Algebraic Recognizability of Languages

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LaBRI, Université Bordeaux-1 and CNRS

MFCS Conference, Prague, August 2004

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- These are sets of discrete structures, arising from the modeling of computations, data structures,...
- Since the origins of TCS, words = finite sequences of letters
- Need to discuss more models: trees, traces, pomsets, graphs

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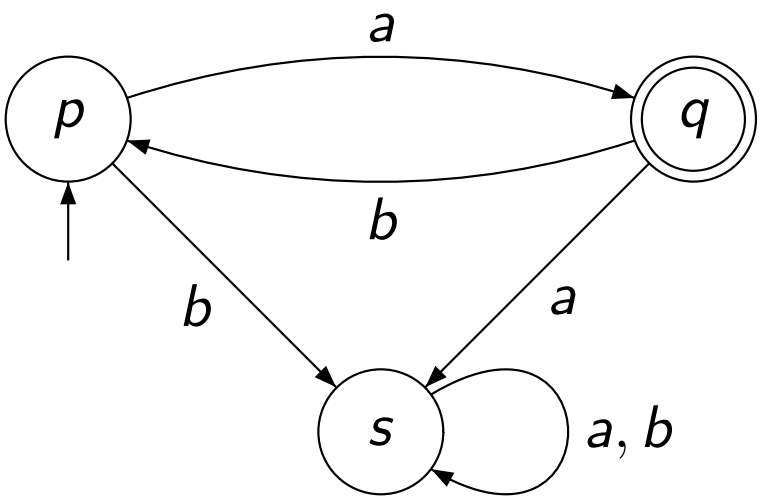

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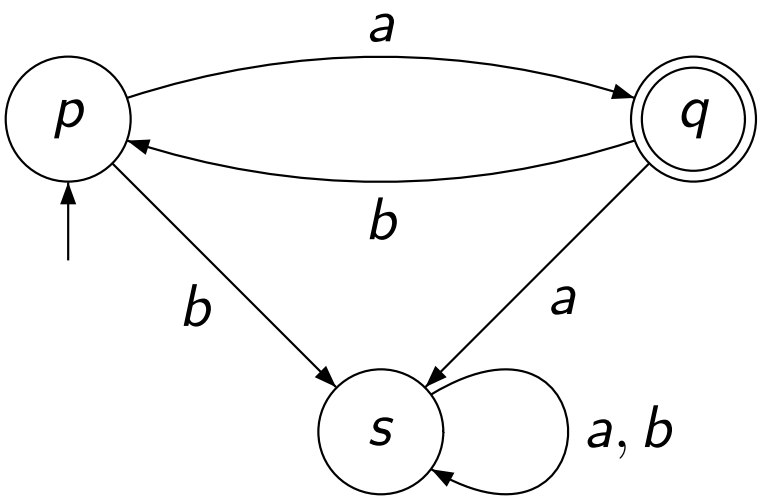

$(ab)^*a$

$\forall x\ \forall y$
$\neg(\exists z\ x < z < y) \Rightarrow (R_a x \Leftrightarrow R_b y)$
$\wedge\ R_a\mathrm{min}\ \wedge\ R_a\mathrm{max}$

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- analyzing the word case
- discussing what we can expect in other settings
- and reviewing a number of situations where interesting results have emerged

Syntactic monoid vs. minimal automaton

L, recognizable language: 2 *computable* invariants

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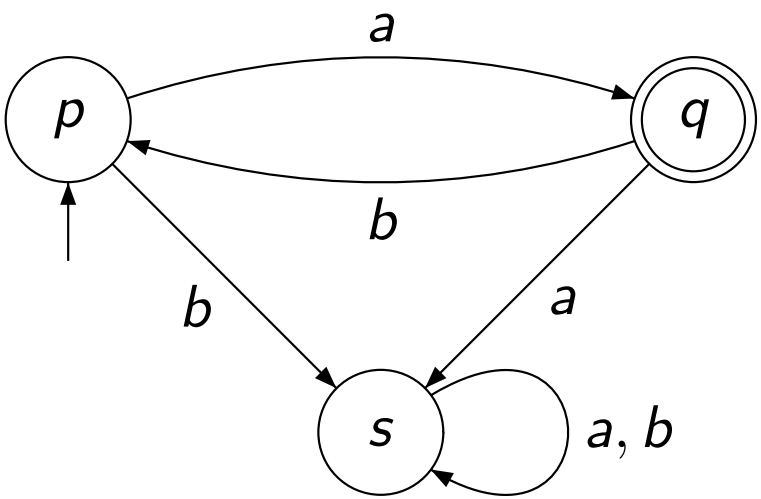

