Algebraic extensions of finitely generated subgroups of free groups[1]

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CSL, Lisboa, July 2005

[1] Joint work with E. Ventura (Barcelona) and A. Myasnikov (Montréal)

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- $a_n \mapsto a^n b^n$ is injective from F_∞ to F_2

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Thus: *two free factors of F are congruent if and only if they have the same rank*

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but this algorithm is not completely satisfactory: if H and K are not congruent, it does not always tell you why – we would like a classification of the congruence classes of subgroups; ideally, a computable list of invariants of a subgroup H, characterizing its congruence class

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- describes the way H sits in larger subgroups of F
- is related with Takahasi's theorem (1951)

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- so let me digress on the representation of finitely generated subgroups

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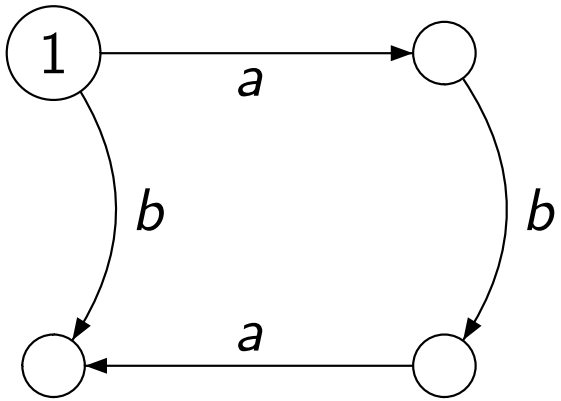

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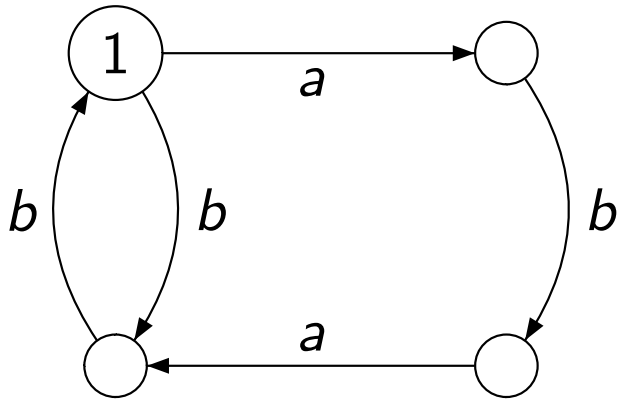

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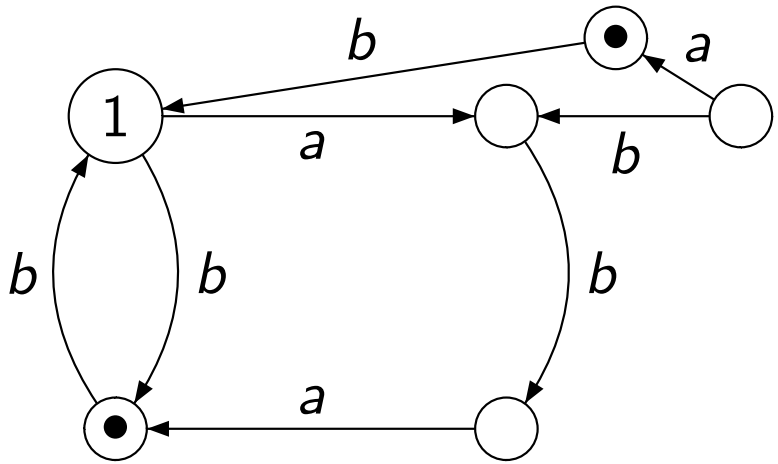

