

The Pin-Reutenauer algorithm for classes of aperiodic semigroups

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Outline

- A. Profinite semigroups and closures: some notation.
- B. The Pin-Reutenauer algorithm.
- C. Proof ideas and main ingredients.

Pseudovarieties

- ▶ **Pseudovariety**: class of finite semigroups closed under
 - ▶ finite direct products,
 - ▶ subsemigroup,
 - ▶ quotient.
- ▶ **S**: all finite semigroups.
- ▶ **G**: all finite **groups**.
- ▶ **A**: all finite **aperiodic** (group-free) semigroups.
- ▶ **R**: all finite **\mathcal{R} -trivial** semigroups.
- ▶ **V**: a generic pseudovariety.

Relatively V -free profinite semigroups

- ▶ X : fixed finite alphabet.
- ▶ A semigroup S **separates** $u, v \in X^+$ if there is a homomorphism $\varphi : X^+ \rightarrow S$ such that $\varphi(u) \neq \varphi(v)$.
- ▶ Define a pseudo-metric d_V :

$$\begin{cases} r_V(u, v) &= \min\{|S| : S \in V \text{ and } S \text{ separates } u \text{ and } v\}. \\ d_V(u, v) &= 2^{-r_V(u, v)}. \end{cases}$$

- ▶ $u \sim_V v$ if and only if $d_V(u, v) = 0$ defines a congruence.
- ▶ Relatively V -free profinite semigroup $\overline{\Omega}_X V$: completion of $(X^+ / \sim_V, d_V)$. Elements of $\overline{\Omega}_X V$ are called pseudowords.

Implicit signatures

- ▶ Implicit signature σ : set of pseudowords, containing the multiplication.
- ▶ **Example:** $\kappa = \{_{-}\cdot_{-}, _{\omega^{-1}}\}$.
- ▶ Each element of σ can be interpreted on a profinite semigroup.
- ▶ Given σ , a profinite semigroup S has a structure of “ σ -semigroup” obtained by evaluating each operation of σ in S .
- ▶ $\Omega_X^\sigma V$ is the σ -subsemigroup of $\overline{\Omega}_X V$ generated by X .

Notation: Closures for profinite topologies

- ▶ $L \subseteq S$ topological semigroup: $\text{cl}_S(L)$ denotes the closure of L in S .

$$\text{cl}(L) \stackrel{\text{def}}{=} \text{cl}_{\overline{\Omega}_X S}(L) \qquad \text{cl}_\sigma(L) \stackrel{\text{def}}{=} \text{cl}_{\Omega_X^\sigma S}(L)$$

$$\text{cl}_V(L) \stackrel{\text{def}}{=} \text{cl}_{\overline{\Omega}_X V}(L) \qquad \text{cl}_{\sigma, V}(L) \stackrel{\text{def}}{=} \text{cl}_{\Omega_X^\sigma V}(L)$$

- ▶ The topology on $\Omega_X^\sigma V$ is the induced topology in $\overline{\Omega}_X V$:

$$\text{cl}_{\sigma, V}(L) = \text{cl}_V(L) \cap \Omega_X^\sigma V.$$

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- ▶ We abusively use the above notation for $L \subseteq X^+$: eg, we write $\text{cl}_{\sigma, V}(L)$ instead of $\text{cl}_{\sigma, V}(p_V(\iota(L)))$, where $p_V : \overline{\Omega}_X S \rightarrow \overline{\Omega}_X V$ is the canonical projection and $\iota : X^+ \rightarrow \overline{\Omega}_X S$ the canonical embedding.

Notation: algebraic closures

- ▶ Let σ be an implicit signature, S be a σ -semigroup, and $L \subseteq S$.

$\langle L \rangle_\sigma = \sigma$ -subsemigroup of S generated by L .

(in practice in $L \subseteq \Omega_X^\sigma S$)

$$\langle L \rangle_{\sigma, \mathbf{V}} = \langle p_{\mathbf{V}}(L) \rangle_\sigma$$

The Pin-Reutenauer algorithm

- ▶ The Pin-Reutenauer algorithm holds for V and σ if, for all rational languages $K, L \subseteq X^+$, the following equations hold:

$$\text{cl}_{\sigma, V}(KL) = \text{cl}_{\sigma, V}(K) \cdot \text{cl}_{\sigma, V}(L),$$

$$\text{cl}_{\sigma, V}(L^*) = \langle \text{cl}_{\sigma, V}(L) \rangle_{\sigma}.$$

- ▶ Makes it possible to “compute” the closure of any rational language in the relatively V -free σ -semigroup $\Omega_X^{\sigma}V$.