	p	q	s	
a	q	s	s	
b	s	p	s	
a^2	s	s	s	
ab	p	s	s	
ba	s	q	s	
b^2	s	s	s	$b^2 = a^2 = 0$
aba	q	s	s	$aba = a,\ bab = b$

$$S((ab)^*a) = \{a, b, ab, ba, 0\}$$

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Classification of recognizable languages: characterize and decide significant classes

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essentially the same property; provides decidability

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Extending FO with modulo quantifiers

FO+MOD: allow quantification $\exists x\ \varphi(x)$ and $\exists^{\bmod q} x\ \varphi(x)$, *i.e.*, $\mathrm{card}\{x \mid \varphi(x) \text{ holds}\}$ is a multiple of q.

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Σ_2: $\exists x_1 \cdots \exists x_n \, \forall y_1 \cdots \forall y_m \, \varphi$ where φ is quantifier-free
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provides decidability

Dot-depth hierarchy and quantifier alternation

Σ_n, Π_n and $\mathrm{Bool}(\Sigma_n)$ are also characterized algebraically and in terms of the complexity of a star-free expression (dot-depth),

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... but for $n \geq 2$ or 3, we do not know any decision algorithm

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that is, if $e = e^2$, then $(ese)^2 = ese$ and $esete = etese$: a decidable property

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- but deciding FO is PSPACE-complete (Cho, Huynh)

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MSO-formulas, interpreted on the nodes, using the parent-child predicate, the order between siblings, and the Σ-labeling

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consider morphisms from T_Σ into finite Σ-algebras, or equivalently, finite-index congruences on T_Σ

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- *L is equational*

i.e., L is a component of a solution of a system of equations in the Σ-algebra T_Σ (can be expressed also in terms of context-free grammars)

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- deterministic bottom-up tree automata provide an efficient implementation

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Disappointing points:

- deterministic bottom-up tree automata and finite Σ-algebras are essentially the same thing
- no classification results: in particular, we do not know how to characterize or decide FO-definable tree languages

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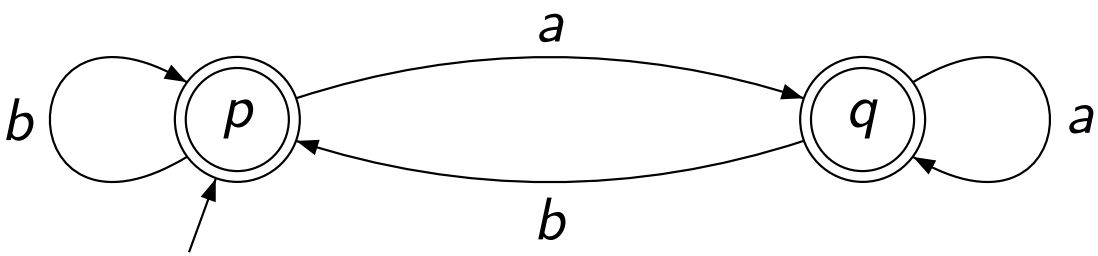

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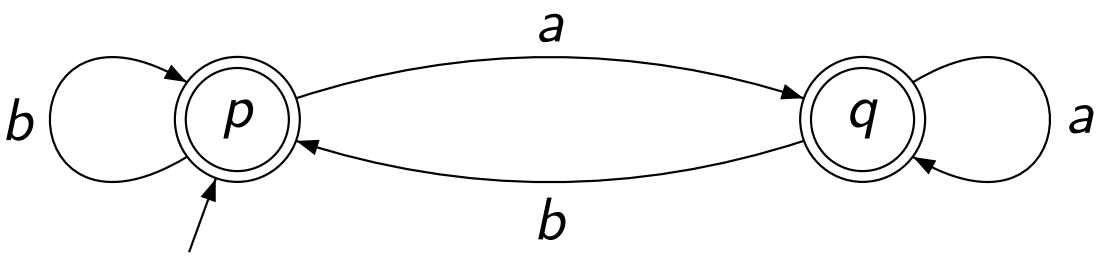
Büchi automata, Muller automata, Rabin automata, are equivalent to MSO-definability, and to ω-regular expressions, *i.e.*, finite unions of KL^{ω} (K, L recog. languages of finite words)