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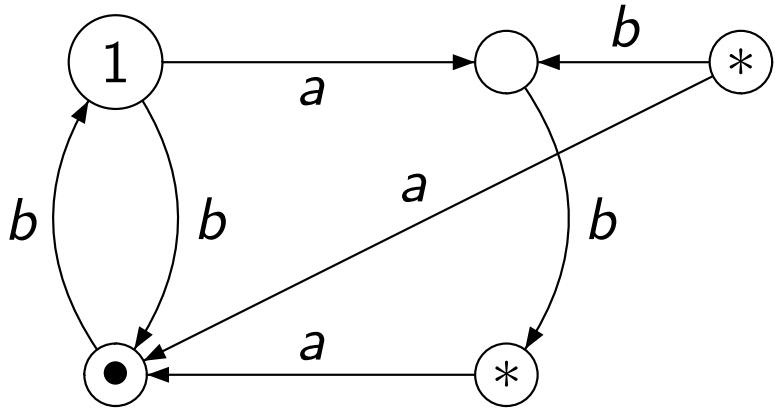

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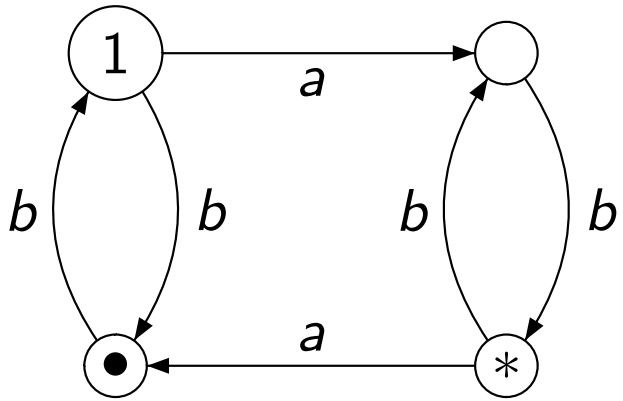

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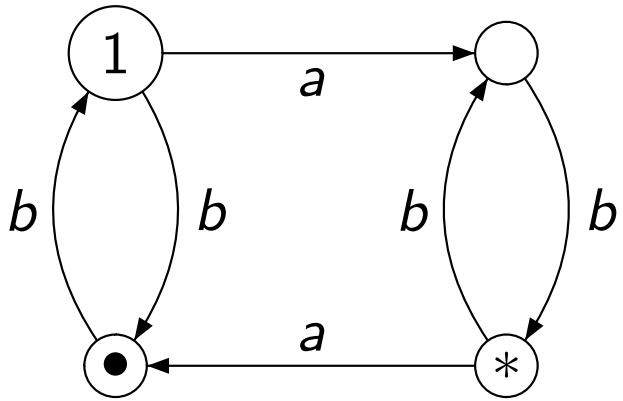

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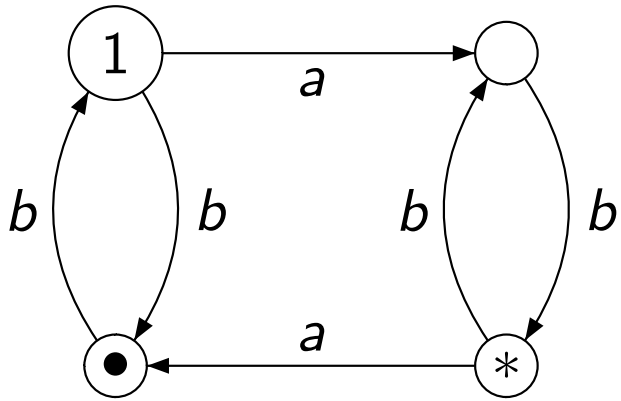
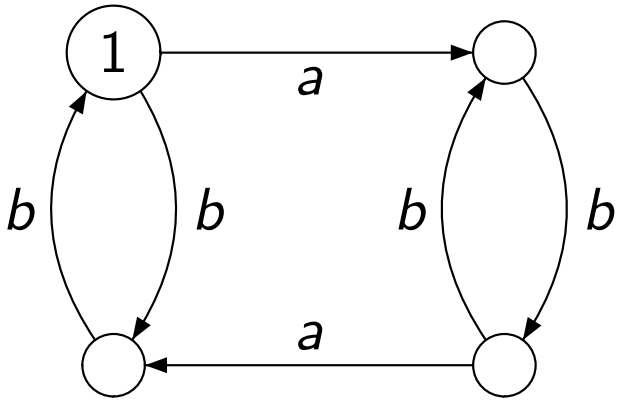


- a deterministic, co-deterministic automaton $\Gamma_A(H)$, with a distinguished vertex, and every non-distinguished vertex has valence at least 2, denoted $\Gamma_A(H)$
- $\Gamma_A(H)$ depends on H, not on the given generators of H – but depends on A!