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- ▶ Makes it possible to “compute” the closure of any rational language in the relatively V -free σ -semigroup $\Omega_X^{\sigma}V$.
- ▶ Note: $\text{cl}_{\sigma, V}(KL) \supseteq \text{cl}_{\sigma, V}(K) \cdot \text{cl}_{\sigma, V}(L)$ always hold true (multiplication is continuous).

The Pin-Reutenauer algorithm holds for G and κ

- ▶ In the free group $\Omega_X^\kappa G$ endowed with the profinite topology, for $K, L \subseteq X^+$ regular:

$$\begin{aligned} \text{cl}_{\kappa, G}(KL) &= \text{cl}_{\kappa, G}(K) \cdot \text{cl}_{\kappa, G}(L), \\ \text{cl}_{\kappa, G}(L^*) &= \langle L \rangle_\kappa. \end{aligned} \tag{1}$$

It is actually not necessary to propagate the closure in (1).

- ▶ Conjectured by Pin and Reutenauer, reduced to another conjecture proved by Ribes and Zalesskiĭ.
- ▶ Equivalent to Rhodes' type II conjecture, proved by Ash.

The Pin-Reutenauer algorithm holds for A and κ

Theorem [Almeida, J.C. Costa, Z.]

The Pin-Reutenauer procedure holds for A and κ :

$$\text{cl}_{\kappa,A}(KL) = \text{cl}_{\kappa,A}(K) \cdot \text{cl}_{\kappa,A}(L), \quad (2)$$

$$\text{cl}_{\kappa,A}(L^*) = \langle \text{cl}_{\kappa,A}(L) \rangle_{\kappa}. \quad (3)$$

Proof ideas and ingredients: σ -fullness (Almeida, Steinberg '00)

- ▶ The following always hold:

$$\text{cl}_{\sigma, \mathbf{V}}(L) = \rho_{\mathbf{V}}(\text{cl}(L)) \cap \Omega_X^{\sigma} \mathbf{V}.$$

- ▶ A pseudovariety \mathbf{V} is σ -full if for every regular $L \subseteq X^+$:

$$\text{cl}_{\sigma, \mathbf{V}}(L) = \rho_{\mathbf{V}} \left(\text{cl}(L) \cap \Omega_X^{\sigma} S \right)$$

- ▶ One can show this is equivalent to: for every regular $L \subseteq X^+$,

$$\text{cl}_{\sigma, \mathbf{V}}(L) = \rho_{\mathbf{V}}(\text{cl}_{\sigma}(L)).$$

- ▶ To compute the closure in $\Omega_X^{\sigma} \mathbf{V}$, one can compute it in $\Omega_X^{\sigma} S$ and project onto the free pro- \mathbf{V} semigroup.

σ -fullness and inheritance of the PR-algorithm

Proposition [ACZ]

Let V and W be pseudovarieties such that

1. $V \subseteq W$,
2. Both V and W are σ -full,
3. The Pin-Reutenauer algorithm holds for W .

Then the Pin-Reutenauer algorithm also holds for V .

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Easy proof. Eg, if product and closure commute for W :

$$\begin{aligned} \text{cl}_{\sigma, V}(KL) &= p_V(\text{cl}_{\sigma}(KL)) && \text{since } V \text{ is } \sigma\text{-full} \\ &= p_{W, V}[p_W(\text{cl}_{\sigma}(KL))] \\ &= p_{W, V}[\text{cl}_{\sigma, W}(KL)] && \text{since } W \text{ is } \sigma\text{-full} \\ &= p_{W, V}[\text{cl}_{\sigma, W}(K) \cdot \text{cl}_{\sigma, W}(L)] && \text{by hypothesis} \\ &= p_{W, V}[\text{cl}_{\sigma, W}(K)] \cdot p_{W, V}[\text{cl}_{\sigma, W}(L)] \end{aligned}$$

and back to $\text{cl}_{\sigma, V}(K)\text{cl}_{\sigma, V}(L)$.

The Pin-Reutenauer for A : Case of the product

- ▶ $\text{cl}_{\kappa, A}(KL) \supseteq \text{cl}_{\kappa, A}(K) \cdot \text{cl}_{\kappa, A}(L)$ by continuity of multiplication.
- ▶ For the reverse implication, use the fact that A is κ -factorial.
Every factor in $\overline{\Omega}_X A$ of an element of $\Omega_X^\kappa A$ is again in $\Omega_X^\kappa A$.

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 - ▶ By compactness, one can assume (x_n) and (y_n) convergent to $x \in \text{cl}_A(K)$ and $y \in \text{cl}_A(L)$.

