

TAMENESS OF SOME LOCALLY TRIVIAL PSEUDOVARITIES

JORGE ALMEIDA
Centro de Matemática
Departamento de Matemática Pura
Faculdade de Ciências, Universidade do Porto
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
jalmeida@fc.up.pt

MARC ZEITOUN
LIAFA
Université Paris 7
2, place Jussieu
75251 Paris Cedex 05
mz@liafa.jussieu.fr

Abstract

Tameness is a strong property of semigroup pseudovarieties related to the membership problem. Let κ be the signature comprising semigroup multiplication and the omega implicit operation. We prove the κ -tameness of the pseudovarieties **N**, **D**, **K** and **LI**.

1 Introduction

Given a class \mathbf{V} of finite semigroups, solving the membership problem for \mathbf{V} consists in devising an algorithm to test whether a given semigroup lies in \mathbf{V} . The class is said to be decidable when such an algorithm exists. The motivation of this notion originates from a seminal result of Eilenberg [1] which states a correspondence between certain classes of semigroups, named pseudovarieties, and varieties of formal languages. In this way, syntactic properties of a rational language can be tested algorithmically on its syntactic semigroup by solving a membership problem. Several connections also

link pseudovarieties to other domains of mathematics and computer science: twenty years after Eilenberg's theorem, fruitful correspondences between finite semigroups, rational languages, first order logic, second order monadic logic and temporal logic still motivate intensive work. See [2, 3, 4, 5, 6] for different presentations of these relationships.

Pseudovarieties are frequently defined by means of unary or binary operators. Three of these operators, the join, the semidirect product and the Mal'cev product do not preserve decidability: Albert, Baldinger and Rhodes [7] exhibited two decidable pseudovarieties whose join is not decidable; in the same paper, they announced that their result could be translated to prove that the semidirect product does not preserve decidability, a task achieved by Rhodes [8]. Rhodes also gave in this paper an example of two decidable pseudovarieties whose Mal'cev product is undecidable. The proofs rely on *ad hoc* constructions and coding of Minsky machines. For other operators (such as the power), there is no such result at present. Still, it is often a challenge to show that a pseudovariety constructed from decidable pseudovarieties is decidable, even if it is very unlikely that *usual* pseudovarieties give rise to undecidable ones. See [9].

To overcome this problem, a new idea which emerged quite recently is to study stronger conditions, expecting them to be more robust than decidability. Ideally, such a condition

- should imply decidability,
- should provide ways to show that, at least in particular cases, it is preserved by operators,
- should be reasonable: most of the interesting, useful pseudovarieties should enjoy it.

With the introduction of pointlike sets, Henckell [10] proposed in 1988 such an approach. He proved that the pseudovariety \mathbf{A} of aperiodic (group-free) semigroups has a strong property which implies decidability. Shortly after, in 1991, Ash introduced in an outstanding paper [11] another concept involving graphs to answer Rhodes's type II conjecture. His definition was well-suited for this purpose, but unfortunately, it was designed to fit the very particular framework of finite groups. The first author [12] found an appropriate generalization of Ash's definition which makes sense for all semigroup pseudovarieties. This generalization still implies decidability, and he proposed

to name it *hyperdecidability*. It has in fact several variations, depending on which operators and pseudovarieties are involved. It turns out that Henckell's definition is also a particular case of hyperdecidability.

Hyperdecidability already proved to be very helpful. The intent of the first author was primarily to decide semidirect products, but results concerning joins can also be obtained using hyperdecidability [12, 13, 14]. Steinberg and the first author [15] refined the notion of hyperdecidability to what they called tameness [16], a property in which the essential decidability component is a word problem.

A drawback of the notion of tameness is that it involves elaborate techniques. Proving tameness of a simple pseudovariety might be difficult (at least with the currently available tools), while decidability of any usual pseudovariety is trivial. Yet, some important pseudovarieties are already known to be tame (for instance \mathbf{G} , the pseudovariety of all finite groups: this is Ash's result) and it is expected that most common pseudovarieties are tame. Thus, proving tameness of pseudovarieties is a nontrivial and useful endeavor.

Here we address the question of the tameness of \mathbf{N} , \mathbf{D} , \mathbf{K} and \mathbf{LI} , four well-known and simple pseudovarieties. They are defined as follows: \mathbf{N} is the class of all nilpotent semigroups (whose unique idempotent is a zero); \mathbf{D} (resp. \mathbf{K}), is the class of all semigroups in which any idempotent is a right (resp. a left) zero, and \mathbf{LI} is the smallest pseudovariety containing both \mathbf{D} and \mathbf{K} . We prove that they are all tame.

The paper is organized as follows. In Section 2, we lay down some notation which will be helpful in the proofs, we set up the terminology and we recall definitions concerning words, semigroups, graphs and pseudovarieties. We next prove that the word problem involved in the definition of κ -tameness is decidable for \mathbf{N} , \mathbf{D} , \mathbf{K} and \mathbf{LI} in Section 3. Finally, we prove that \mathbf{N} , \mathbf{K} , \mathbf{D} and \mathbf{LI} are κ -tame in Subsections 4.1, 4.2, 4.3, and 4.4 respectively.

2 Prerequisites, notation

We assume that the reader is acquainted with finite semigroup theory. He is referred to [17] for an introduction to pseudovarieties, and to [15, 16] for connections between decidability and tameness.

Words

In the sequel, A denotes a finite alphabet. The free semigroup (resp. the free monoid, the free group) generated by A is denoted by A^+ (resp. by A^* , by $FG(A)$). The length of a word x is denoted by $|x|$. The empty word is denoted by 1 .

The set of all right (resp. left) infinite words is denoted by A^ω (resp. by ${}^\omega A$). We let $A^\infty = A^+ \cup A^\omega$ and ${}^\infty A = A^+ \cup {}^\omega A$. A word of $A^\omega \cup {}^\omega A$ is *infinite*, whereas a word of A^* is *finite*. By convention, we index the first letter of any word by 1 . Of course, the first letter of a left infinite word is its rightmost letter. The k^{th} letter of x will be denoted by $(x)_k$, or by x_k for short.

