

Upward-closed objectives are exactly the positive boolean combinations of Büchi objectives

Aymeric Vincent

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In this small note, we show that upward-closed objectives defined in [1] are exactly the positive boolean combinations of Büchi objectives. We follow their notations.

Upward-closed objectives are closed under disjunction and conjunction

By definition, this is easy to see: let Φ and Φ' be two upward-closed objectives.

- Let $w \in \Phi \cap \Phi'$, and $v \in C^\omega$ such that $\text{Inf}(w) \subseteq \text{Inf}(v)$.
Because Φ is upward-closed, we have $v \in \Phi$, and the same for Φ' . Thus $v \in \Phi \cap \Phi'$.
- Let $w \in \Phi \cup \Phi'$, and $v \in C^\omega$ such that $\text{Inf}(w) \subseteq \text{Inf}(v)$.
 - If $w \in \Phi$, because Φ is upward-closed, $v \in \Phi$
 - If $w \in \Phi'$, because Φ' is upward-closed, $v \in \Phi'$

Thus, in either case (or both), $v \in \Phi \cup \Phi'$.

Combined with the fact that Büchi objectives are upward-closed, any positive boolean combination of Büchi objectives is upward-closed.

Any upward-closed objective is a positive boolean combination of Büchi objectives

Let Φ be an upward-closed objective. We consider the finite lattice of subsets of colors which we note $\mathcal{P}(C)$. Each element X in the lattice corresponds to a class of words $\{v \mid \text{Inf}(v) = X\}$.

Now, we consider the image Ψ of Φ by Inf : let $\Psi = \{X \in \mathcal{P}(C) \mid \exists u \in \Phi, X = \text{Inf}(u)\}$.

We note $\min(\Psi)$ the set of minimal elements of Ψ , and for any element X of the lattice, we note $\uparrow X$ the set of supersets of X (the set of elements which are greater in the lattice).

1. Ψ is uniquely determined by its minimal elements, and moreover $\Psi = \bigcup_{M \in \min(\Psi)} \uparrow M$
 - $\Psi \subseteq \bigcup_{M \in \min(\Psi)} \uparrow M$
Let $X \in \Psi$. X belongs to a set $\uparrow M$ with $M \in \min(\Psi)$, otherwise, X is an element of Ψ which is smaller than any minimal element. (!)
 - $\bigcup_{M \in \min(\Psi)} \uparrow M \subseteq \Psi$
Let $X \in \uparrow M$ such that $M \in \min(\Psi)$. Because M is in Ψ and Ψ is upward-closed, X belongs to Ψ .
2. For every set $M \in \min(\Psi)$, $\uparrow M = \bigcap_{m \in M} \uparrow \{m\}$.
 - $\uparrow M \subseteq \bigcap_{m \in M} \uparrow \{m\}$
Let $X \in \uparrow M$ and $m \in M$. It follows that $m \in X$, and thus $X \in \uparrow \{m\}$.
 - $\bigcap_{m \in M} \uparrow \{m\} \subseteq \uparrow M$
Let $X \in \bigcap_{m \in M} \uparrow \{m\}$. For every $m \in M$, $m \in X$, i.e. $M \subseteq X$, and hence $X \in \uparrow M$.

The sets $\{x\}$, for a given colour x represent exactly the images by Inf of Büchi objectives. (1) and (2) say that upward-closed objectives can be represented as disjunctions of conjunctions of simple Büchi objectives.

References

- [1] Krishnendu Chatterjee, Luca de Alfaro, and Thomas A Henzinger. Trading memory for randomness. In *QEST: International Conference on Quantitative Evaluation of Systems*, pages 206–217. IEEE Computer Society, 2004.