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 - ▶ By compactness, one can assume (x_n) and (y_n) convergent to $x \in \text{cl}_A(K)$ and $y \in \text{cl}_A(L)$.
 - ▶ Since A is κ -factorial and $w = xy$, we get $x, y \in \Omega_X^\kappa A$.
 - ▶ So $x \in \text{cl}_{\kappa, A}(K)$, and $y \in \text{cl}_{\kappa, A}(L)$, whence $w \in \text{cl}_{\kappa, A}(K) \cdot \text{cl}_{\kappa, A}(L)$.

Another ingredient: star-free languages separating elements of $\Omega_X^k A$.

Theorem (McCammond'2001)

Using the rewriting following system, there is a procedure to transform any ω -word into a **normal form**: two ω -words are equal over $\overline{\Omega}_X A$ if and only if they have the same normal form.

1. $(x^\omega)^\omega \longleftrightarrow x^\omega$;
2. $(x^k)^\omega \longleftrightarrow x^\omega$ for $k \geq 2$;
3. $x^\omega x^\omega \longleftrightarrow x^\omega$;
4. $x^\omega x \longleftrightarrow x^\omega \longleftrightarrow xx^\omega$;
5. $(xy)^\omega x \longleftrightarrow x(yx)^\omega$.

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The **rank** of $w \in \Omega_X^k A$ is the maximum nesting of ω -powers in the term in normal form representing w .

Neighborhood bases of star-free languages

- ▶ For $L \subseteq X^+$, let $L^{>n} = L^n L^+$.
- ▶ Given an ω -term w (term built from X using concatenation and ω -power), let $L_n(w)$ be the (regular) language obtained from w by replacing all " ω " by " $> n$ ".
- ▶ Example:

$$L_2(a^\omega abb^\omega) = a^2 a^+ abb^2 b^+,$$

$$L_2((a^\omega b)^\omega) = (a^2 a^+ b)^2 (a^2 a^+ b)^+.$$

- ▶ Informally, $L_n(w)$ is obtained from w by replacing ω -powers by **large** iterations (more than n times).

Key properties of the languages $L_n(w)$

Theorem [ACZ]

If w is in normal form, then

1. $L_n(w)$ is star-free for n large enough, depending only on w .
2. $p_A^{-1}(w) = \bigcap_n \text{cl}(L_n(w))$

Families $L_n(w)$ separate ω -terms, in the sense that for two ω -terms u, v :

$$(\forall n L_n(u) \cap L_n(v) \neq \emptyset) \implies p_A(u) = p_A(v).$$

and

$$p_A\left(\bigcap_n \text{cl}(L_n(u))\right) = \{p_A(u)\} = \bigcap_n p_A(\text{cl}(L_n(u))).$$

PR algorithm for A: handling iteration:

$$\text{cl}_{\kappa, A}(L^+) = \langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}$$

- ▶ The inclusion $\langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A} \subseteq \text{cl}_{\kappa, A}(L^+)$ is easy: since $\text{cl}_{\kappa, A}(L^+)$ contains $\text{cl}_{\kappa, A}(L)$, it suffices to show that $\text{cl}_{\kappa, A}(L^+)$ is a κ -semigroup.

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- ▶ We want to represent $w \in \text{cl}_{\kappa, A}(L^+)$ by a κ -term on $\text{cl}_{\kappa, A}(L)$.
- ▶ Use induction on the rank and "length" of w .
- ▶ Proof sketch for a normal form $w = v^\omega$ of rank $n \geq 1$.

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- ▶ $w = v^\omega \in \text{cl}_{\kappa, A}(L^+)$.
- ▶ Since $L_n(w)$ is star-free for n large enough, $\text{cl}_A(L_n(w))$ is clopen.
- ▶ Since $w \in \text{cl}_{\kappa, A}(L^+)$, there exists $w_n \in L_n(w) \cap L^+$.
- ▶ Since $w_n \in L_n(w)$, the sequence $(w_n)_n$ converges to w .
- ▶ **Easy case:** there is a subsequence $(w_{i_n})_n$ of w_n and a fixed N such that $w_{i_n} \in L^N$. Then use the product case:

$$w \in \text{cl}_{\kappa, A}(L^N) \subseteq (\text{cl}_{\kappa, A}(L))^N \subseteq \langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}.$$

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$$\text{inclusion } \text{cl}_{\kappa, A}(L^+) \subseteq \langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}$$

- ▶ **Otherwise:** write

$$w_n = w_{1,n} w_{2,n} \cdots w_{k_n,n}, \quad w_{j,n} \in L.$$

(with k_n unbounded.)

Main problem: reduce to a bounded number of factors, while still converging to w .

