

The join of the pseudovarieties of idempotent semigroups and locally trivial semigroups

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Abstract

This article solves a problem proposed by Almeida: the computation of the join of two well-known pseudovarieties of semigroups, namely the pseudovariety of bands and the pseudovariety of locally trivial semigroups. We use a method developed by Almeida, based on the theory of implicit operations.

1. Introduction

The problem solved in this paper was originally motivated by the classification of rational languages.

The description of the lattice of varieties of semigroups is a famous problem in semigroup theory. Important results in this direction were given by Biryukov [15], Fennemore [18, 19] and Gerhard [20], who described the lattice of varieties of bands, and by Polák [26, 27, 28] who described the lattice of varieties of completely regular semigroups.

There is an analogous problem for pseudovarieties, which is mainly motivated by considerations of automata theory. Indeed, Eilenberg's variety theorem [31] gives a bijective correspondence between varieties of rational languages and pseudovarieties of semigroups. In particular, the join $\mathbf{V} \vee \mathbf{W}$ of two pseudovarieties of semigroups \mathbf{V} and \mathbf{W} corresponds to the smallest variety of languages containing the varieties corresponding to \mathbf{V} and \mathbf{W} .

Little is known on the join of two pseudovarieties, even when they are simple. In contrast, its definition is straightforward, and it appears in a very natural way, when considering parallel computations of automata.

Most results concerning this problem are recent. Several kinds of partial or particular answers were given:

- Some computations can be done explicitly. The main technique to compute a join is to use the theory of implicit operations. This theory was developed and used in numerous applications by Almeida [7, 4]. Reiterman's theorem [29] is the starting point of the equational theory of pseudovarieties: it states that pseudovarieties are defined by pseudoidentities, just as varieties are defined by identities. Using this

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theorem and some *ad hoc* facts, Almeida, Azevedo and Weil [3, 2, 8, 10, 11, 14, 13] computed some non-trivial joins for which algebraic methods failed.

- Sometimes, one can only determine whether the join has a finite basis of pseudoidentities. For instance, Almeida [6] proved that the commutative pseudovarieties are finitely based, while Volkov [33], Trotter and Volkov [32] and the author [35] gave examples of non finitely based joins.

- In most cases, however, nothing can be proved about the basis describing the pseudovariety. The main problem is then the decidability of the pseudovariety \mathbf{V} : given a finite semigroup S , is the membership of S in \mathbf{V} decidable? Once again, the problem is very difficult. A recent result of Albert, Baldinger and Rhodes [1] states that even the join of two decidable pseudovarieties might not be decidable.

Rhodes [30], Almeida [7] and Kharlampovich and Sapir [23] proposed a list of problems on pseudovarieties, most of which are still open. In this article, we solve problem **23** of Almeida [7]: “Compute $\mathbf{LI} \vee \mathbf{B}$ ”, where \mathbf{LI} is the pseudovariety of semigroups whose local subsemigroups are trivial, and \mathbf{B} is the pseudovariety of bands. We give a basis and a simple algebraic characterization for this join. We also give a description of the implicit operations on $\mathbf{LI} \vee \mathbf{B}$.

The paper is organized as follows: after a brief section on basic definitions, the main result: the computation of $\mathbf{LI} \vee \mathbf{B}$ (Theorem 3.1 below) is stated and proved in section 3. Section 4 is devoted to the computation of some subpseudovarieties of $\mathbf{LI} \vee \mathbf{B}$. Finally, in section 5, we analyse the structure of the implicit operations on $\mathbf{LI} \vee \mathbf{B}$.

2. Notations and background

We briefly review the main definitions and some useful facts about semigroups, pseudovarieties and implicit operations. For more details, the reader is referred to the books of Almeida [7], Pin [25], to the original article of Almeida [4] or for a brief introduction, to the surveys of Almeida and Weil [5, 10, 11, 34]. We assume the reader to be familiar with some basic notions of topology, universal algebra and semigroup theory (see for instance [16, 17, 25]).

2.1.. Semigroups and pseudovarieties

In the sequel, A_n denotes the finite alphabet $\{x_1, \dots, x_n\}$. For a word $u \in A_n^+$, the *content* $c(u)$ of u is the set of all letters appearing in u . The *length* $|u|$ of u is the number of letters of u .