The product of two elements x, y of A^∞ is defined as follows: if $x, y \in A^+$, then xy is defined in the usual way; right infinite words are left zeros; finally, if x is finite and y is infinite, then xy is the right infinite word defined by

$$(xy)_k = \begin{cases} x_k & \text{if } k \leq |x| \\ y_{k-|x|} & \text{otherwise} \end{cases}$$

The multiplication of elements of ${}^\infty A$ is defined symmetrically. It is a straightforward observation that A^ω , ${}^\omega A$, A^∞ and ${}^\infty A$ are semigroups.

Recall that $u \in A^*$ is a *prefix* of $v \in A^\infty$ if there exists w such that $v = uw$. Symmetrically, $u \in A^*$ is a *suffix* of $v \in {}^\infty A$ if there exists w such that $v = wu$.

A right infinite word is *ultimately periodic* if it is of the form $yx^\omega = yxxx \cdots$, with $x \in A^+$ and $y \in A^*$. Ultimately periodic left infinite words are defined symmetrically.

Semigroups

Given a semigroup S , we let S^1 be the semigroup S itself if it is a monoid, or the disjoint union $S \uplus \{1\}$ where 1 acts as a neutral element otherwise. Given an element s of a finite semigroup (resp. of a compact topological semigroup), the subsemigroup (resp. the closed subsemigroup) generated by s contains a unique idempotent, denoted by s^ω . The set of idempotents of a semigroup S is denoted by $E(S)$.

Pseudovarieties, implicit operations

A semigroup *pseudovariety* is a class of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products. In what follows, \mathbf{V} denotes a pseudovariety.

By a *profinite semigroup*, we mean a compact semigroup which is residually finite. Equivalently, profinite semigroups are projective limits of finite semigroups. For a pseudovariety \mathbf{V} , a *pro- \mathbf{V} semigroup* is a profinite semigroup which is residually in \mathbf{V} . The semigroup $\overline{\Omega}_A \mathbf{V}$ is defined to be a pro- \mathbf{V} semigroup which is relatively free with respect to \mathbf{V} in the sense that, for every mapping $\xi : A \rightarrow S$ into a pro- \mathbf{V} semigroup S , there is a unique continuous homomorphism $\hat{\xi} : \overline{\Omega}_A \mathbf{V} \rightarrow S$ extending ξ . This leads to a natural interpretation of every $\pi \in \overline{\Omega}_A \mathbf{V}$ as an operation $\pi_S : S^A \rightarrow S$ which maps $\xi \in S^A$ to $\pi \hat{\xi}$. Elements of $\overline{\Omega}_A \mathbf{V}$ are called *implicit operations on \mathbf{V}* . Elements of the subsemigroup of $\overline{\Omega}_A \mathbf{V}$ generated by A are said to be *explicit*. The reader is referred to [17, 18] for more information on pseudovarieties and implicit operations.

For $\pi \in \overline{\Omega}_A \mathbf{V}$, one can check that $\pi_S : S^A \rightarrow S$ commutes with continuous homomorphisms between pro- \mathbf{V} semigroups, in the sense that if $\varphi : S \rightarrow T$ is such a continuous homomorphism, then $\pi_S \varphi = \varphi^A \pi_T$ (where $\varphi^A : S^A \rightarrow T^A$ maps (x_1, \dots, x_n) to $(x_1 \varphi, \dots, x_n \varphi)$). Conversely, any collection $(\pi_S)_{S \text{ pro-}\mathbf{V}}$ of such mappings commuting with continuous homomorphisms uniquely determines an implicit operation of $\overline{\Omega}_A \mathbf{V}$. Now, for any $\pi \in \overline{\Omega}_A \mathbf{V}$, the mapping $\pi_S^\omega : S \rightarrow S$ which associates to each $s \in S$ the idempotent $(\pi_S(s))^\omega$ commutes with continuous homomorphisms. Thus, the collection (π_S^ω) defines an implicit operation denoted by π^ω . One can also verify that π^ω is the idempotent of the closed subsemigroup generated by π in $\overline{\Omega}_A \mathbf{V}$.

The pseudovariety of all finite semigroups is denoted by \mathbf{S} . We denote by $q_{\mathbf{V}} : \overline{\Omega}_A \mathbf{S} \rightarrow \overline{\Omega}_A \mathbf{V}$ the continuous homomorphism mapping the generators of $\overline{\Omega}_A \mathbf{S}$ to the generators of $\overline{\Omega}_A \mathbf{V}$.

Graphs

In this paper, all graphs considered are finite and directed. This will be understood in the sequel. More precisely, a finite *graph* $\Gamma = \mathcal{V} \uplus \mathcal{E}$ is given by two disjoint finite sets \mathcal{V} and \mathcal{E} , together with two mappings $\alpha, \omega : \mathcal{E} \rightarrow \mathcal{V}$. Elements of \mathcal{V} are the *vertices* of Γ , elements of \mathcal{E} are its *edges*, and for any $e \in \mathcal{E}$, $e\alpha$ (resp. $e\omega$) is the *beginning* (resp. the *end*) of the edge e . We shall also write $\Gamma = (\mathcal{V}, \mathcal{E})$ and $\mathcal{V} = \mathcal{V}(\Gamma)$, $\mathcal{E} = \mathcal{E}(\Gamma)$.

A path is a sequence of edges e_1, \dots, e_k such that $e_i\omega = e_{i+1}\alpha$ for $1 \leq i \leq k-1$. Two paths e_1, \dots, e_k and f_1, \dots, f_ℓ are *coterminal* if $e_1\alpha = f_1\alpha$ and $e_k\omega = f_\ell\omega$.

A graph is *strongly connected* if for any of its vertices v and w , there is a

path from v to w and a path from w to v . It is *connected* if for any two vertices, there is a non-oriented path between them. The *connected component* (resp. *strongly connected component*) of a vertex is the largest connected subgraph (resp. strongly connected subgraph) containing that vertex.

κ -reducibility

We recall in this subsection some fundamental notions due to Steinberg and the first author [15].