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- an Eilenberg-style variety theorem holds, with many interesting classes characterized (Perrin, Pin, Carton)

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- MSO formulas are clearly defined once the structures are decided upon
- but algebraic recognizability depends on the choice of an algebraic structure

Usually, the structures are given, and we need to invent the abstract algebraic structure: monoids for words, Σ-algebras for Σ-trees, ω-semigroups for A^∞, etc

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On the other hand, if we add the unary shift operation $a_1 \cdots a_{n-1} a_n \longmapsto a_n a_1 \cdots a_{n-1}$, then $a^* b$ is not recognizable

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Too few operations give too many recognizable languages: if there are no operations, then every partition is a congruence, every subset is recognizable

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Converse:
if we have a *MS-definable parsing* mapping, parse: $S \to$ Terms such that val(parse(x)) $= x$, then recognizability implies CMSO-definability

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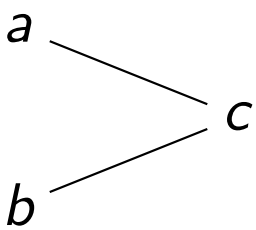

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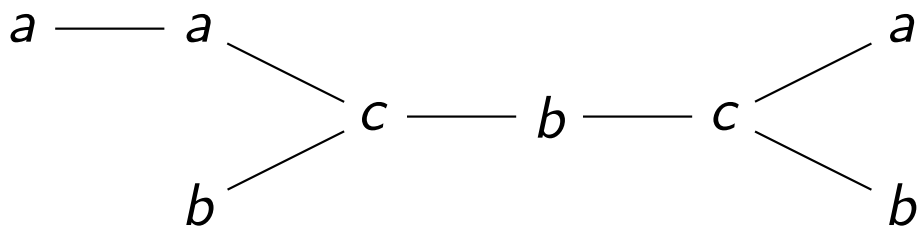

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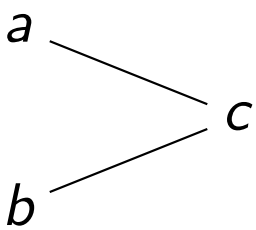

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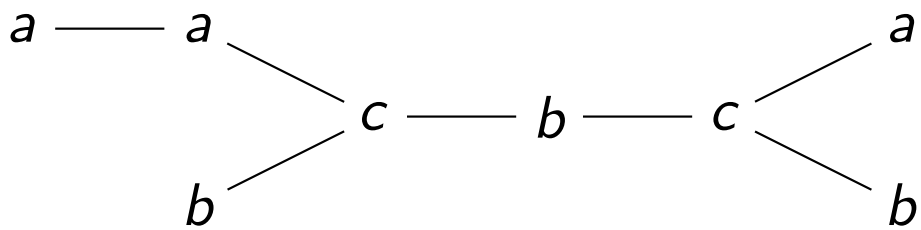

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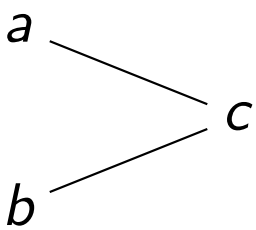

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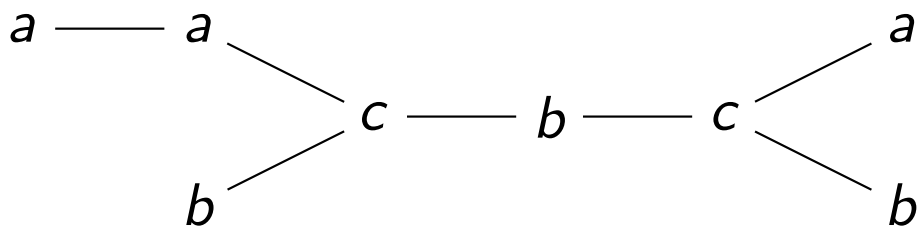

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- but few other classes of recognizable trace languages have been characterized that way...