Spanning tree, rank and basis



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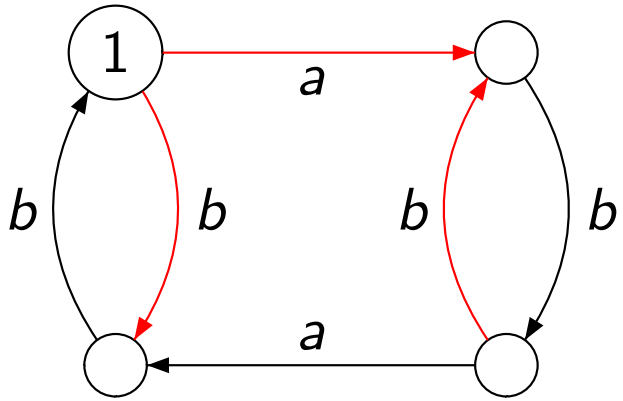

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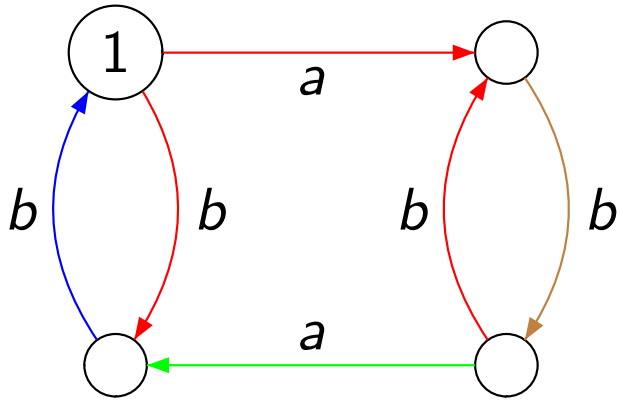

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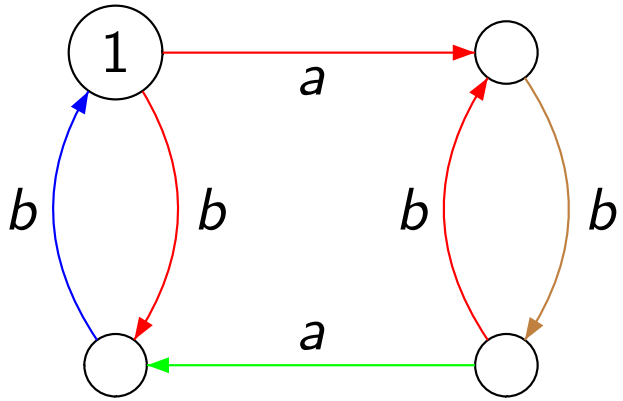

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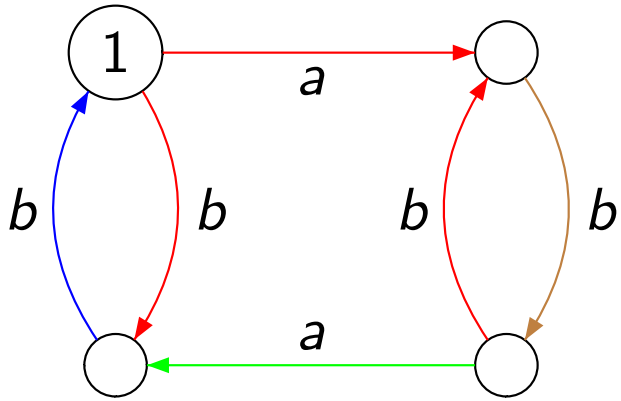