The subset of idempotent elements of a semigroup S is denoted by $E(S)$. If S is finite, the *exponent* of S is the least integer k such that s^k is idempotent for every $s \in S$. Recall that the local subsemigroup of S associated to the idempotent e is the subsemigroup eSe of S .

A *pseudovariety* of semigroups is a class of finite semigroups closed under formation of finite direct product, homomorphic image and subsemigroup. Here is a list of some important pseudovarieties:

- The pseudovariety of all finite semigroups is denoted by \mathbf{S} .
- The pseudovariety of all finite aperiodic (or group-free) semigroups is denoted by

A.

- An idempotent semigroup is called a *band*. The pseudovariety of finite bands is denoted by **B**.
- The pseudovariety of finite commutative bands (that is, of finite semilattices) is denoted by **SI**.
- **LI** denotes the pseudovariety of all semigroups whose local subsemigroups are trivial. It has some well-known subpseudovarieties:
 - for every $k \in \mathbb{N}$, **LI_k** is the pseudovariety of all finite semigroups satisfying the identity $x_1 \dots x_k y z_1 \dots z_k = x_1 \dots x_k z_1 \dots z_k$.
 - **K** (resp. **D**) is the pseudovariety of all finite semigroups in which idempotents are left zeros (resp. right zeros).
 - for every $k \in \mathbb{N}$, **K_k** (resp. **D_k**) is the pseudovariety of all finite semigroups such that every product of k elements of S is a left zero (resp. a right zero).
 - **N** is the pseudovariety of all finite nilpotent semigroups, that is, semigroups with a zero which is the unique idempotent.
 - for every $k \in \mathbb{N}$, **N_k** is the pseudovariety of all finite nilpotent semigroups S such that every product of k elements of S is zero.
- Finally, for any pseudovariety **V**, **DV** is the pseudovariety of all semigroups whose regular **D**-classes are semigroups of **V**. In particular, **DS** consists of all finite semigroups whose regular **D**-classes are semigroups, and **DA** of all finite semigroups whose regular **D**-classes are aperiodic semigroups.

It is important to keep in mind the fact that a pseudovariety containing **N**, like **LI**, cannot satisfy any non-trivial identity. Indeed, **N_k** does not satisfy any non-trivial identity $u = v$ where $|u| < k$ and $|v| < k$. This simple property will be used several times in the sequel.

Recall that the free band over A_n is a finite semigroup, quotient of the free semigroup A_n^+ by the congruence generated by $x = x^2$ [22, 24]. We denote this congruence over A_n^+ by \equiv_n .

A relational morphism τ from a semigroup S to a semigroup T is a relation that associates to each $s \in S$ a non empty subset $s\tau$ of T such that

$$\forall s, s' \in S, \quad (s\tau)(s'\tau) \subseteq (ss')\tau$$

The Mal'cev product $\mathbf{V} \circledast \mathbf{W}$ of two pseudovarieties **V** and **W** is the pseudovariety of all finite semigroups S satisfying the following property: there exists a relational morphism τ from S into a semigroup of **W** such that the inverse image of every idempotent of S lies in **V**.

2.2.. Implicit operations

Let **V** be a given pseudovariety. A semigroup S *separates* two words u and v of A_n^+ if there exists a semigroup morphism $\varphi : A_n^+ \rightarrow S$ such that $u\varphi \neq v\varphi$. One defines a pseudo-metric $e_{\mathbf{V}}$ on A_n^+ as follows:

$$r_{\mathbf{V}}(u, v) = \min\{|S| \mid S \in \mathbf{V} \text{ and separates } u \text{ and } v\}$$

and

$$e_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u, v)}$$

By convention, $\min \emptyset = \infty$ and $2^{-\infty} = 0$.

Thus, $e_{\mathbf{V}}(u, v) \leq 2^{-k}$ if and only if u and v have the same evaluation on every semigroup of \mathbf{V} whose size is less than k . It is not difficult to verify the following, for all $u, v, w \in A_n^+$:

1. $e_{\mathbf{V}}(u, u) = 0$,
2. $e_{\mathbf{V}}(u, v) = e_{\mathbf{V}}(v, u)$,
3. $e_{\mathbf{V}}(u, v) \leq \max(e_{\mathbf{V}}(u, w), e_{\mathbf{V}}(w, v))$,
4. $e_{\mathbf{V}}(uw, vw) \leq e_{\mathbf{V}}(u, v)$ and $e_{\mathbf{V}}(wu, wv) \leq e_{\mathbf{V}}(u, v)$.

