

Decision problems for classes of semigroups and rational languages

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Outline

1 Motivations and context

Pseudovarieties of semigroups: decidability issues

Equation systems and reducibility

The case of R : proof outline

Framework: classification of regular languages

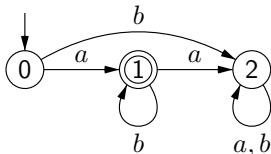
- From logical definability/combinatorial properties to algebraic properties.

High-level description
of rational language L

$$e = ab^*, \varphi = a \wedge XGb$$

Logics,
combinatorics.

Minimal automaton $\mathcal{A}_{\min}(L)$



Syntactic semigroup $M(L)$
(finite, [K56])

	0	1	2
a	1	2	2
b	2	1	2

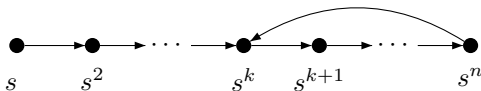
$$\{a, b, a^2\}$$
$$ab = a, ba = a^2,$$
$$b^2 = b$$

Algebra

- Is it possible to recover some properties of L from $S(L)$?

Semigroups

- ▶ Semigroup: (S, \cdot) where \cdot is associative.
- ▶ Examples: A^+ (free semigroup), A^* , square matrices,...
- ▶ Idempotent $e \in S$: $e^2 = e$.
- ▶ In a finite semigroup, $s^\omega =$ unique idempotent of $\{s, s^2, s^3, \dots\}$.



High-level vs. algebraic properties

Language L	$\mathcal{A}_{\min}(L)$	$M(L)$	Logical definability
Star free	Counter free	A	FO($<$), LTL
[Sch65,McN-P71,K68]			
$\bigcup_f \Pi$ non ambiguous	2-way part. ord.	DA	FO ₂ ($<$), UTL, $\Sigma_2 \cap \Pi_2$
[Sch76,SchThV01,ThW98,EVW97,PW97]			
Loc. threshold testable	Forbidden patterns	ACom * LI	FO(\leq)
[S85,ThW85,BP89]			
$\bigcup_f \Pi$ left det.	Very weak	R	
[Ei74]			
Piecewise testable	Very weak + ...	J	Bool(Σ_1)

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Motivations and context

- 2 Pseudovarieties of semigroups: decidability issues

Equation systems and reducibility

The case of R : proof outline

Decidability through operators

- ▶ Pseudovarieties arising from natural properties are usually decidable.
- ▶ Common operators do **not** preserve decidability [ABR92,R99]. $\text{Com} \vee V$.
- ▶ **Idea**: strengthen decidability property to gain preservation through operators.
- ▶ Several successive attempts during the last decade.
- ▶ **Tameness** [AS98,A02]
 - ▶ a property of “uniform resolution” of equation systems.
 - ▶ a word problem.

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The case of R : proof outline

How systems of equations appear

- ▶ The *Basis Theorem* for semidirect products gives

$$\mathbf{SI} * \mathbf{V} = \llbracket wu^2 = wu, wuv = wvu : \mathbf{V} \models wu = wv = w \rrbracket.$$

- ▶ To check that a finite semigroup S belongs to $\mathbf{SI} * \mathbf{V}$, it suffices to verify:

If $\bar{w}, \bar{u}, \bar{v} \in S$ are such that $\bar{w}\bar{u}^2 \neq \bar{w}\bar{u}$ or $\bar{w}\bar{u}\bar{v} \neq \bar{w}\bar{v}\bar{u}$, then there are no pseudowords $w, u, v \in \widehat{F}_A S$ and evaluation of the generators A in S such that:

1. w, u, v are evaluated to $\bar{w}, \bar{u}, \bar{v}$, respectively;
2. $\mathbf{V} \models wu = wv = w$.

Thus, we have the system of equations

$$zx = zy = z$$

upon whose variables x, y, z we impose constraints in the semigroup S .

- ▶ We want to decide whether there is a *solution of the system modulo \mathbf{V}* .
- ▶ Equation systems also appear for other operators. Mal'cev products:

$$\mathbf{SI} \textcircled{m} \mathbf{V} = \llbracket u^2 = u, uv = vu : \mathbf{V} \models u^2 = u = v \rrbracket.$$

Semi-algorithm for non-solvability

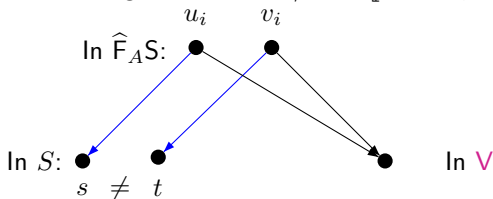
- ▶ If the system has a solution in $\widehat{F}_A S$ modulo V then it also has a solution modulo any A -generated semigroup from V .
- ▶ By a **compactness argument**, the converse is also true.
- ▶ For a specific A -generated semigroup T from V , existence of solutions modulo T can be determined by checking a **finite number of candidates**.
- ▶ Semi-algorithm to enumerate non-solvable systems of equations:
 - ▶ enumerate all $(A, X, (u_i = v_i)_{i \in I}, S, \psi, (s_x)_{x \in X}, T)$, where
 - ▶ A and X are finite sets,
 - ▶ $(u_i = v_i)_{i \in I}$ is a system of word equations,
 - ▶ S is an A -generated finite semigroup and $\psi : A^+ \rightarrow S$
 - ▶ s_x is a constraint for variable $x \in X$,
 - ▶ T is an A -generated finite semigroup from V .
 - ▶ for each such tuple, test whether the system has a solution mod T .