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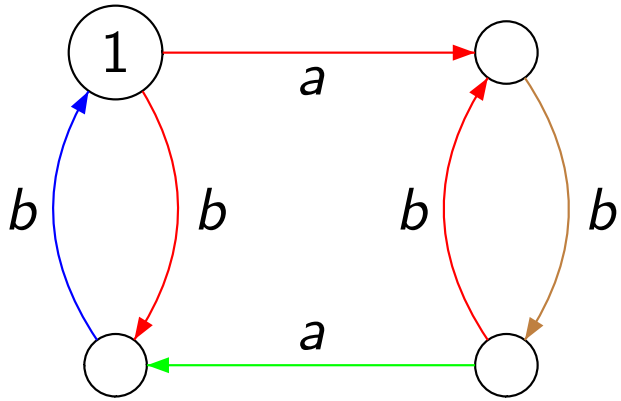

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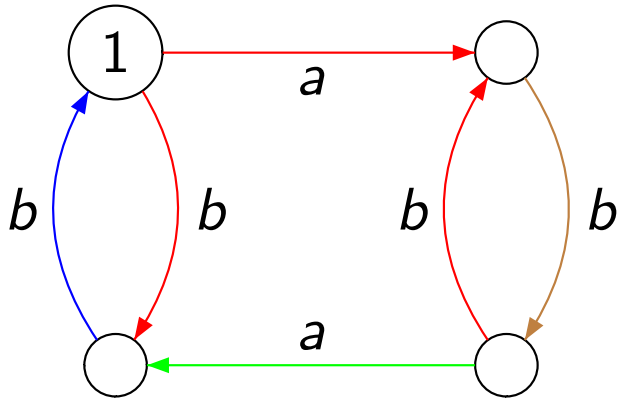

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A spanning tree has $|V| - 1$ edges, so $\mathrm{rk}(H) = |E| - |V| + 1$

Morphisms of automata

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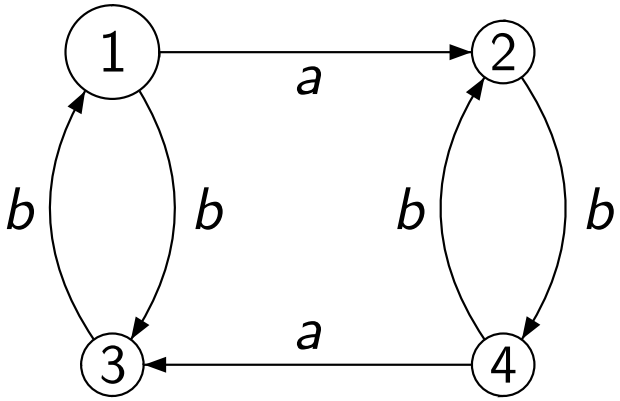