It is then straightforward to see that the relation $\sim_{\mathbf{V}}$ defined by

$$u \sim_{\mathbf{V}} v \iff e_{\mathbf{V}}(u, v) = 0$$

is a congruence. The quotient $A_n^+/\sim_{\mathbf{V}}$ is a semigroup, denoted by $F_n(\mathbf{V})$. In fact, $F_n(\mathbf{V})$ is the free semigroup of the variety generated by \mathbf{V} . Furthermore, if \mathbf{V} is not trivial, there exists in \mathbf{V} a semigroup with at least two elements that separates x_i and x_j for $i \neq j$. Hence, the function $x_i \mapsto x_i/\sim_{\mathbf{V}}$ is a bijection. We assume now that \mathbf{V} is not trivial and we identify x_i with $x_i/\sim_{\mathbf{V}}$ and A_n with $A_n/\sim_{\mathbf{V}}$.

By properties 1. to 3., $e_{\mathbf{V}}$ induces an ultrametric distance function $d_{\mathbf{V}}$ over $F_n(\mathbf{V})$. By property 4., the multiplication in $F_n(\mathbf{V})$ is uniformly continuous for this metric, so that $F_n(\mathbf{V})$ is a topological semigroup. The completion of the metric space $(F_n(\mathbf{V}), d_{\mathbf{V}})$ is denoted by $\overline{F}_n(\mathbf{V})$. Thus, $\overline{F}_n(\mathbf{V})$ is a compact totally disconnected topological semigroup, and $F_n(\mathbf{V})$ is dense in $\overline{F}_n(\mathbf{V})$. The elements of $\overline{F}_n(\mathbf{V})$ are called the n -ary *implicit operations*. Implicit operations that lie in fact in $F_n(\mathbf{V})$ are said to be *explicit*.

When $F_n(\mathbf{V})$ is finite, then $\overline{F}_n(\mathbf{V}) = F_n(\mathbf{V})$. This happens for $\mathbf{V} = \mathbf{SI}$ and for $\mathbf{V} = \mathbf{B}$, for instance. $F_n(\mathbf{SI})$ is the free semigroup of the variety of semilattices: it is isomorphic to the semigroup $(\mathbf{P}(A_n), \cup)$, where $\mathbf{P}(A_n)$ is the power set of A_n . Thus, a subset of A_n determines a unique implicit operation on $\overline{F}_n(\mathbf{SI})$. For example, the operations $x_4x_3^3x_1$ and $x_1x_3x_4$ are equal because they have the same content: $\{x_1, x_3, x_4\}$.

Every explicit operation $u = u(x_1, \dots, x_n) \in F_n(\mathbf{V})$ defines for every semigroup $S \in \mathbf{V}$ a function u_S from S^n into S : $u_S(s_1, \dots, s_n) = u(s_1, \dots, s_n)$. The function u_S is called the evaluation of u over S . For instance, the evaluation of $u = x_1x_4^2x_3 \in F_4(\mathbf{V})$ is the function u_S from S^4 into S such that $u_S(a, b, c, d) = ad^2c$. It is easy to see that one can define the evaluation π_S of an implicit operation $\pi \in \overline{F}_n(\mathbf{V})$ over a semigroup $S \in \mathbf{V}$ as follows: if $(u_k)_k$ is a sequence of explicit operations whose limit is π , then the sequence $((u_k)_S)_k$ converges to a value that does not depend on the sequence $(u_k)_k$. By definition, π_S is this limit value.

A pair $(\pi, \rho) \in \overline{F}_n(\mathbf{V}) \times \overline{F}_n(\mathbf{V})$ is called a *pseudoidentity*. Let (π, ρ) be a pseudoidentity. A semigroup $S \in \mathbf{V}$ satisfies (π, ρ) if and only if $\pi_S = \rho_S$. We

will then write $S \models \pi = \rho$. If Σ is a set of pseudoidentities on a pseudovariety \mathbf{V} , S satisfies Σ if S satisfies every pseudoidentity of Σ , and a class \mathbf{C} of semigroups satisfies Σ if every semigroup of \mathbf{C} satisfies Σ . We will then write $\mathbf{C} \models \Sigma$. The class $[[\Sigma]]_{\mathbf{V}}$ of all semigroups of \mathbf{V} satisfying Σ is a pseudovariety. In the sequel, we shall write $[[\Sigma]]$ instead of $[[\Sigma]]_{\mathbf{S}}$.

