

Structural properties of the free profinite aperiodic semigroup

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Motivations

- ▶ Many classes of regular languages correspond to **pseudovarieties** of semigroups.
- ▶ Example [Schützenberger 65]
 - Star-free languages \longleftrightarrow aperiodic semigroups.

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- ▶ Many classes of regular languages correspond to **pseudovarieties** of semigroups.
- ▶ Example [Schützenberger 65]
Star-free languages \longleftrightarrow aperiodic semigroups.
- ▶ [Bir37,Lav63,Rei82,Ban83,Alm89]: associate to each pseudovariety a **profinite semigroup**, capturing properties of that pseudovariety.
- ▶ Not much known on profinite semigroups for “large” pseudovarieties.
- ▶ We focus on **A**, the pseudovariety of **aperiodic** semigroups.

A related language-theoretic question

- ▶ Let \mathcal{L} be a class of regular languages.
- ▶ Two languages K_1, K_2 are \mathcal{L} -separated if there exist $L_1, L_2 \in \mathcal{L}$ st.

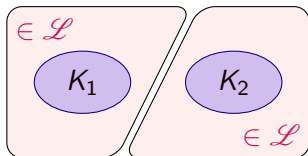
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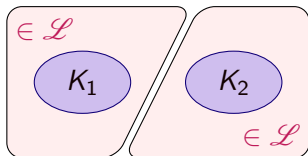
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- ▶ Separation Problem.

Input K_1, K_2 regular.

Question Are K_1, K_2 separated by star-free languages?

Pseudovarieties

- ▶ **Pseudovariety**: class of finite semigroups closed under
 - ▶ finite direct products,
 - ▶ subsemigroup,
 - ▶ quotient.

G: all finite **groups**.

A: all finite **aperiodic** (group-free) semigroups.

V: a generic pseudovariety.

To a pseudovariety, we associate a profinite semigroup, as follows.

The metric d_V induced by V on X^+

- ▶ A semigroup S separates $u, v \in X^+$ if there is a homomorphism

$$\varphi : X^+ \rightarrow S \text{ with } \varphi(u) \neq \varphi(v).$$

- ▶ Assumption on V : $u \neq v \implies u, v$ can be separated by some $S \in V$.
- ▶ Define a metric d_V :

$$\begin{aligned} r_V(u, v) &= \min \left\{ |S| : S \in V \text{ and } S \text{ separates } u \text{ and } v \right\}. \\ d_V(u, v) &= 2^{-r_V(u, v)}. \end{aligned}$$

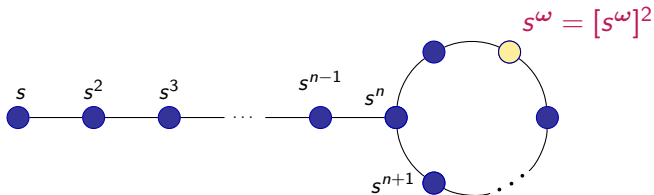
2 words are close if we need a large semigroup of V to separate them.

The pro- V free profinite semigroup

Reutenauer 1979

For $x \in X^+$, the sequence $x^{n!}$ is a Cauchy sequence since it evaluates to $\varphi(x^{n!}) = \varphi(x)^{n!} = \varphi(x)^\omega$ in all semigroup S with less than n elements.

Does not converge in X^+ .

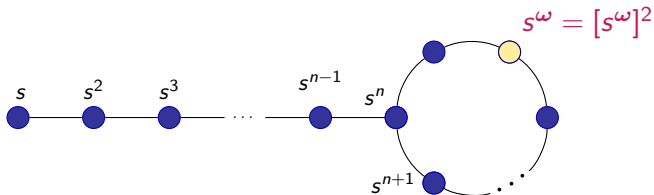


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Does not converge in X^+ . Converges in $\overline{\Omega}_X \mathbf{V}$ to an idempotent noted x^ω .



- ▶ The metric space $(X^+, d_{\mathbf{V}})$ is not complete.
- ▶ Free pro- \mathbf{V} semigroup $\overline{\Omega}_X \mathbf{V}$: topological completion of $(X^+, d_{\mathbf{V}})$.
- ▶ Note. Equation $z = z^2$: no solution in X^+ , but solutions in $\overline{\Omega}_X \mathbf{V}$.