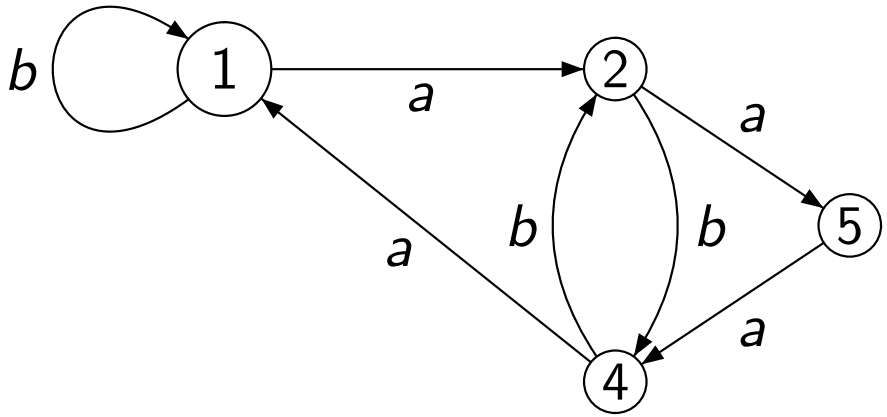

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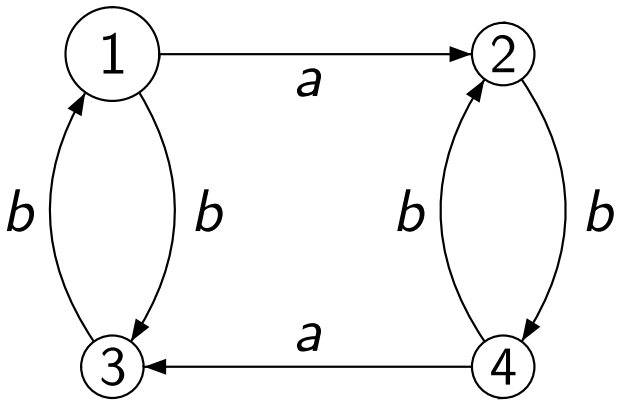

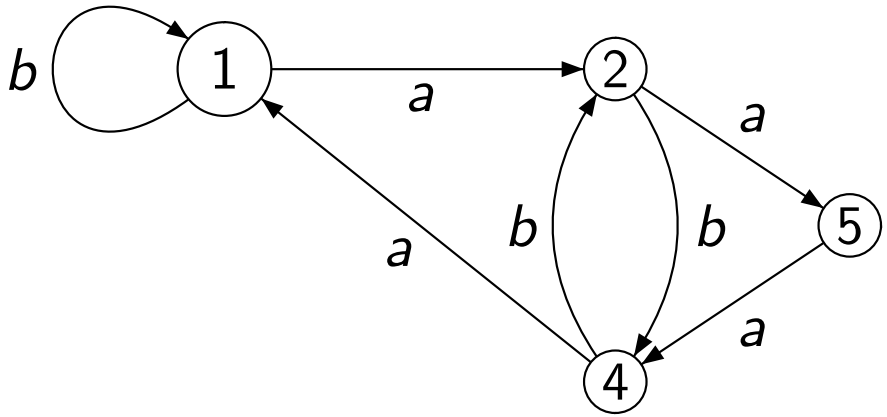

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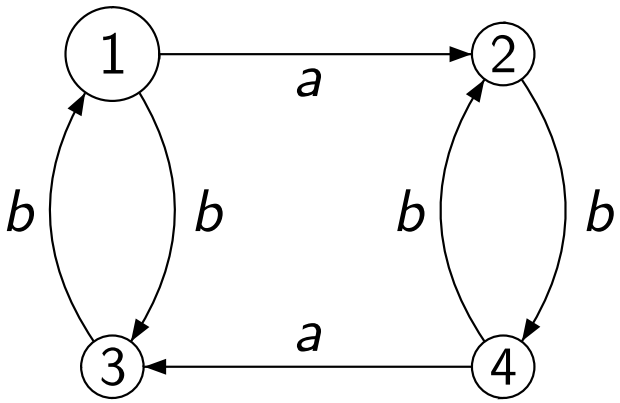