The following proposition comes from the definitions and will be used many times:

Proposition 2.1. *Let \mathbf{V} be a pseudovariety, and $(u_m)_m$ a sequence of explicit operations of $F_n(\mathbf{V})$. Then, this sequence converges if and only if*

$$\forall S \in \mathbf{V}, \exists m \in \mathbb{N}, \forall p, q \geq m, (u_p)_S = (u_q)_S$$

and it converges to an implicit operation π if and only if

$$\forall S \in \mathbf{V}, \exists m \in \mathbb{N}, \forall p \geq m, (u_p)_S = \pi_S \quad \blacksquare$$

This proposition has an important consequence: given an implicit operation π , the sequence $(\pi^{k!})_k$ converges. The limit of this sequence is denoted by π^ω . For any letter x , the evaluation of x^ω on an element s of a finite semigroup is the unique idempotent of the subsemigroup generated by s .

The following characterization of pseudovarieties is the fundamental theorem of the equational theory of pseudovarieties; notice the analogy with Birkhoff's completeness theorem:

Theorem 2.2. Reiterman [29] *Let \mathbf{V} be a pseudovariety of semigroups and let \mathbf{W} be a subclass of \mathbf{V} . Then, \mathbf{W} is a pseudovariety if and only if there exists a set of pseudoidentities $\Sigma \subseteq \overline{F}_n(\mathbf{V}) \times \overline{F}_n(\mathbf{V})$ such that $\mathbf{W} = [[\Sigma]]_{\mathbf{V}}$. \blacksquare*

For example, \mathbf{B} is defined by the pseudoidentity $x = x^2$. In the same way, $\mathbf{Sl} = [[x = x^2, xy = yx]] = [[xy = yx]]_{\mathbf{B}}$. More generally, \mathbf{A} is equal to $[[x^\omega = x^{\omega+1}]]$. Let us now see some other examples. It is straightforward to check that a semigroup belongs to \mathbf{K} if and only if every idempotent is a left zero. Therefore, the pseudovariety \mathbf{K} can be defined by the pseudoidentity $x^\omega y = x^\omega$. Similarly, $\mathbf{D} = [[yx^\omega = x^\omega]]$.

Notice also that if \mathbf{V}_1 (resp. \mathbf{V}_2) is defined by Σ_1 (resp. Σ_2), then $\mathbf{V}_1 \cap \mathbf{V}_2$ is defined by $\Sigma_1 \cup \Sigma_2$. Thus, $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$ is defined by $x^\omega y = x^\omega = yx^\omega$. For convenience, we will write $\pi = 0$ instead of $\pi x_i = x_i \pi = 0$ if x_i is a letter. Thus, $\mathbf{N} = [[x^\omega = 0]]$. This example shows that the computation of a basis of an intersection $\mathbf{V}_1 \cap \mathbf{V}_2$ is straightforward when bases of \mathbf{V}_1 and \mathbf{V}_2 are known.

In contrast, it is difficult to find a basis for the join $\mathbf{V}_1 \vee \mathbf{V}_2$. Indeed, the pseudoidentities satisfied by $\mathbf{V}_1 \vee \mathbf{V}_2$ are exactly those satisfied by both \mathbf{V}_1 and \mathbf{V}_2 . In general, this characterization does not give much information about the exact form of these pseudoidentities. However, for $\mathbf{LI} = \mathbf{K} \vee \mathbf{D}$, it is easy to get such a basis: \mathbf{LI} is defined by $x^\omega y x^\omega = x^\omega$, or alternatively by $x^\omega y z^\omega = x^\omega z^\omega$. Indeed, to say that a semigroup S satisfies $x^\omega y x^\omega = x^\omega$ means that for every $e \in E(S)$

and every $s \in S$, the elements ese and e are equal: the local subsemigroup eSe is trivial.

