawing orbits

Write a Python function `logistic_orbit(r, x0, itermin, itermax)` that returns the list $[x_{\text{itermin}}, \dots, x_{\text{itermax}-1}]$ of the orbit of x_0 under the map g_r .

Write a Python function `plot_logistic_orbit(r, x0, itermin, itermax)` that draws this chunk of orbit as the function $\{\text{itermin}, \dots, \text{itermax} - 1\} \rightarrow [0, 1], n \mapsto f^n(x_0)$.

Hint: you can have a look at the `point2d` and `line2d` functions.

Write a Python function `logistic_cobweb(r, x0, itermin, itermax)` that draws this chunk of orbit directly on the graph of the map $g_r : [0, 1] \rightarrow [0, 1]$, as a *cobweb plot* (see the left part of the picture below).

Hint: you can have a look at this [wikipedia article](#).

The following interact allows to change the values of the four parameters easily and observe the different behaviours of the orbits.

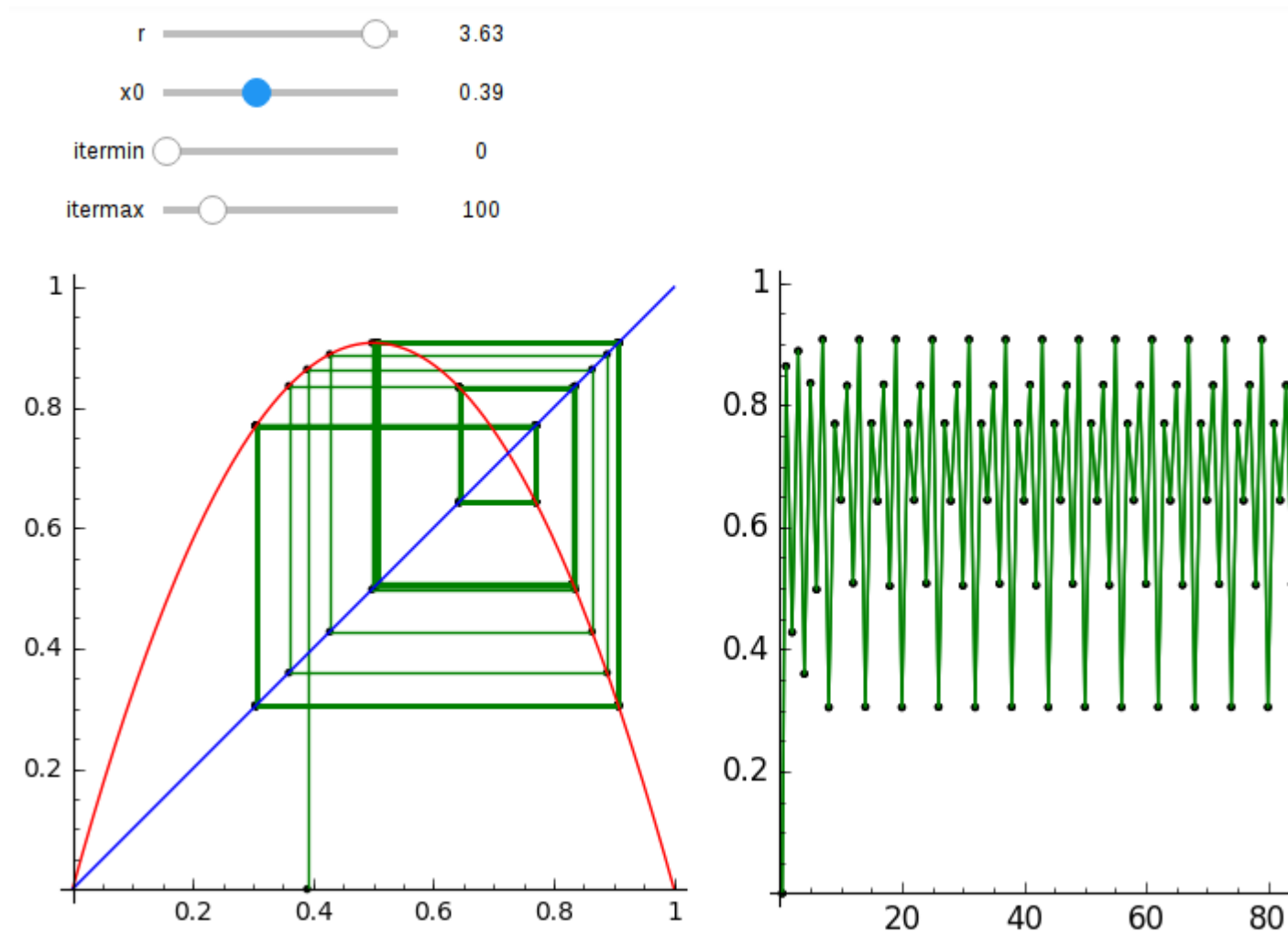
Hint: to align images, you can tune the `aspect_ratio` option of the plotting functions in the previous questions.

```

sage: @interact
....: def _(r = slider(0.0, 4, step_size=0.01, label='r'), x0 = slider(0, 1, step_s
....:     graphics_array(((logistic_cobweb(r, x0, itermin, itermax), plot_logistic_o

```

The interact could look like this:



Here, we can observe the first 100 iterates of the orbit of the point $x_0 = 0.39$ for the map $g_{3.63}$. It is approaching a periodic orbit of period 6.