Reducibility and ω -word problem vs. decidability

- ▶ Sufficient condition for decidability, more robust under operators.
- ▶ If V is recursively enumerable, reducible, and has a decidable κ -word problem, then V is decidable (in fact, much more [AS98]).

Proof

- ▶ Semi-alg. to decide $S \in V$ since V is recursively enumerable.
- ▶ Semi-alg. to decide $S \notin V = \llbracket u_i = v_i, i \in I \rrbracket$: $S \notin V$ iff $\exists i, S \not\models u_i = v_i$.



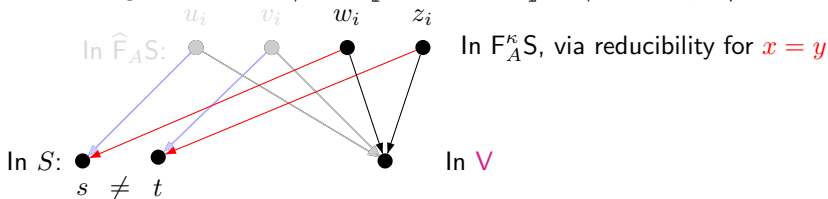
- ▶ Enumerate tuples $(w, z, s, t) \in F_A^{\kappa^2} \times S^2$.
- ▶ Test equality on V (κ -word problem) and S .

Reducibility and ω -word problem vs. decidability

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- ▶ Test equality on \mathbf{V} (κ -word problem) and S .

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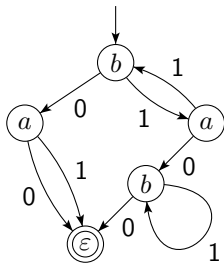
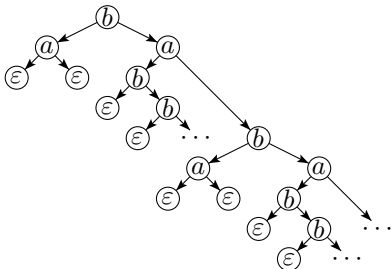
Equation systems and reducibility

4 The case of R: proof outline

Word problem for κ -terms over \mathbf{R}

- ▶ Iterate left-basic factorization on κ -terms to test equality.
- ▶ Construction of a characteristic DFA from a κ -term.

Exemple $(ab^\omega a)^\omega$.



Theorem The word problem over \mathbf{R} for κ -terms u, v can be solved in $O(|A|(|u| + |v|))$ time.

No rewriting system found.

Long words [BC01]

Labeled posets endowed with

- ▶ series concatenation $P \cdot Q$. Order : $P < Q$.
- ▶ ω -power, $P^\omega = \omega \times P$. Lexicographic order.

$W(A)$ = subalgebra generated by singletons $\{a\}$, $a \in A$.

Long words

- ▶ The variety \mathcal{V} generated by $W(A)$ is defined
 - ▶ $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
 - ▶ $(x \cdot y)^\omega = x \cdot (y \cdot x)^\omega$
 - ▶ $(x^n)^\omega = x^\omega$ pour $n \geq 2$.
- ▶ The variety \mathcal{V} has **no finite basis**.
- ▶ **Word problem for ω -terms** : $O(|u|^2|v|^2)$ -time decidable.

\mathbf{R}^κ

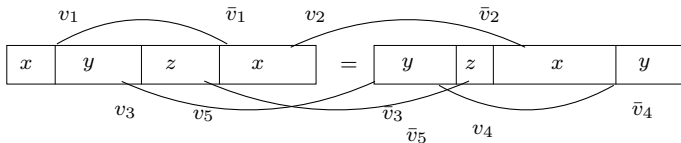
- ▶ The variety \mathbf{R}^κ of ω -semigroups generated by \mathbf{R} has the following basis.
 - ▶ $(xy)^\omega x^\omega = (xy)^\omega x = x(yx)^\omega = (xy)^\omega$
 - ▶ $(x^\omega)^\omega = x^\omega$
 - ▶ $(x^r)^\omega = x^\omega$ pour $n \geq 2$
- ▶ The variety \mathbf{R}^κ has **no finite basis**.

Complete κ -reducibility of \mathbf{R} : proof technique

Theorem [J. Almeida, J.C. Costa, MZ]

The pseudovariety \mathbf{R} is completely κ reducible.

- ▶ Values of variables in different positions are matched in pairs:



- ▶ Each box is identified by the position of its beginning together with the new variable that determines it:

i_0						v_0	i_4							\bar{v}_0			
i_0	v_1	i_1		v_3	i_2	v_5	i_3	v_2	i_4	v_4	i_5	\bar{v}_5	i_6		\bar{v}_2	i_7	\bar{v}_4
							i_3	\bar{v}_1	i_4	\bar{v}_3							

- ▶ A quadruple of the form (i, v, j, \bar{v}) is called a *boundary equation*.