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- an automaton model is known to match recognizability under the same hypothesis (Lodaya, Weil)

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- For MSCs, recognizability has been considered only with reference to a monoid structure (i.e. considering linearizations), and several logics have been proposed

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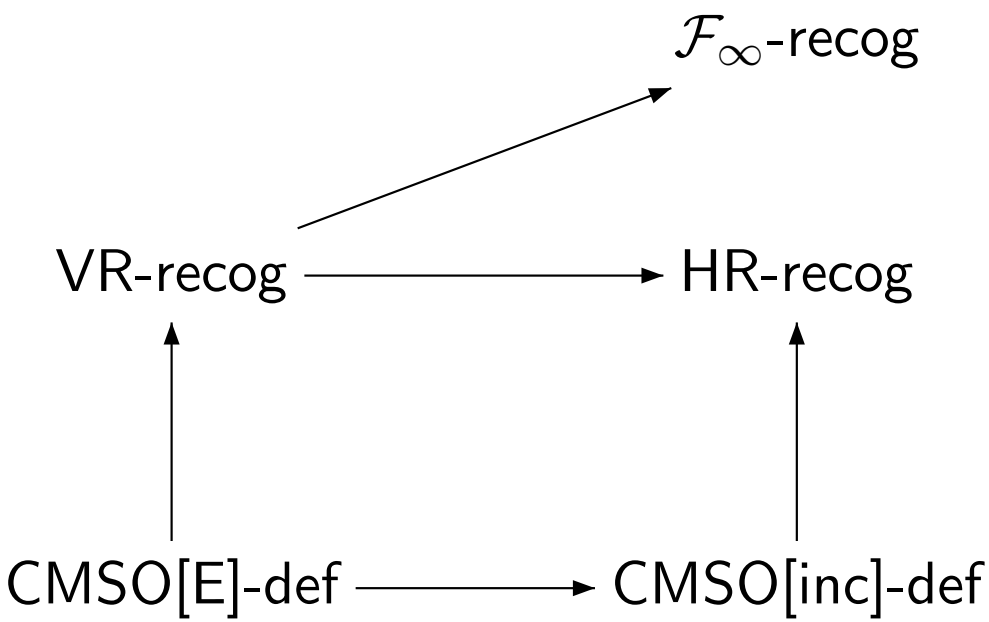
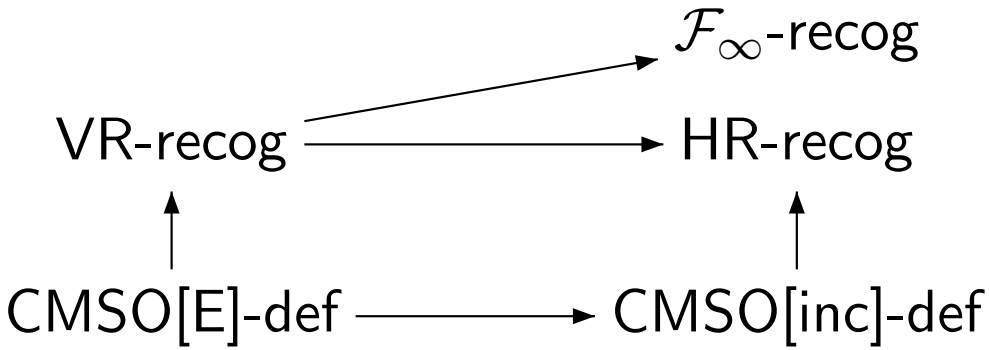
- CMSO[inc] – the domain is 2-sorted (V, E), and inc is the incidence predicate

General implications

The following implications are known, and strict (Courcelle, Weil)



Finiteness conditions yield equivalence results



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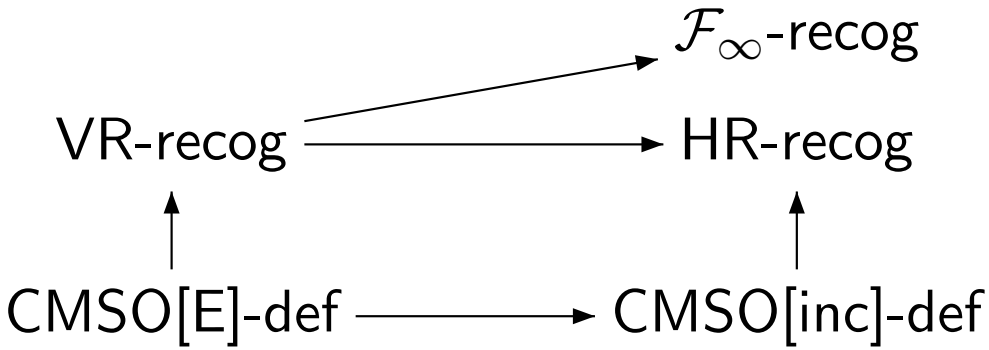