PR algorithm for A: handling iteration:

$$\text{inclusion } \text{cl}_{K,A}(L^+) \subseteq \langle \text{cl}_{K,A}(L) \rangle_{K,A}$$

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- ▶ Group factors of L .
- ▶ If necessary, use periodic repetitions: replace w_n by

$$\tilde{w}_n = \tilde{w}_{1,n} \tilde{w}_{2,n} \cdots (\tilde{w}_{i,n} \cdots \tilde{w}_{j,n})^\omega \cdots \tilde{w}_{K,n}, \quad \tilde{w}_{j,n} \in L.$$

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- ▶ $w_n \in L_n(w) = L_n(v^\omega) = [L_n(v)]^{>n}$, so we get another factorization

$$w_n = v_{1,n} v_{2,n} \cdots v_{p_n,n}, \quad p_n > n \text{ and } v_{j,n} \in L_n(v)$$

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$$v_{j,n} \in L_n(v), \text{ etc.}$$

- ▶ Consider a morphism $\varphi : X^* \rightarrow M$ recognizing L and $\{1\}$.
- ▶ Build a finite graph Γ_n as follows:
 - ▶ **Vertices:** $\{\hat{\ }, \$\} \cup \{(s, t) \in M \times M : L_n(v) \cap \varphi^{-1}(s)L^*\varphi^{-1}(t) \neq \emptyset\}$
 - ▶ **Edge** $\hat{\ } \rightarrow (s, t)$ if $(L_n(v))^* \varphi^{-1}(s) \cap L \neq \emptyset$.
 - ▶ **Edge** $(s, t) \rightarrow \$$ dually.
 - ▶ **Edges** $(s_1, t_1) \rightarrow (s_2, t_2)$ if $\varphi^{-1}(t_1)(L_n(v))^* \varphi^{-1}(s_2) \cap L \neq \emptyset$.
- ▶ The 2 factorizations define a path γ_n from $\hat{\ }$ to $\$$ in the graph.

PR algorithm for A: handling iteration:

$$\text{inclusion } \text{cl}_{\kappa, A}(L^+) \subseteq \langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}$$

- ▶ Since the number of vertices is fixed, one can assume that the set of vertices and edges (“support”) used by the paths γ_n is **constant**.
- ▶ **First case**: this support if paths γ_n has no cycle. In this case, all paths γ_n are the same simple path from $\hat{}$ to $\$$.
- ▶ We deduce for each n sequences of the length of that path $(x_{i,n})_i$ and $(y_{i,n})_i$ corresponding to edges and vertices of the path.
- ▶ $x_{i,n} \in L$ so $\lim_n x_{i,n} = x_i \in \text{cl}_A(L)$,
- ▶ $y_{i,n} \in L^* \cap X^* L_n(v) X^*$, so it converges to $y_i \in \text{cl}_A(L^*)$ and **has rank less than that of w** . **Induction**: $y_i \in \langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}$
- ▶ Therefore $w = x_{1,n} y_{1,n} x_{2,n} y_{2,n} \cdots$ is also in $\langle \text{cl}_{\kappa, A}(L) \rangle_{\kappa, A}$.

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- ▶ Since the number of vertices is fixed, one can assume that the set of vertices and edges (“support”) used by the paths γ_n is **constant**.
- ▶ **Second case**: this support has a **loop**. Extracting if necessary, one can assume that all γ_n have the same prefix up to the **same simple loop**.

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 - ▶ **cut** other loops while staying in $L^+ \Rightarrow$ bounded number of factors.

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 - ▶ **pump** this loop with all factors staying in L^+ : replace the loop by its ω -power. \Rightarrow as many copies of words of $L_n(v)$ as wanted, ensuring convergence to w .

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Proposition

The pseudovariety \mathbf{R} is κ -full.

Proof by induction on X , using the algebraic structure of $\overline{\Omega}_X \mathbf{R}$.

Corollary

The Pin-Reutenauer algorithm holds for the pseudovariety \mathbf{R} and the canonical signature κ .

Proof using the inheritance theorem for κ -full pseudovarieties.

Two natural questions

1. Automata for term languages.

- ▶ (Henckell's algorithm) Given regular $K, L \subseteq X^+$, one can decide whether

$$\text{cl}_{\mathbf{A}}(K) \cap \text{cl}_{\mathbf{A}}(L) = \emptyset.$$

- ▶ By a weak form of κ -reducibility for \mathbf{A} , this is equivalent

$$\text{cl}_{\kappa, \mathbf{A}}(K) \cap \text{cl}_{\kappa, \mathbf{A}}(L) = \emptyset.$$

Is it possible to test it using automata accepting languages in $\Omega_X^\kappa \mathbf{A}$?

- ## 2. The pseudovariety \mathbf{S} of all finite semigroups is σ -full, for every σ . Does the Pin-Reutenauer algorithm hold for \mathbf{S} and κ ?