An *implicit signature* is a set σ of implicit operations containing the basic semigroup multiplication, \cdot . In this paper, we consider a canonical such signature, namely:

$$\kappa = \{ \cdot, \omega \}$$

Via the natural interpretation of implicit operations as operations on profinite semigroups, every profinite semigroup S has a natural structure of a σ -semigroup. For a pseudovariety \mathbf{V} , we denote by $\Omega_A^\sigma \mathbf{V}$ the free σ -semigroup in the variety of σ -semigroups generated by \mathbf{V} . This turns out to be the σ -subsemigroup of $\overline{\Omega}_A \mathbf{V}$ generated by A . The σ -word problem for \mathbf{V} is the word problem for the relatively free σ -semigroup $\Omega_A^\sigma \mathbf{V}$, that is, the problem of deciding equality of σ -terms in \mathbf{V} .

A *relational morphism* of semigroups $\mu : S \rightarrow T$ is a relation $\mu \subseteq S \times T$ with domain S which is a subsemigroup. For a finite A -generated semigroup S , the *canonical σ -relational morphism with respect to \mathbf{V}* is the composite $\mu_{\mathbf{V}}^\sigma = \varphi^{-1} p_{\mathbf{V}}$ where $\varphi : \Omega_A^\sigma \mathbf{S} \rightarrow S$ and $p_{\mathbf{V}} : \Omega_A^\sigma \mathbf{S} \rightarrow \Omega_A^\sigma \mathbf{V}$ are the homomorphisms of σ -semigroups determined by the choice of generators. In the same setting, the *canonical relational morphism with respect to \mathbf{V}* is the composite $\mu_{\mathbf{V}} = \psi^{-1} q_{\mathbf{V}}$ where $\psi : \overline{\Omega}_A \mathbf{S} \rightarrow S$ and $q_{\mathbf{V}} : \overline{\Omega}_A \mathbf{S} \rightarrow \overline{\Omega}_A \mathbf{V}$ are the continuous homomorphisms determined by the choice of generators.

A *labeling* of a graph Γ by a semigroup S is a mapping γ from Γ into S^1 such that $\mathcal{E}(\Gamma)\gamma \subseteq S$. The *label* of a path e_1, \dots, e_k is by definition $e_1\gamma \cdots e_k\gamma$. A labeling γ is said to be *consistent* if $e\alpha\gamma \cdot e\gamma = e\omega\gamma$ holds for every edge e . Following Ash [11], if $\eta : \mathcal{E}(\Gamma) \rightarrow A^+$ is a function, we define the *label* of a non-oriented path e_1, \dots, e_k to be the reduced form of the group word $(e_1\eta)^{\varepsilon_1} \cdots (e_k\eta)^{\varepsilon_k} \in FG(A)$ where $\varepsilon_i = 1$ if the edge e_i is read in the direction of the path and $\varepsilon_i = -1$ otherwise. We say that such a function η *commutes* if the label of any non-oriented cycle is 1. If η is the restriction to edges of a labeling γ of Γ , then we also say that γ commutes if η commutes.

We say that two labelings γ and δ of a graph Γ respectively by semigroups

S and T are μ -related, where $\mu : S \rightarrow T$ is a relational morphism, if for every $g \in \Gamma$ the pair $(g\gamma, g\delta)$ belongs to $\mu \cup \{(1, 1)\}$. A labeling of a graph Γ by S is said to be μ -inevitable if there is a μ -related consistent labeling of Γ by T .

For a pseudovariety \mathbf{V} , we say that a labeling of a graph by a finite semigroup S is \mathbf{V} -inevitable if it is μ -inevitable for every relational morphism $\mu : S \rightarrow T$ into a semigroup T from \mathbf{V} . By a compactness result due to the first author [19], a labeling of a finite graph by a finite semigroup S is \mathbf{V} -inevitable if and only if it is $\mu_{\mathbf{V}}$ -inevitable. In other words:

Proposition 2.1 *A labeling γ of a finite graph Γ by a finite A -generated semigroup S is \mathbf{V} -inevitable if and only if there exists a labeling δ of Γ by $\overline{\Omega}_A \mathbf{S}$ such that $\delta\psi = \gamma$, where the homomorphism $\psi : \overline{\Omega}_A \mathbf{S} \rightarrow S$ respects the choice of generators and $\delta q_{\mathbf{V}}$ is consistent. \square*

We say that \mathbf{V} is σ -reducible if every \mathbf{V} -inevitable labeling of a finite graph by a finite semigroup S is $\mu_{\mathbf{V}}^{\sigma}$ -inevitable.

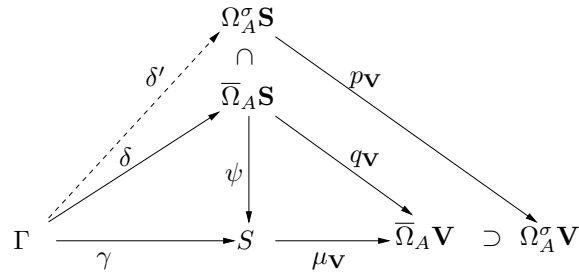


Figure 1: A commutating diagram expressing σ -reducibility

A recursively enumerable pseudovariety \mathbf{V} is said to be σ -tame if it is σ -reducible and the σ -word problem for \mathbf{V} is decidable. We say that \mathbf{V} is tame if it is σ -tame for some recursively enumerable implicit signature σ consisting of computable implicit operations.

Basic lemmas

This section gathers elementary statements that we use in the paper.

Reiterman's theorem [20] states that a pseudovariety is defined by *pseudoidentities* — formal identities whose members are implicit operations. It is analogous to Birkhoff's theorem on varieties. See [18] for full developments.

A finite semigroup S satisfies a pseudoidentity $\pi = \rho$ if $\pi_S = \rho_S$. The class of finite semigroups satisfying a set of pseudoidentities Σ is denoted by $[[\Sigma]]$. We are interested in several pseudovarieties:

- $\mathbf{N} = [[xy^\omega = y^\omega = y^\omega x]]$: the class of all finite nilpotent semigroups.
- $\mathbf{D} = [[xy^\omega = y^\omega]]$: the class of all finite semigroups in which any idempotent is a right zero.
- $\mathbf{K} = [[y^\omega x = y^\omega]]$: the class of all finite semigroups in which any idempotent is a left zero.
- $\mathbf{LI} = [[y^\omega xz^\omega = y^\omega z^\omega]]$: the class of all finite locally trivial semigroups.