Main properties of the free profinite semigroup

- ▶ The metric d_V can be extended by continuity to $\overline{\Omega_X V}$.
- ▶ A sequence of words converges in the metric space $(\overline{\Omega_X V}, d_V)$ iff its projection on every finite semigroup of V is **ultimately constant**.
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- ▶ **Universal** property:

$$\begin{array}{ccc} X^+ & \xrightarrow{\iota} & \overline{\Omega_X V} \\ & \searrow \phi & \\ & & S \in V \end{array}$$

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Examples of elements of $\overline{\Omega_X}V$

- ▶ $\overline{\Omega_X}V$ is in general **uncountable**. Yet, it contains **friendly elements**:
 - ▶ words,
 - ▶ elements that can be represented by **ω -terms**.

$$x \in \overline{\Omega_X}V : x^\omega = \lim x^{n!}$$

- ▶ Example of ω -terms: $(a^\omega b)^\omega$, $a(a^\omega ba)^\omega a^\omega b$.

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$$x \in \overline{\Omega}_X \mathbf{V} : \quad x^{\omega-1} = \lim x^{n!-1}$$

- ▶ In a finite group, $x^{\omega-1}$ evaluates to $\hat{\varphi}(x^{\omega-1}) = \hat{\varphi}(x)^\omega \hat{\varphi}(x)^{-1} = \hat{\varphi}(x)^{-1}$.
 \rightsquigarrow In $\overline{\Omega}_X \mathbf{G}$, the **free group on X** is represented by terms built from letters using concatenation and $(\omega - 1)$ -power.

Two problems for $\overline{\Omega_X A}$

1. **Word problem:** do 2 ω -terms represent the same element of $\overline{\Omega_X A}$?

In $\overline{\Omega_X A}$, we have $x^\omega = x^{\omega+1}$: if $s = \hat{\varphi}(x)$



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2. **Separation problem**: given $K_1, K_2 \subseteq X^+$ regular
 K_1, K_2 are separated by star-free languages

\Updownarrow [Almeida 96–99]

The closures of K_1, K_2 in $\overline{\Omega_X A}$ intersect.

Computing closures

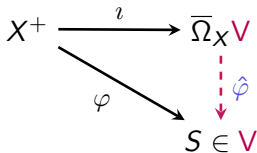
Theorem [Alm90]

Closures of \mathbf{V} -recognizable languages form a basis of the pro- \mathbf{V} topology.

Proposition [Alm90]

If L is recognized by $\varphi : X^+ \rightarrow S \in \mathbf{V}$, then the closure of L in $\overline{\Omega}_X \mathbf{V}$ is

$$\text{cl}(L) = \hat{\varphi}^{-1}(\varphi(L)).$$



Corollary

If L_1, L_2 are \mathbf{V} -recognizable and $L_1 \cap L_2 = \emptyset$, then $\text{cl}(L_1) \cap \text{cl}(L_2) = \emptyset$.

Word problem for $\Omega_X^\omega A$

- ▶ Two distinct words can be separated by an aperiodic semigroup.
- ▶ But $(b(aa)^\omega)^{\omega+1}b$ and $b(a^\omega b)^\omega$ represent the same element of $\Omega_X^\omega A$.

Theorem (McCammmond '01)

- ▶ A procedure to transform any ω -term into a **normal form**.

1. $(x^\omega)^\omega \longleftrightarrow x^\omega$
2. $(x^k)^\omega \longleftrightarrow x^\omega$ for $k \geq 2$
3. $x^\omega x^\omega \longleftrightarrow x^\omega$
4. $(xy)^\omega x \longleftrightarrow x(yx)^\omega$
5. $x^\omega x \longleftrightarrow x^\omega \longleftrightarrow xx^\omega$

- ▶ Two ω -words are equal over $\bar{\Omega}_X A$ iff they have the same normal form.

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- ▶ Two ω -words are equal over $\bar{\Omega}_X A$ iff they have the same normal form.

Induction parameter: **rank** = nesting depth of ω -powers in the normal form.

Neighborhood bases of star-free languages

- ▶ For $L \subseteq X^+$, let $L^{>n} = L^n L^+$.
- ▶ Given an ω -term α , let $L_n(\alpha)$ be the language obtained from α by replacing all " ω " by " $> n$ ".
- ▶ Example:

$$L_2(a^\omega abb^\omega) = a^2 a^+ abb^2 b^+,$$

$$L_2((a^\omega b)^\omega) = (a^2 a^+ b)^2 (a^2 a^+ b)^+.$$

- ▶ Informally, $L_n(\alpha)$ is obtained from α by replacing ω -powers by large iterations (more than n times).
- ▶ Easy. The element of $\overline{\Omega}_X \mathbf{A}$ represented by α is in $\text{cl}(L_n(\alpha))$.