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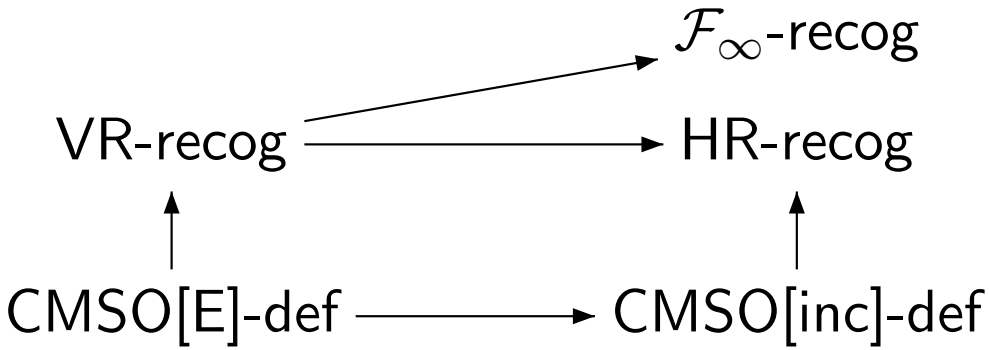

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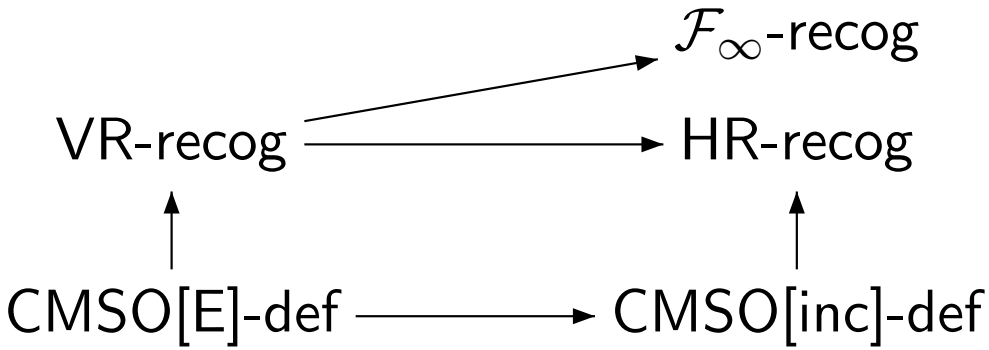

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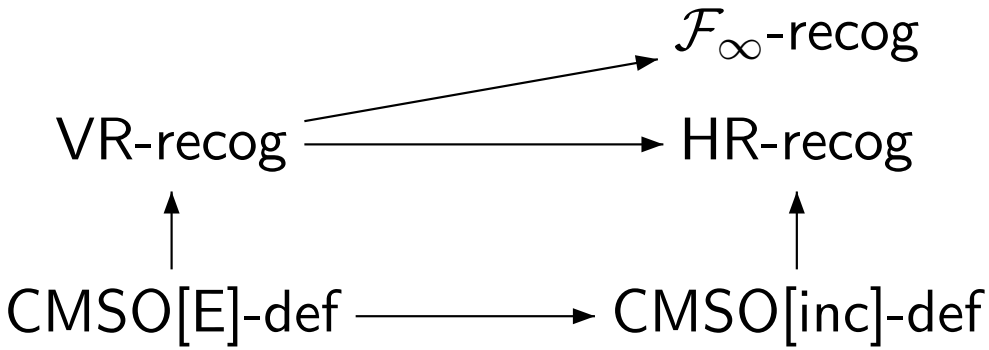

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- and if $\mathcal{F}$ is weakly rigid, this is equivalent to CMSO[E]-definability (Weil)

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- Unavoidably very complex if we want to discuss graph or poset languages without restrictions, but adding finiteness conditions often helps

Thank you for your attention!