Note that \mathbf{LI} is the join of \mathbf{K} and \mathbf{D} (that is, the smallest pseudovariety containing both \mathbf{K} and \mathbf{D}). Another important fact is that $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$, and so each of these pseudovarieties contains \mathbf{N} . The following lemma summarizes basic algebraic properties of our pseudovarieties. See [21] for details.

Lemma 2.2 *Let \mathbf{V} be one of the pseudovarieties \mathbf{N} , \mathbf{K} , \mathbf{D} or \mathbf{LI} .*

1. *The semigroup of explicit operations on \mathbf{V} is isomorphic to A^+ .*
2. *An element of $\overline{\Omega}_A \mathbf{V}$ is either explicit or idempotent.* □

We now describe the idempotents, how they multiply and how they relate with explicit operations.

Lemma 2.3 *Let \mathbf{V} be one of the pseudovarieties \mathbf{N} , \mathbf{K} , \mathbf{D} or \mathbf{LI} . Then $E(\overline{\Omega}_A \mathbf{V})$ is an ideal. In particular, it is a semigroup. Furthermore,*

1. *The unique idempotent of $\overline{\Omega}_A \mathbf{N}$ is a zero.*
2. *$E(\overline{\Omega}_A \mathbf{K})$ is isomorphic to A^ω , and $\overline{\Omega}_A \mathbf{K}$ is isomorphic to A^∞ .*
3. *$E(\overline{\Omega}_A \mathbf{D})$ is isomorphic to ${}^\omega A$, and $\overline{\Omega}_A \mathbf{D}$ is isomorphic to ${}^\infty A$.*
4. *$E(\overline{\Omega}_A \mathbf{LI})$ is isomorphic to $A^\omega \times {}^\omega A$.* □

In view of statement 1 of Lemma 2.2, we shall identify explicit operations (on \mathbf{N} , \mathbf{K} , \mathbf{D} or \mathbf{LI}) with words. In view of Lemma 2.3 (statements 2 and 3), we may also identify idempotents of \mathbf{K} (resp. of \mathbf{D}) with right infinite words (resp. with left infinite words). Note that in $\overline{\Omega}_A\mathbf{K}$, the right infinite word $yxxx\cdots$ ($y \in A^*$, $x \in A^+$) corresponds to the implicit operation yx^ω . We can therefore use the notation x^ω safely. The dual remark holds for \mathbf{D} .

The next result follows immediately from [17, Corollary 5.6.2 (c)] and from the fact that A^+ is dense in $\overline{\Omega}_A\mathbf{S}$:

Lemma 2.4 *Let $\pi \in \overline{\Omega}_A\mathbf{S}$ be non-explicit and let S be a finite semigroup. Then there exist $x, y, z \in A^*$ with $y \neq 1$ such that S satisfies $\pi = xy^\omega z$. \square*

There is a partial action of $FG(A)$ on ${}^\infty A = \overline{\Omega}_A\mathbf{D}$ defined by $\pi \cdot a = \pi a$ and $(\pi a) \cdot a^{-1} = \pi$ ($a \in A$, $\pi \in \overline{\Omega}_A\mathbf{D}$).

Lemma 2.5 *Let Γ be a connected graph, and let δ be a labeling of Γ by $\overline{\Omega}_A\mathbf{D}$ such that*

1. δ is consistent,
2. δ maps edges to explicit operations,
3. δ does not commute.

Then, δ maps any vertex to an implicit operation of the form $u^\omega v$, where $u \neq 1$ and v are words, with u fixed.

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_k$ be a non-oriented cycle having a label $h \neq 1$. Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k = \mathbf{v}_0$ be the sequence of vertices along this cycle. By consistency of δ , we have $\mathbf{v}_{i+1}\delta = \mathbf{v}_i\delta \cdot (\mathbf{e}_i\delta)^{\epsilon_i}$, where $\epsilon_i = 1$ if the edge \mathbf{e}_i is read in the direction of the path and $\epsilon_i = -1$ otherwise. Hence, $x = x \cdot h$, where $x = \mathbf{v}_0\delta$. Set $h = a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$, with $a_i \in A$ and $\epsilon_i \in \{-1, 1\}$.

We first show that x is ultimately periodic. If h or h^{-1} belongs to A^+ , then there is a word $u \in A^+$ such that $x = x \cdot u$, hence $x = u^\omega$. Otherwise, let $i = \min\{\ell \mid \epsilon_\ell \epsilon_{\ell+1} = -1\}$. Since h is reduced, we have $a_i \neq a_{i+1}$. Observe that there exists no ℓ such that $\epsilon_\ell = 1$, $\epsilon_{\ell+1} = -1$, since otherwise the action of $a_{\ell+1}^{-1}$ on $x \cdot a_1^{\epsilon_1} \cdots a_\ell^{\epsilon_\ell}$ would not be defined. So $\epsilon_1 = \cdots = \epsilon_i = -1$, and $\epsilon_{i+1} = \cdots = \epsilon_\ell = 1$. Therefore, $a_i \cdots a_1$ is a suffix of x , so let $x = x' a_i \cdots a_1$. We then have $x \cdot h = x$, so

$$x' a_i \cdots a_1 = x' a_{i+1} \cdots a_n$$

We cannot have $i = n - i$, otherwise a_i would be equal to a_{i+1} . Therefore, the previous equality implies that there exists $u \in A^+$ such that $x' = x'u$, so $x' = u^\omega$ and $x = u^\omega v$ where $v = a_i \cdots a_1$.

