On embedding of finitely generated profinite semigroups into 2-generated profinite semigroups

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Abstract

Koryakov recently asked the following question: when a pseudovariety $\mathbf{V}$ does not satisfy any non-trivial identity, does there exist an embedding from any finitely generated $\mathbf{V}$-free profinite semigroup into the 2-generated $\mathbf{V}$-free profinite semigroup? During the conference "Semigroups, Automata and Languages" in Porto (June 1994 [2]), a positive answer to this question was conjectured. We give here a counterexample to this conjecture.

1 Problem and motivations

The aim of this note is to show that there exists a semigroup pseudovariety $\mathbf{V}$ such that the n-generated $\mathbf{V}$-free profinite semigroup is not embeddable in the 2-generated one for $n \geq 3$, and such that the free semigroup over the variety generated by $\mathbf{V}$ is free.

The motivation for this problem was given by Almeida [1]. It is related to the widely open study of the lattice of pseudovarieties. The original question was the proof of the $\vee$-irreducibility of $\mathbf{S}$, the pseudovariety of all finite semigroups: is it possible to write $\mathbf{S}$ as a non trivial join $\mathbf{V}_1 \vee \mathbf{V}_2$? This kind of problem was already studied for other pseudovarieties, and also for Birkhoff's varieties of semigroups. For instance, Dean and Evans [3] proved that the variety of all semigroups is $\vee$-irreducible.

For finite semigroups, the question remains natural but the answer is more difficult. Until 1993, the strongest result related to this question stated that there are no pseudovarieties covered by $\mathbf{S}$ in the lattice of pseudovarieties. This statement, due to Margolis [5], suggests a positive answer to the $\vee$-irreducibility of $\mathbf{S}$. However, the proof scheme of Dean and Evans cannot be followed directly, since they use an embedding of the n-generated free semigroup into the 2-generated semigroup. In order to show that $\mathbf{S}$ is $\vee$-irreducible, Almeida [1] proposed to find an embedding from the n-generated free *profinite* semigroup $\overline{\mathrm{F}}_n(\mathbf{S})$ into the 2-generated one $\overline{\mathrm{F}}_2(\mathbf{S})$. Following Almeida's track, Koryakov [4] proved the following theorem:

Theorem 1.1 *There exists an embedding from* $\overline{\mathrm{F}}_n(\mathbf{S})$ *into* $\overline{\mathrm{F}}_2(\mathbf{S})$. ∎

For free semigroups, it is straightforward to find such an embedding: just map letter a_i on $a_1 a_2^i$. For profinite free semigroups, the result is difficult. Margolis, Sapir and Weil [6] recently established other coding results to prove that some pseudovarieties are indecomposable for some operators: the join, the Mal'cev product and the semidirect product. It follows in particular from their work that the pseudovariety $\mathbf{S}$ is $\vee$-irreducible.

During the conference "Semigroups, Automata and Languages", Koryakov conjectured a generalization of Theorem 1.1. Following him, let us say that a pseudovariety is *big* if it does not satisfy any non-trivial identity. Given a pseudovariety $\mathbf{V}$, denote by $\overline{\mathrm{F}}_n(\mathbf{V})$

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the n-generated $\mathbf{V}$-free profinite semigroup (called the set of *implicit operations*). The proposed generalization is the following:

Conjecture 1.2 Let $\mathbf{V}$ be a big semigroup pseudovariety. Then there exists an embedding from $\overline{\mathrm{F}}_n(\mathbf{V})$ into $\overline{\mathrm{F}}_2(\mathbf{V})$.

2 The result

Theorem 1.1 holds for some big pseudovarieties, such as the pseudovariety of nilpotent semigroups or the pseudovariety of groups. However, we give in this section examples of big pseudovarieties $\mathbf{V}$ such that $\overline{\mathrm{F}}_n(\mathbf{V})$ is not embeddable into $\overline{\mathrm{F}}_2(\mathbf{V})$ for any $n \geq 3$.