Attractive fixed points

Prove that g_r has two fixed points, namely 0 and $1 - 1/r$ (if it belongs to $[0, 1]$).

Observe that the graph of g_r intersects the line $y = x$ exactly at the fixed points

A *fixed point* p of g_r is said to be *attractive* if $|g'_r(p)| < 1$ and *repulsive* if $|g'_r(p)| > 1$.

When a fixed point p is attractive, there exists a neighborhood N of p such that the orbit of every x_0 in N converges to p . The set of points x_0 whose orbits converge to p is called the *basin of attraction* of p .

For which values of r is 0 attractive? **For which** values of r is $1 - 1/r$ attractive?

Check this behaviour for $r = 0.6$, $r = 1.8$ and $r = 2.2$ with your interact.

Prove that, when 0 is an attractive fixed point, its basin of attraction is $[0, 1] \setminus \{1 - 1/r\}$.

Prove that, when $1 - 1/r$ is an attractive fixed point, its basin of attraction is $(0, 1)$.

When $1 - 1/r$ is an attractive periodic point, the orbit $g_r^n(1/2)$ converges to $1 - 1/r$. **Find** the speed of convergence

When $1 - 1/r$ is repulsive, **find** the speed at which a point close to 0.5 drifts away from 0.5

A periodic point p is called *super-attractive* if $|g_r'(p)| = 0$. Find the parameter for which $1 - 1/r$ is a super-attractive fixed point.

For this parameter, find the convergence speed

Attractive periodic orbits

This definition of attraction and repulsion extend to periodic points. Let p be a periodic point of g_r with period n . That is $g_r^n(p) = p$ and $g_r^m(p) \neq p$ for $m < n$. In order to see the behavior of orbits in a neighborhood of a fixed point one need to study the derivative of g_r^n which equals

$$(g_r^n)'(x) = g_r'(x)g_r'(g_r(x)) \dots g_r'(g_r^{n-1}(x)).$$

Hence, a periodic orbit $p, g_r(p), \dots, g_r^{n-1}(p)$ of g_r is said to be respectively *attractive*, *super-attractive* or *repulsive* if $|g_r'(p)g_r'(g_r(p)) \dots g_r'(g_r^{n-1}(p))|$ is < 1 , $= 0$ or > 1 .

Check that, for $r = 3.3$, there is no attractive fixed point, but an attractive periodic orbit of period 2.

Compute an exact expression (with radicals) of the two points of this attractive orbit (you can also give their minimal polynomial).

Bifurcation diagram

Up to now we worked with a fixed parameter r and studied the behaviour of the iterations of g_r . We will now try to understand how the dynamics evolve with r . We will hence work in the parameter space $[0, 4]$.

The *bifurcation diagram* B of the family $\{g_r\}$ is the subset of $[0, 4] \times [0, 1]$ of points (r, x) such that x is an accumulation point of the orbit of $1/2$ for g_r . We will denote by B_r the slice at r , that is the set of accumulation point of the orbit of $1/2$ for g_r . We hence have

$$B = \cup_{r \in [0, 4]} \{r\} \times B_r.$$

Compute an approximation of the slices B_r for various values of r

Check that the set of accumulation point of a random orbit is also B_r .

Hint: you can have a look at `RDF.random_element`.

Write a Python function `logistic_bifurcation_diagram(rmin, rmax, rstep)` that returns an approximation of the slice $B \cap [rmin, rmax] \times [0, 1]$ of the bifurcation diagram, where two consecutive values of r are at distance `rstep`.

Hint: you can have a look at `srange`.

Draw the complete bifurcation diagram

Observe, when r increases, the evolution of attractive periodic orbits of period 1, 2, 4, 8, ...

Islands of stability

Prove that for all $r \in [3, 4]$ the map g_r has a (unique) periodic point of period 2

Prove that the segment $[3, 1 + \sqrt{6}]$ corresponds to the regime where this orbit of period 2 is attractive

Prove that for $r = 1 + \sqrt{5}$ the critical point $1/2$ is part of the periodic orbit. In other words, the orbit of period 2 is super-attractive. We say that $1 + \sqrt{5}$ is the *center of the island of stability* $[3, 1 + \sqrt{6}]$

Draw the bifurcation diagram with vertical lines at the parameters 3 and $1 + \sqrt{5}$ and $1 + \sqrt{6}$.