Suppose now that a vertex \mathbf{v} is labeled by an implicit operation of the form $u^\omega v$. If there is an oriented path from \mathbf{v} to a vertex \mathbf{v}' labeled w , then $\mathbf{v}'\delta = u^\omega v w$. On the other hand, if there is an oriented path from a vertex \mathbf{v}' to \mathbf{v} labeled w , then $\mathbf{v}'\delta \cdot w = u^\omega v$, which implies that w is a suffix of $u^m v$ for a sufficiently large m , and therefore, by canceling w , that $\mathbf{v}'\delta$ is also of the form $u^\omega t$ for some word t .

Since Γ is connected, all vertices are labeled by implicit operations of the form $u^\omega v$ with the same u . \square

Lemma 2.6 *Let $\pi, \rho \in \overline{\Omega}_A \mathbf{S}$ be non-explicit operations such that $\pi q_{\mathbf{D}} = \rho q_{\mathbf{D}}$. Then there exist factorizations $\pi = \pi_1 \pi_2$ and $\rho = \rho_1 \pi_2$ where π_2 is non-explicit. Moreover, if $\pi q_{\mathbf{D}} = u^\omega v$ where $u \neq 1$ and v are explicit operations, then one can choose $\pi_2 = u^\omega v$.*

Proof. Since π and ρ are non-explicit, there exist sequences (u_n) and (v_n) of words of increasing length converging respectively to π and ρ . Since $\pi q_{\mathbf{D}} = \rho q_{\mathbf{D}}$, we can assume that the words u_n and v_n have the same suffix of length n . Furthermore, if $\pi q_{\mathbf{D}} = \rho q_{\mathbf{D}} = u^\omega v$, then one can assume that $u^n v$ is a suffix of both u_n and v_n . Both statements follow then by a standard compactness argument. \square

3 The κ -word problem

In this section, we observe that the solution of the κ -word problem for \mathbf{N} , \mathbf{K} , \mathbf{D} and \mathbf{LI} is very easy.

Let \mathbf{V} be one of the pseudovarieties \mathbf{N} , \mathbf{K} , \mathbf{D} and \mathbf{LI} . We first observe that a κ -term which involves the operation $_{-}^\omega$ is an idempotent over \mathbf{V} . Secondly, by Lemma 2.2, two κ -terms which do not involve the operation $_{-}^\omega$ are equal over \mathbf{V} if and only if they are the same term. Thirdly, by the same lemma, if two κ -terms are equal over \mathbf{V} , then they either both involve the operation $_{-}^\omega$ or neither does. This reduces the κ -word problem for \mathbf{V} to κ -terms involving the operation $_{-}^\omega$. In the case $\mathbf{V} = \mathbf{N}$, all such κ -terms are equal over \mathbf{N} , namely equal to 0. Since \mathbf{LI} is the join of \mathbf{K} and \mathbf{D} , the κ -word problem for \mathbf{LI} reduces to that for \mathbf{K} and for \mathbf{D} . Moreover, since \mathbf{K} and \mathbf{D} are dual, it suffices to solve the κ -word problem for \mathbf{D} .

To solve the κ -word problem for \mathbf{D} , we consider the following reduction rules, where u, v and w denote elements of A^* :

$$vu^\omega \rightarrow u^\omega \quad u, v \neq 1, \quad (1)$$

$$(vu^\omega w)^\omega \rightarrow vu^\omega w \quad u \neq 1, \quad (2)$$

$$(u^n)^\omega \rightarrow u^\omega \quad u \neq 1, \quad (3)$$

$$(uv)^\omega u \rightarrow (vu)^\omega \quad u \neq 1. \quad (4)$$

This is clearly a Noetherian system since the rules reduce the length of terms. Say that a term is *irreducible* if it is not possible to apply a reduction rule to it. By rules 1 and 2, irreducible terms are of the form $u^\omega v$ or v . By rule 3, u has to be a primitive word. By rule 4, no nontrivial prefix of v in $u^\omega v$ can be a prefix of u . By associativity, and since idempotents are right zeros in \mathbf{D} , the two sides of any of the above rules clearly coincide over \mathbf{D} . To conclude, it suffices to show that two irreducible terms $u_1^\omega v_1, u_2^\omega v_2$ are equal over \mathbf{D} if and only if $u_1 = u_2$ and $v_1 = v_2$. This follows from Lemma 2.3 (3).

4 κ -reducibility

In this section, we show that the pseudovarieties \mathbf{N} , \mathbf{K} , \mathbf{D} and \mathbf{LI} are κ -reducible.

Throughout this section, whenever S is a finite A -generated semigroup, we let ψ denote the unique homomorphism from $\overline{\Omega}_A \mathbf{S}$ to S which respects the choice of generators.

4.1 The case of \mathbf{N}

Let γ be an \mathbf{N} -inevitable labeling of a finite graph Γ by a finite A -generated semigroup S . By Proposition 2.1, there is a labeling δ of Γ by $\overline{\Omega}_A \mathbf{S}$ such that $\delta\psi = \gamma$ and $\delta q_{\mathbf{N}}$ is consistent. We have to construct a labeling $\delta' : \Gamma \rightarrow \Omega_A^\kappa \mathbf{S}$ such that $\delta'\psi = \gamma$ and $\delta' p_{\mathbf{N}}$ is consistent (see Figure 1). Note that by Lemma 2.4, the following conditions are equivalent for an element $s \in S$:

- $s = \pi\psi$ for some $\pi \in \overline{\Omega}_A \mathbf{S}$ which is not explicit;
- $s = \pi\psi$ for some $\pi \in \Omega_A^\kappa \mathbf{S}$ which is not explicit.

From here it follows that for any edge and vertex whose label under δ is non-explicit, we can relabel it by a non-explicit κ -term without affecting the equality $\delta\psi = \gamma$ or the consistency of $\delta q_{\mathbf{N}}$. This proves the following theorem:

Theorem 4.1 *The pseudovariety \mathbf{N} is κ -tame.* □

4.2 The case of \mathbf{K}

Let γ be a \mathbf{K} -inevitable labeling of a finite graph Γ by a finite A -generated semigroup S . By Proposition 2.1, there is a labeling δ of Γ by $\overline{\Omega}_A \mathbf{S}$ such that $\delta\psi = \gamma$ and $\delta q_{\mathbf{K}}$ is consistent. We have to construct a labeling $\delta' : \Gamma \rightarrow \Omega_A^{\kappa} \mathbf{S}$ such that $\delta'\psi = \gamma$ and $\delta' p_{\mathbf{K}}$ is consistent (see Figure 1).