The word problem: Key results

For an ω -term α , we define a parameter $\mu(\alpha) \in \mathbb{N}$. (*def. is technical*).

Theorem [ACZ, Star freeness]

If α is an ω -term in normal form and $n \geq \mu(\alpha)$, then $L_n(\alpha)$ is star free.

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Theorem [ACZ, Separation]

If α and β are in normal form and $n > \max\{\mu(\alpha), \mu(\beta)\}$, then

$$L_n(\alpha) \cap L_n(\beta) \neq \emptyset \implies \alpha = \beta.$$

Proof. Induction on the rank.

Application: McCammond's Theorem

Theorem [ACZ, Word problem: McCammond's Theorem reproved]

If α and β are ω -words in normal form that are equal over A , then $\alpha = \beta$.

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If α and β are ω -words in normal form that are equal over A , then $\alpha = \beta$.

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- ▶ Let $n > \max\{\mu(\alpha), \mu(\beta)\}$.
- ▶ By the Separation Theorem, it suffices to show that $L_n(\alpha) \cap L_n(\beta) \neq \emptyset$.
- ▶ Suppose that $L_n(\alpha) \cap L_n(\beta) = \emptyset$.
- ▶ Since $L_n(\alpha)$ and $L_n(\beta)$ are star-free, we get $\text{cl}(L_n(\alpha)) \cap \text{cl}(L_n(\beta)) = \emptyset$.
- ▶ Since $\alpha \in \text{cl}(L_n(\alpha))$ and $\beta \in \text{cl}(L_n(\beta))$, it follows that α and β do not coincide over \mathbf{A} , a contradiction. □

The separation problem for groups

For **groups**, ingredients to solve the separation problem:

$$X^* \leftrightarrow FG(X) \leftrightarrow \overline{\Omega_X G}$$

1. Reducibility

$\text{cl}(L_1) \cap \text{cl}(L_2) \neq \emptyset$ in $\overline{\Omega_X G}$ is witnessed by an element of $FG(X)$.

- ▶ Related to Rhodes' type II conjecture, proved by Ash.
- ▶ And to Pin & Reutenauer conjecture...
...reduced to another conjecture proved by Ribes & Zalesskiĭ.

2. The Pin-Reutenauer algorithm

Computes closures of a regular language in the free group $FG(X)$.

3. Effective computation of intersections and emptiness test in $FG(X)$.

The Pin-Reutenauer algorithm for groups

- ▶ $\text{cl}_{\text{FG}(X)}(L)$: closure of L in $\text{FG}(X)$ for the pro- G topology.

For regular $K, L \subseteq X^*$:

$$\text{cl}_{\text{FG}(X)}(KL) = \text{cl}_{\text{FG}(X)}(K) \cdot \text{cl}_{\text{FG}(X)}(L),$$

$$\text{cl}_{\text{FG}(X)}(L^*) = \langle L \rangle \quad (\text{subgroup generated by } L \text{ in } \text{FG}(X)).$$

- ▶ Gives an algorithm to **effectively compute** the closure of a rational language in the free group (or free monoid) for the pro- G topology.

The Pin-Reutenauer algorithm for aperiodics

- ▶ Structure to replace the free group? use ω -terms!
- ▶ Each ω -term can be interpreted in $\overline{\Omega}_X A$.
- ▶ $\Omega_X^\omega A$ is the subsemigroup of $\overline{\Omega}_X A$ generated by all ω -terms.
= ω -terms modulo equality on aperiodic semigroups.

$$\begin{array}{lclcl} X^* & \hookrightarrow & \text{FG}(X) & \hookrightarrow & \overline{\Omega}_X G \\ X^+ & \hookrightarrow & \Omega_X^\omega A & \hookrightarrow & \overline{\Omega}_X A \end{array}$$

- ▶ $\Omega_X^\omega A$ is nice: countable, representation by terms, word problem OK.

Algebraic and topological closures

For $L \subseteq X^+$:

$\text{cl}(L)$ = closure of L in $\overline{\Omega}_X A$.

$\text{cl}_\omega(L)$ = closure of L in $\Omega_X^\omega A$.

Proposition [Reducibility, Henckell]

For all rational languages $K, L \subseteq X^+$,

$$\begin{aligned} \text{cl}(K) \cap \text{cl}(L) \neq \emptyset \\ \iff \\ \text{cl}_\omega(K) \cap \text{cl}_\omega(L) \neq \emptyset. \end{aligned}$$

The Pin-Reutenauer algorithm for A

Let $\langle L \rangle_\omega = \omega$ -subsemigroup of $\Omega_X^\omega A$ generated by L .