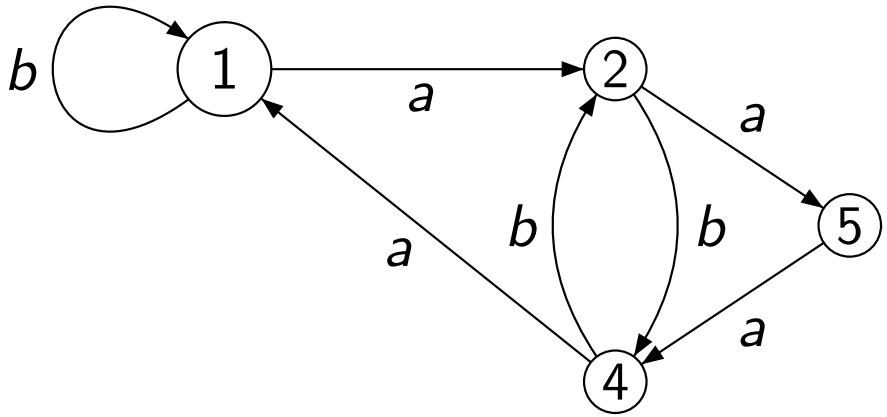

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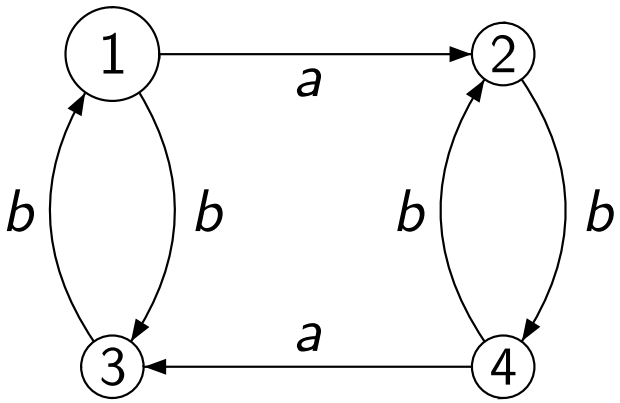

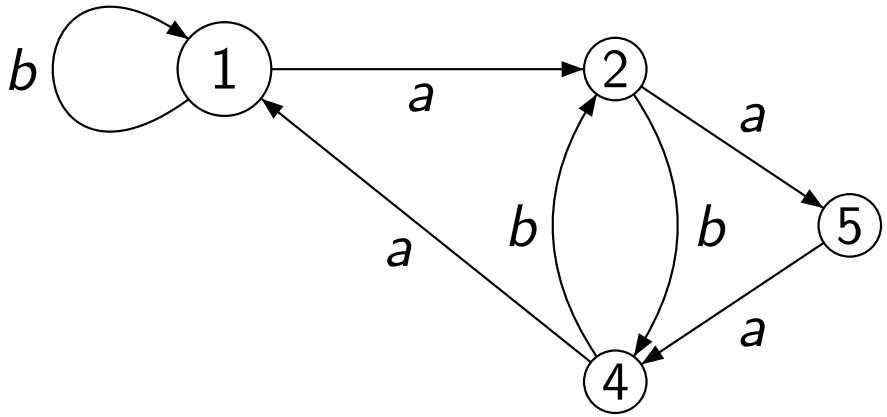

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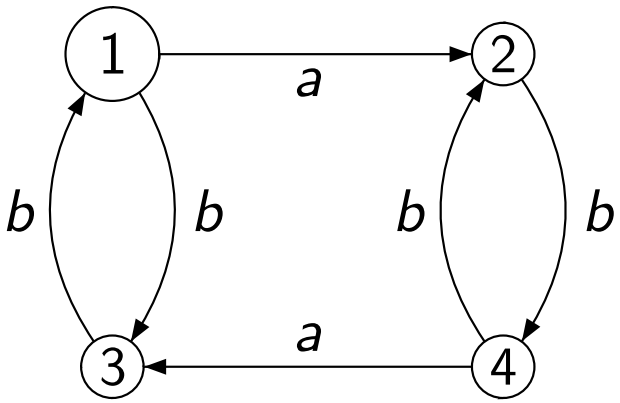

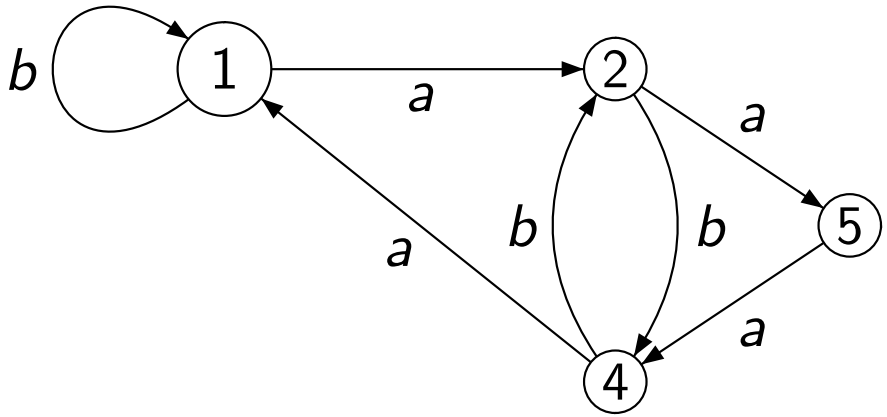

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- If $\varphi_{H,K}$ is one-to-one, then $H \leq_{\mathrm{ff}} K$
- If $\varphi_{H,K}$ is onto, we say that K is an *overgroup* of H (a finite number of them, which depends on A)