Let \mathcal{V}_{ω} be the subset of $\mathcal{V}(\Gamma)$ consisting of all vertices labeled under δ by a non-explicit operation. Let θ be the equivalence relation on \mathcal{V}_{ω} generated by the relation

$$\{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v}, \mathbf{w} \in \mathcal{V}_{\omega} \text{ and there is an edge from } \mathbf{v} \text{ to } \mathbf{w}\}$$

and denote by $\mathbf{v}\theta$ the θ -class of \mathbf{v} . Finally, let $\mathcal{E}_{\mathbf{v}\theta}$ be the set of all edges such that $\mathbf{e}\alpha \notin \mathcal{V}_{\omega}$ and $\mathbf{e}\omega \theta \mathbf{v}$.

Let $\mathbf{v} \in \mathcal{V}_{\omega}$. For any edge $\mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}$, $\mathbf{e}\alpha\delta$ is explicit and $\mathbf{e}\delta$ is not. Let

$$m_{\mathbf{v}\theta} = \max\{|\mathbf{e}\alpha\delta| \mid \mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}\}$$

and let $u_{\mathbf{v}\theta}$ be the prefix of length $m_{\mathbf{v}\theta}$ of $\mathbf{v}\delta q_{\mathbf{K}}$. Then by consistency of $\delta q_{\mathbf{K}}$ and by the choice of $m_{\mathbf{v}\theta}$, the non-explicit operation $\pi_{\mathbf{v}\theta} = u_{\mathbf{v}\theta}^{-1}(\mathbf{v}\delta q_{\mathbf{K}})$ is a suffix which is common to $\mathbf{v}\delta q_{\mathbf{K}}$ and to all $\mathbf{e}\delta q_{\mathbf{K}}$ such that $\mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}$, so that

$$\mathbf{v}\delta q_{\mathbf{K}} = u_{\mathbf{v}\theta} \pi_{\mathbf{v}\theta}; \tag{5}$$

moreover, for each $\mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}$, there is a factorization

$$\mathbf{e}\delta q_{\mathbf{K}} = u_{\mathbf{e}} \pi_{\mathbf{v}\theta} \tag{6}$$

such that

$$\mathbf{e}\alpha\delta \cdot u_{\mathbf{e}} = u_{\mathbf{v}\theta} \text{ for every edge } \mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}. \tag{7}$$

In view of (5) and (6), and by the dual of Lemma 2.6, there exist implicit operations $\rho_{\mathbf{e}}$, $\rho_{\mathbf{v}\theta}$ and a non-explicit operation $\sigma_{\mathbf{v}\theta}$ such that

$$\begin{aligned} \mathbf{v}\delta &= u_{\mathbf{v}\theta} \cdot \sigma_{\mathbf{v}\theta} \cdot \rho_{\mathbf{v}\theta} \\ \mathbf{e}\delta &= u_{\mathbf{e}} \cdot \sigma_{\mathbf{v}\theta} \cdot \rho_{\mathbf{e}} \text{ for every edge } \mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}. \end{aligned}$$

By Lemma 2.4, there exist words $w_{\mathbf{v}\theta}, w_{\mathbf{e}}, x_{\mathbf{v}\theta}, y_{\mathbf{v}\theta}$ and $z_{\mathbf{v}\theta}$ such that S satisfies $\sigma_{\mathbf{v}\theta} = x_{\mathbf{v}\theta}y_{\mathbf{v}\theta}^{\omega}z_{\mathbf{v}\theta}$, $\rho_{\mathbf{v}\theta} = w_{\mathbf{v}\theta}$ and $\rho_{\mathbf{e}} = w_{\mathbf{e}}$. We define δ' to be the labeling of Γ by $\Omega_A^{\kappa}\mathbf{K}$ as follows :

- δ' coincides with δ on vertices and edges labeled by explicit operations,
- $\mathbf{v}\delta' = u_{\mathbf{v}\theta}x_{\mathbf{v}\theta}y_{\mathbf{v}\theta}^{\omega}z_{\mathbf{v}\theta}w_{\mathbf{v}\theta}$ for any vertex $\mathbf{v} \in \mathcal{V}_{\omega}$,
- $\mathbf{e}\delta' = u_{\mathbf{e}}x_{\mathbf{v}\theta}y_{\mathbf{v}\theta}^{\omega}z_{\mathbf{v}\theta}w_{\mathbf{e}}$ for any edge $\mathbf{e} \in \mathcal{E}_{\mathbf{v}\theta}$,
- $\mathbf{e}\delta'$ is any explicit operation which coincides with $\mathbf{e}\delta$ over S , for any \mathbf{e} such that $\mathbf{e}\alpha \in \mathcal{V}_{\omega}$.

By construction, $\delta'\psi = \gamma$ and, by (7), $\delta'q_{\mathbf{K}}$ is consistent. This proves the following theorem:

Theorem 4.2 *The pseudovariety \mathbf{K} is κ -tame.* □

Remark 4.3 *Observe that by construction, for any $g \in \Gamma$, $g\delta'$ is explicit if and only if $g\delta$ is. In this case, $g\delta = g\delta'$. This remark will be used in Sec. 4.4.*

4.3 The case of \mathbf{D}

For any two labelings δ, δ' of the same graph $\Gamma = (\mathcal{V}, \mathcal{E})$ define the condition $\mathcal{C}(\delta, \delta')$ as:

$$\forall \mathbf{v} \in \mathcal{V}, \quad (\mathbf{v}\delta \text{ is explicit} \iff \mathbf{v}\delta' \text{ is explicit}) \quad \mathcal{C}(\delta, \delta')$$

Let γ be a \mathbf{D} -inevitable labeling of a finite graph Γ by a finite A -generated semigroup S . By Proposition 2.1, there is a labeling δ of Γ by $\overline{\Omega}_A\mathbf{S}$ such that $\delta\psi = \gamma$ and $\delta q_{\mathbf{D}}$ is consistent. We have to construct a labeling $\delta' : \Gamma \rightarrow \Omega_A^{\kappa}\mathbf{S}$ such that $\delta'\psi = \gamma$ and $\delta'p_{\mathbf{D}}$ is consistent (see Figure 1). In fact, for a technical reason, we want to construct δ' which also satisfies $\mathcal{C}(\delta, \delta')$.