We shall assume a knowledge of the basic notions of finite and profinite semigroup theory, and we borrow the notations of [1]. Let A_n be the alphabet $\{a_1, \ldots, a_n\}$. We denote by $\mathbf{N}$ the pseudovariety of finite nilpotent semigroups, by $\mathbf{Sl}$ the pseudovariety of finite semilattices, and by $\mathbf{DS}$ the pseudovariety of finite semigroups whose regular $\mathscr{J}$-classes form subsemigroups. The result is stated in the following theorem.

Theorem 2.1 *Let $\mathbf{V}$ be a pseudovariety between $\mathbf{N} \vee \mathbf{Sl}$ and $\mathbf{DS}$. Then $\mathbf{V}$ is a big pseudovariety, but for any $n \geq 3$, there is no injective morphism from $\overline{\mathrm{F}}_n(\mathbf{V})$ into $\overline{\mathrm{F}}_2(\mathbf{V})$.*

It is well-known that any pseudovariety containing $\mathbf{N}$ is big. See for instance [1, pp. 88–89]. Therefore, $\mathbf{V}$ is big. In order to complete the proof, we shall use an additional statement [1, pp. 219–220]. For any pseudovariety $\mathbf{V}$ containing $\mathbf{Sl}$, we denote by c the content morphism, that is, the canonical projection from $\overline{\mathrm{F}}_n(\mathbf{V})$ onto $\overline{\mathrm{F}}_n(\mathbf{Sl}) = 2^{A_n}$. Let us recall that this morphism is continuous if we endow $\overline{\mathrm{F}}_n(\mathbf{V})$ with the profinite topology.

Theorem 2.2 *Let $\mathbf{V}$ be a pseudovariety between $\mathbf{Sl}$ and $\mathbf{DS}$. Let e and f be idempotents of $\overline{\mathrm{F}}_n(\mathbf{V})$. Then e and f are $\mathscr{J}$-equivalent if and only if $c(e) = c(f)$.* ∎

The idempotent ordering $<$ is defined by $e < f$ if and only if $e = ef = fe$ and $e \neq f$. We deduce from Theorem 2.2 the following result.

Corollary 2.3 *Let $\mathbf{V}$ be a pseudovariety between $\mathbf{Sl}$ and $\mathbf{DS}$. Then the longest chain of idempotents of $\overline{\mathrm{F}}_n(\mathbf{V})$ in the idempotent ordering has length n.*

Proof. If $e < f$, then $e = ef = fe$ so $c(f) \subseteq c(e)$. Assume that $c(f) = c(e)$. Then Theorem 2.2 implies that $e \mathscr{J} f$. Therefore, e and f are $\mathscr{J}$-equivalent idempotent which are $\mathscr{R}$- and $\mathscr{L}$-comparable, so they are equal, a contradiction with $e < f$. Hence $e < f$ implies $c(f) \subsetneq c(e)$, so the longest chain of idempotents of $\overline{\mathrm{F}}_n(\mathbf{V})$ in the idempotent ordering has length at most n.

It remains to show that there exists such a chain of length n. For $i = 1, \ldots, n$, let $e_i = (a_1 \cdots a_i)^\omega$. Let $f_1 = e_1$ and define recursively f_i by $f_i = (f_{i-1} e_i f_{i-1})^\omega$ for $2 \leq i \leq n$. Since f_{i-1} is idempotent, we have immediately $f_i f_{i-1} = f_{i-1} f_i = f_i$. Since $c(f_i) = \{a_1, \ldots, a_i\}$, we also have $f_i \neq f_{i-1}$, so $f_i < f_{i-1}$. ∎

Let us now conclude the proof of Theorem 2.1. Let $n \geq 3$ and assume that there exists an injective morphism $\varphi : \overline{\mathrm{F}}_n(\mathbf{V}) \to \overline{\mathrm{F}}_2(\mathbf{V})$. If e, f are idempotents of $\overline{\mathrm{F}}_n(\mathbf{V})$ such that $e < f$, then $e' = \varphi(e)$ and $f' = \varphi(f)$ are idempotents such that $e' = e'f' = f'e'$. Moreover, $e' \neq f'$ since φ is injective. Hence, φ maps any chain of idempotents of $\overline{\mathrm{F}}_n(\mathbf{V})$ to a chain of idempotents of $\overline{\mathrm{F}}_2(\mathbf{V})$, a contradiction with Corollary 2.3. ∎

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