Proof of Takahasi's theorem

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- That's it! The set of overgroups of H is a Takahasi family for H: if $H \leq K$, then K is a free extension of an overgroup of H – besides, H has finitely many overgroups, and they are computable.

Proof of Takahasi's theorem

Theorem
Let $H \leq_{fg} F$. There exist $H_1, \ldots, H_n$, finitely many subgroups such that: if $H \leq K \leq F$, then $H \leq H_i \leq_{ff} K$ for some $1 \leq i \leq n$

If $H \leq K$, then there exists $\varphi_{H,K} : \Gamma_A(H) \to \Gamma_A(K)$. Let L such that $\Gamma_A(L) = \varphi_{H,K}(\Gamma_A(H))$. Then L is an overgroup of H, and $L \leq_{\mathrm{ff}} K$.

- That's it! The set of overgroups of H is a Takahasi family for H: if $H \leq K$, then K is a free extension of an overgroup of H – besides, H has finitely many overgroups, and they are computable.
- But it is not a canonical family

Algebraic extensions

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- If K is $\leq_{\text{ff}}$-minimal in $\mathcal{L}$ and $H \leq L \leq_{\text{ff}} K$, then $H \leq M \leq_{\text{ff}} L$ for some $M \in \mathcal{L}$, so $M \leq_{\text{ff}} K$, and by minimality, $M = L = K$, that is, $K \in \text{AE}(H)$

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- If $H \leq_{\text{ff}} F$, then $\text{AE}(H) = \{H\}$. For instance, no non-trivial overgroup of $\langle ab, aca \rangle$ is an algebraic extension

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- ... using the Whitehead algorithm, which determines whether a subgroup is a free factor in another

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Note that the property of being K-algebraic over H, depends on K

Composition of extensions

Let $H \leq K$, and let $H \leq K_i \leq K$, $i = 1, 2$

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This extension L is called *the algebraic closure of H in K*

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- If in addition $\mathcal{P}$ is decidable, then $\mathrm{cl}_{\mathcal{P}}(H)$ is computable

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- However, it is not always computable
- Note that $\mathrm{rk}(\mathrm{cl}_{\mathsf{V}}(H)) \leq \mathrm{rk}(H)$, but it is not known whether this is because $H \leq_{\mathrm{ealg}} \mathrm{cl}_{\mathsf{V}}(H)$

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- an equation is an element e of the countably generated free group $F(X, Y_1, \ldots, Y_n, \ldots)$

Equations over subgroups

- Let $H \leq F$. An H-equation is a word $e \in H * \langle X \rangle$, where X is a new symbol (the unknown). A solution of H is an element x such that $e(x) = 1$
- Thus x is a solution of some H-equation if and only if the homomorphism $H * \langle X \rangle \to \langle H, x \rangle$ is not one-to-one, if and only $\langle H, x \rangle$ has rank at most $\mathrm{rk}(H)$, if and only if $H \leq_{\mathrm{alg}} \langle H, x \rangle$
- an equation is an element e of the countably generated free group $F(X, Y_1, \ldots, Y_n, \ldots)$
- Let $\mathcal{E}$ be a set of equations: say that H is $\mathcal{E}$-closed if H contains every solution of every particularization over H of an equation in $\mathcal{E}$.

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- Similarly, let $\mathcal{E}_{pure} = \{X^nY_1 \mid n \geq 2\}$. Then H is $\mathcal{E}_{pure}$-closed if and only if H is pure.

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Thus the malnormal closure of $H \leq_{\text{fg}} F$ has rank at most $\text{rk}(H)$, as well as the pure closure. . .