We first reduce to the case in which edges are labeled by explicit operations under δ . If an edge \mathbf{e} is labeled by a non-explicit operation, then by consistency of $\delta q_{\mathbf{D}}$, we have $\mathbf{e}\delta q_{\mathbf{D}} = \mathbf{e}\omega\delta q_{\mathbf{D}}$. By Lemma 2.6, there are factorizations $\mathbf{e}\delta = \pi\rho$ and $\mathbf{e}\omega\delta = \pi'\rho$ with ρ non-explicit. Drop the edge \mathbf{e} and replace the labels under γ and δ of the vertex $\mathbf{e}\omega$ by $\rho\psi$ and ρ , respectively, to obtain labelings γ_1 and δ_1 of $\Gamma \setminus \{\mathbf{e}\}$ such that $\delta_1\psi = \gamma_1$ and $\delta_1 q_{\mathbf{D}}$ is consistent. Assume that there is a labeling δ'_1 of $\Gamma \setminus \{\mathbf{e}\}$ by $\Omega_A^{\kappa}\mathbf{S}$ such that

a) $\delta'_1\psi = \gamma_1$,

b) $\delta'_1 q_{\mathbf{D}}$ is consistent,

c) $\mathcal{C}(\delta_1, \delta'_1)$ holds.

Let δ' be the extension of $\delta'_1|_{\Gamma \setminus \{e, e\omega\}}$ to Γ obtained by letting $e\delta' = u(e\omega\delta'_1)$ and $e\omega\delta' = u'(e\omega\delta'_1)$, where u and u' are words such that $u\psi = \pi\psi$ and $u'\psi = \pi'\psi$. Then, δ' satisfies

a) $\delta'\psi = \gamma$,

b) $\delta' q_{\mathbf{D}}$ is consistent

c) $\mathcal{C}(\delta, \delta')$ holds.

Indeed, $\delta'\psi = \gamma$ is straightforward from the choices of u and u' . Since $e\omega\delta_1 = \rho$ is non-explicit and since $\mathcal{C}(\delta_1, \delta'_1)$ holds, $e\omega\delta'_1$ is also non-explicit. This implies immediately the consistency of $\delta' q_{\mathbf{D}}$, as well as the validity of $\mathcal{C}(\delta, \delta')$.

By induction on the number of edges labeled by non-explicit operations under δ , we may therefore assume that all edges are labeled by explicit operations.

Since every edge is labeled by an explicit operation, if $e\alpha\delta$ (resp. $e\omega\delta$) is labeled by a non-explicit operation for a certain edge $e \in \mathcal{E}(\Gamma)$ then, by consistency of $\delta q_{\mathbf{D}}$, we also have $e\omega\delta$ (resp. $e\alpha\delta$) non-explicit. Since we may assume that Γ is connected, it follows that either all vertices of Γ are labeled under δ by explicit operations, or they are all labeled by non-explicit operations. If they are labeled by explicit operations, then δ is already a labeling of Γ by $\Omega_A^k \mathbf{S}$ which is $\mu_{\mathbf{D}}^k$ -related to γ , such that $\delta q_{\mathbf{D}}$ is consistent, and $\mathcal{C}(\delta, \delta)$ holds trivially. So assume that all vertices are labeled by non-explicit operations under δ .

The next step reduces to the case where δ (in fact, the restriction of δ to edges) commutes. If there is a non-oriented cycle labeled by an element $h \neq 1$ of $FG(A)$, then by Lemma 2.5 the labels by $\delta q_{\mathbf{D}}$ of all vertices are of the form $u^\omega v$ ($u \neq 1$). By Lemma 2.6 and by the fact that, over a finite semigroup, every implicit operation coincides with an explicit one, we can set $v\delta' = wu^\omega v \in \Omega_A^k \mathbf{S}$, where $v\delta q_{\mathbf{D}} = u^\omega v$, so that $v\delta'\psi = v\gamma$. Since $\delta q_{\mathbf{D}} = \delta' q_{\mathbf{D}}$, clearly $\delta' q_{\mathbf{D}}$ is consistent and by construction $\delta'\psi = \gamma$. Since for any $v \in \mathcal{V}$, neither $v\delta'$ nor $v\delta$ is explicit, the condition $\mathcal{C}(\delta, \delta')$ clearly holds.

Let now Γ be a graph and δ be a commuting labeling. Let m be the maximal length of labels of paths of a spanning tree of Γ . Since δ commutes,

the label of a non-oriented path only depends on its initial and terminal vertices. Therefore, the length of the label of any path in Γ is at most m .

Lemma 4.4 *Under the above assumptions and with the above notation, there exist a fixed $\pi \in \overline{\Omega}_A \mathbf{D}$ which is not explicit and words $u_v \in A^+$ ($v \in \mathcal{V}(\Gamma)$) such that $v\delta q_{\mathbf{D}} = \pi u_v$ and such that the labeling η of Γ by A^* defined by $v\eta = u_v$ ($v \in \mathcal{V}(\Gamma)$) and $e\eta = e\delta$ ($e \in \mathcal{E}(\Gamma)$) is consistent.*

Proof. Select a vertex v_0 . Since $v_0\delta$ is not explicit by assumption, we can factorize the implicit operation $v_0\delta q_{\mathbf{D}}$ as $\pi \cdot u_{v_0}$, where $|u_{v_0}| = m$ and where π is not explicit. For any other vertex v , select a non-oriented path from v_0 to v , and let h be the label of this path; put $u_v = u_{v_0}h$. Since the length of h is at most m , and since the action of h on u_{v_0} is defined, u_v belongs to A^* . Note that the equality $u_v = u_{v_0}h$ shows that η is consistent. Finally, observe that by consistency of $\delta q_{\mathbf{D}}$, the image of v under $\delta q_{\mathbf{D}}$ is precisely πu_v . \square