Find the upper bound of the island of stability corresponding to the periodic orbit of period 4 starting from $1 + \sqrt{5}$ as well as its center

Draw the bifurcation diagram together with vertical lines delimiting the islands of stability for period 2 and 4 as well as their centers

Could you compute the center for the next bifurcations with period 8, 16,, ...

The behavior of the sequence of islands of stability for this sequence of period doubling was intensively studied on computers in the 70's by Feigenbaum. Let us denote by c_n the center of the island of stability of period 2^n . He observed the existence of a constant δ so that as $n \rightarrow \infty$ we have a convergence

$$\frac{c_{n+1} - c_n}{c_n - c_{n-1}} \rightarrow \delta$$

where $\delta \simeq 4.6692\dots$ is called the *Feigenbaum constant*. This was later proven by Lanford (1982), Eckmann-Wittwer (1987) and generalized by Lyubitch (1999).

Could you compute a better approximation of the Feigenbaum constant?

Prove that if for some parameter r there is an attractive orbit, then for an interval around r there is an attractive orbit with the same period and a center with a super attractive orbit.

Show that there is a unique island of stability with period 3

Hint: solve the equation satisfied by the center

Could you find an explicit rational number r that belongs to this island?

Chaotic parameters

A famous result of Jakobson (1981) claims that in the space of parameters $[0, 4]$ if we remove all islands of stability, there remain a set of positive Lebesgue measure with interesting dynamics. More precisely, there are maps g_r with invariant measures absolutely continuous with respect to Lebesgue.

Plot an histogram of the orbit of a random point for g_4

Could you find an explicit formula for the shape that you see?

Prove that if $r_3 \simeq 3.6785$ is twice the maximal real root of the polynomial $x^3 - x^2 - x - 1$, then $g_{r_3}^3(1/2) = 1 - 1/r$.

Plot an histogram of the orbit of a random point, with 300 bins, and 100000 iterates.

We say that a parameter r is *post-critically finite* if there exists $m < n$ so that $g_r^n(1/2) = g_r^m(1/2)$ (in other words, the orbit of the critical point $1/2$ terminates into a periodic orbit). We will consider the simple case where $n = m + 1$ that is when the critical point lands into the fixed point. The example r_3 above is a particular case of this situation with $m = 3$.

Solve the equation $g_r^n(1/2) = 1 - 1/r$ in r for $n = 2, 3, 4, 5, 6$.

Draw for each of the parameters found in the above question, the histogram of the orbit of $1/2$

Observe that the peaks of the density can be seen on the bifurcation diagram.

Symbolic coding

Given a map g_r with $r \in [0, 4]$ we can give a *coding* to the orbits. More precisely, given a point $x \in [0, 1]$, we associate the sequence $\pi(x) = (w_0, w_1, w_2, \dots) \in \{L, R, C\}^{\mathbb{N}}$, where $w_i = L$ if $g_r^i(x) \in [0, 1/2)$, $w_i = R$ if $g_r^i(x) \in (1/2, 1]$, and $w_i = C$ if $g_r^i(x) = 1/2$.

Note that $\pi \circ g_r = S \circ \pi$, where $S: \{L, R, C\}^{\mathbb{N}} \rightarrow \{L, R, C\}^{\mathbb{N}}$ is the *shift map* $(w_0, w_1, w_2, \dots) \mapsto (w_1, w_2, \dots)$.

This coding provides a dictionary between dynamical properties of orbits of g_r and combinatorial properties of the produced words. For example, the coding of a periodic orbit is an infinite periodic word.

Write a Python function `logistic_coding(r, x0, itermin, itermax)` that returns the sequence $(w_{itermin}, w_{itermin+1}, \dots, w_{itermax-1})$.

Recall that on the left of the bifurcation diagram we have a sequence of islands of stability whose associated maps present an attractive orbit of period 2^n . Moreover, each of this island has an associated center.

For $n = 1, 2, 3, 4$ determine how the coding of the periodic orbit vary as we move inside the islands

Could you find an explicit construction for the coding of all the period doubling sequence?

We now study the parameter $r = 4$. What is the coding of the orbit of the critical point $x = 1/2$ under g_4 ?

Prove that any sequence of L and R is the coding of a unique element $x \in [0, 1]$

Write a function `logistic_coding_to_point(seq)` that given a sequence `seq` of L and R returns the unique point x so that its associated sequence is the period word `seq seq seq ...`