Theorem [ACZ, Pin-Reutenauer procedure for A]

For all nonempty rational languages $K, L \subseteq X^+$:

$$\text{cl}_\omega(KL) = \text{cl}_\omega(K) \cdot \text{cl}_\omega(L),$$

$$\text{cl}_\omega(L^+) = \langle \text{cl}_\omega(L) \rangle_\omega.$$

Remarks

- ▶ Yields decidability of separation, based on two semi-algorithms.
- ▶ Requires more work to test **efficiently** emptiness of $\text{cl}(L) \cap \text{cl}(K)$.
- ▶ **Easy** inclusions from right to left: multiplication, ω -power continuous.

The Pin-Reutenauer for A : Case of the product

- ▶ $\text{cl}_\omega(KL) \supseteq \text{cl}_\omega(K) \cdot \text{cl}_\omega(L)$ by continuity of multiplication.
- ▶ For the reverse implication, use the fact that A is ω -factorial:
Every factor in $\overline{\Omega}_X A$ of an element of $\Omega_X^\omega A$ is again in $\Omega_X^\omega A$.

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- ▶ Write $w_n = x_n y_n$ with $x_n \in K$ and $y_n \in L$.
- ▶ **Compactness:** assume (x_n) and (y_n) converge to $x \in \text{cl}(K)$ and $y \in \text{cl}(L)$.
- ▶ Since A is ω -factorial and $w = xy$, we get $x, y \in \Omega_X^\omega A$.
- ▶ So $x \in \text{cl}_\omega(K)$, and $y \in \text{cl}_\omega(L)$, whence $w \in \text{cl}_\omega(K) \cdot \text{cl}_\omega(L)$.

Fullness (Almeida, Steinberg '00)

- ▶ S : pseudovariety of **all finite** semigroups.
- ▶ $p_V : \overline{\Omega}_X S \rightarrow \overline{\Omega}_X V =$ continuous morphism mapping $a \in X$ to itself.
- ▶ A pseudovariety V is **full** if:

Closure of L in $\Omega_X^\omega V =$ projection of the closure of L in $\Omega_X^\omega S$.

$$\text{cl}_{\omega, V}(L) = p_V \left(\text{cl}_{\omega, S}(L) \right)$$

Fullness and inheritance of the PR-algorithm

Proposition [ACZ]

Let V and W be pseudovarieties such that

1. $V \subseteq W$,
2. Both V and W are full,
3. The Pin-Reutenauer algorithm holds for W .

Then the Pin-Reutenauer algorithm also holds for V .

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Then the Pin-Reutenauer algorithm also holds for V .

Easy proof. Eg, if product and closure commute for W :

$$\begin{aligned} \text{cl}_{\omega, V}(KL) &= p_V(\text{cl}_{\omega, S}(KL)) && \text{since } V \text{ is full} \\ &= p_{W, V}[p_W(\text{cl}_{\omega, S}(KL))] \\ &= p_{W, V}[\text{cl}_{\omega, W}(KL)] && \text{since } W \text{ is full} \\ &= p_{W, V}[\text{cl}_{\omega, W}(K) \cdot \text{cl}_{\omega, W}(L)] && \text{by hypothesis} \\ &= p_{W, V}[\text{cl}_{\omega, W}(K)] \cdot p_{W, V}[\text{cl}_{\omega, W}(L)] \end{aligned}$$

and back to $\text{cl}_{\omega, V}(K)\text{cl}_{\omega, V}(L)$.

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Proof. Using again star-free languages $L_n(\alpha)$.

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Proposition

- ▶ [ACZ] The pseudovariety of \mathcal{R} -trivial semigroups is full.
- ▶ [Costa] The pseudovariety of locally semilattices is full.

Corollary

The Pin-Reutenauer algorithm holds for these pseudovarieties.

Proof. By inheritance theorem for full pseudovarieties.

Some short-term perspectives

1. Automata for term languages.

- ▶ [Henckell] Given regular $K, L \subseteq X^+$, one can decide whether

$$\text{cl}(K) \cap \text{cl}(L) = \emptyset.$$

- ▶ By ω -reducibility for \mathbf{A} , this is equivalent

$$\text{cl}_\omega(K) \cap \text{cl}_\omega(L) = \emptyset.$$

Is it possible to test it using automata accepting languages in $\Omega_X^\omega \mathbf{A}$?

- ## 2. The pseudovariety \mathbf{S} of all finite semigroups is obviously full. Does the Pin-Reutenauer algorithm hold for \mathbf{S} ?

Thank you!