It follows by Lemma 2.6 that each $v\delta$ has a factorization of the form $\rho_v \pi' u_v$, where π' is a fixed non-explicit operation, and u_v is given by Lemma 4.4. Now, by Lemma 2.4, there exist words w_v, x, y and z such that S satisfies $\rho_v = w_v$ and $\pi' = xy^\omega z$. We define δ' to coincide with δ on edges, and $v\delta' = w_v xy^\omega z u_v$. By construction, we have $\delta'\psi = \gamma$ and $\delta'p_{\mathbf{D}}$ is consistent because the labeling η of Lemma 4.4 is consistent. Again, δ' maps any vertex to a non-explicit operation and $\mathcal{C}(\delta, \delta')$ holds. This proves the following theorem:

Theorem 4.5 *The pseudovariety \mathbf{D} is κ -tame.* \square

Remark 4.6 *Again, $g\delta'$ is explicit (and equal to $g\delta$) if and only if $g\delta$ is.*

4.4 The case of \mathbf{LI}

Let γ be an \mathbf{LI} -inevitable labeling of a finite graph Γ by a finite A -generated semigroup S . By Proposition 2.1, there is a labeling δ of Γ by $\overline{\Omega}_A \mathbf{S}$ such that $\delta\psi = \gamma$ and $\delta q_{\mathbf{LI}}$ is consistent. Since \mathbf{LI} is the join of \mathbf{K} and \mathbf{D} , this is equivalent to $\delta q_{\mathbf{K}}$ and $\delta q_{\mathbf{D}}$ being both consistent. Again, we just have to construct a labeling $\delta' : \Gamma \rightarrow \Omega_A^{\kappa} \mathbf{S}$ such that $\delta'\psi = \gamma$ and $\delta'p_{\mathbf{LI}}$ is consistent.

For any $g \in \Gamma$ such that $g\delta$ is not explicit, we choose a factorization $g\delta = (g\delta)_{\mathbf{K}}(g\delta)_{\mathbf{D}}$ of $g\delta$ in two non-explicit operations on \mathbf{S} . Since $\delta\psi = \gamma$, this induces a factorization $(g\gamma)_{\mathbf{K}}(g\gamma)_{\mathbf{D}}$ of $g\gamma$, with $(g\gamma)_{\mathbf{K}} = (g\delta)_{\mathbf{K}}\psi$ and $(g\gamma)_{\mathbf{D}} = (g\delta)_{\mathbf{D}}\psi$.

The S -labeling $\gamma_{\mathbf{K}}$ of Γ defined by $g\gamma_{\mathbf{K}} = g\gamma$ if $g\delta$ is explicit and $g\gamma_{\mathbf{K}} = (g\gamma)_{\mathbf{K}}$ otherwise is \mathbf{K} -inevitable. Indeed, let $\delta_{\mathbf{K}}$ be the labeling of Γ by $\overline{\Omega}_A \mathbf{S}$ which maps g to $g\delta$ if $g\delta$ is explicit and to $(g\delta)_{\mathbf{K}}$ otherwise. Then, by definition of $\gamma_{\mathbf{K}}$ and $\delta_{\mathbf{K}}$, the equality $\delta\psi = \gamma$ yields $\delta_{\mathbf{K}}\psi = \gamma_{\mathbf{K}}$; and $\delta_{\mathbf{K}}q_{\mathbf{K}}$ is consistent, since \mathbf{K} satisfies $g\delta = g\delta_{\mathbf{K}}$ by the choice of $\delta_{\mathbf{K}}$, and since $\delta q_{\mathbf{K}}$ is consistent by hypothesis. By Theorem 4.2, there is a labeling $\delta'_{\mathbf{K}}$ of Γ by $\Omega_A^{\kappa} \mathbf{S}$ such that $\delta'_{\mathbf{K}}p_{\mathbf{K}}$ is consistent and such that $\delta'_{\mathbf{K}}\psi = \gamma_{\mathbf{K}}$.

Dually, we can define $\gamma_{\mathbf{D}}, \delta_{\mathbf{D}}$ and by Theorem 4.5, there exists a labeling $\delta'_{\mathbf{D}}$ of Γ by $\Omega_A^{\kappa} \mathbf{S}$ such that $\delta'_{\mathbf{D}}p_{\mathbf{D}}$ is consistent and such that $\delta'_{\mathbf{D}}\psi = \gamma_{\mathbf{D}}$. By remarks 4.3 and 4.6, $\delta'_{\mathbf{K}}$ and $\delta'_{\mathbf{D}}$ both coincide with δ on elements sent to explicit operations by δ , and map other elements of Γ on non-explicit operations. We can now define the labeling δ' by $\Omega_A^{\kappa} \mathbf{S}$ as follows: $g\delta' = g\delta$ if $g\delta$ is explicit and $g\delta' = g\delta'_{\mathbf{K}} \cdot g\delta'_{\mathbf{D}}$ otherwise.

We claim that $\delta'\psi = \gamma$: if $g\delta$ is explicit, then $g\delta'\psi = g\delta\psi = g\gamma$. Otherwise, $g\delta'\psi = (g\delta'_{\mathbf{K}} \cdot g\delta'_{\mathbf{D}})\psi = g\delta'_{\mathbf{K}}\psi \cdot g\delta'_{\mathbf{D}}\psi = g\gamma_{\mathbf{K}} \cdot g\gamma_{\mathbf{D}} = g\gamma$. Finally, the consistence of $\delta'_{\mathbf{K}}p_{\mathbf{K}} = \delta'p_{\mathbf{K}}$ and of $\delta'_{\mathbf{D}}p_{\mathbf{D}} = \delta'p_{\mathbf{D}}$ implies the consistence of $\delta'p_{\mathbf{LI}}$. This proves the following theorem:

Theorem 4.7 *The pseudovariety \mathbf{LI} is κ -tame.* □

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