Some definitions on words can now be extended for explicit or implicit operations. For instance, let \mathbf{V} be a pseudovariety containing \mathbf{N} . Since \mathbf{V} does not satisfy any non-trivial identity, the length $|u|$ of any explicit operation u can again be defined as the number of letters of u .

One can also define the content of an implicit operation. For instance, it is natural to say that the content of $x^\omega y t^\omega$ is $\{x, y, z\}$. Notice however that $x^\omega y t^\omega$ is equal to $x^\omega t^\omega$ on $\overline{\mathbf{F}}_n(\mathbf{LI})$. In fact, the content can be defined in a satisfactory way for pseudovarieties containing \mathbf{SI} .

Theorem 2.3. Almeida, Azevedo [7, 13] *Let \mathbf{V} be a pseudovariety containing \mathbf{SI} . Then, two words representing an explicit operation u on $\mathbf{F}_n(\mathbf{V})$ have the same content. We denote this content by $c(u)$. Then, the function $c : \mathbf{F}_n(\mathbf{V}) \rightarrow \mathbf{F}_n(\mathbf{SI})$ can be extended in a unique way in a continuous homomorphism from $\overline{\mathbf{F}}_n(\mathbf{V})$ into $\overline{\mathbf{F}}_n(\mathbf{SI})$.* ■

Thus, for such a pseudovariety, the content of $x^\omega y t^\omega$ is $\{x, y, z\}$.

Implicit operations are in general very difficult to handle. An important simplification has been discovered by Almeida and Azevedo: on \mathbf{DS} , implicit operations have a nice factorization. The structure of the \mathbf{J} classes of $\overline{\mathbf{F}}_n(\mathbf{DS})$ plays an important role. It is described in the next statement:

Theorem 2.4. Azevedo [13] *Let π and ρ be two regular elements of $\overline{\mathbf{F}}_n(\mathbf{DS})$. Then, $\pi \mathbf{J} \rho \iff c(\pi) = c(\rho)$. Furthermore, \mathbf{DS} is the largest pseudovariety satisfying this property.* ■

The fundamental theorem is the following:

Theorem 2.5. Almeida, Azevedo [7, 9, 13, 12] *Every implicit operation $\pi \in \overline{\mathbf{F}}_n(\mathbf{S})$ admits a factorization of the form $\pi = \pi_1 \dots \pi_k$ where π_i is either explicit or its restriction to \mathbf{DS} is regular.* ■

Most non trivial computations of pseudovarieties obtained using the theory of implicit operations were based on this theorem. See [7] for examples.

3. The pseudovariety $\mathbf{LI} \vee \mathbf{B}$

This section is devoted to the proof of the main theorem:

Theorem 3.1. *The pseudovariety $\mathbf{LI} \vee \mathbf{B}$ is determined by*

$$\mathbf{LI} \vee \mathbf{B} = \llbracket (x^\omega y) = (x^\omega y)^2, \quad (y x^\omega) = (y x^\omega)^2 \rrbracket = \mathbf{B} \circledast \mathbf{N} \cap \llbracket x^\omega y z^\omega = x^\omega y^2 z^\omega \rrbracket$$

This statement consists of two equalities. The second one is easier, and will be proved in Lemma 3.2. For the first equality, the inclusion from left to right is easy: notice that if S is a semigroup that belongs to \mathbf{LI} or \mathbf{B} , the set $E(S)$ of idempotents of S is an ideal. Therefore, the pseudovariety $\llbracket (x^\omega y) = (x^\omega y)^2, (yx^\omega) = (yx^\omega)^2 \rrbracket$ is a natural upper bound for $\mathbf{LI} \vee \mathbf{B}$. The proof of the opposite inclusion is based on Reiterman's theorem.

Let us now show the second equality of the theorem:

Lemma 3.2. *The pseudovariety $\llbracket (x^\omega y) = (x^\omega y)^2, (yx^\omega) = (yx^\omega)^2 \rrbracket$ is equal to the Mal'cev product $\mathbf{B} \circledast \mathbf{N}$.*

Proof. Let S be a semigroup and assume that $E(S)$ is an ideal. Then, $S/E(S)$ is nilpotent and the canonical morphism $\tau : S \rightarrow S/E(S)$ satisfies $\tau^{-1}(0) \in \mathbf{B}$. Conversely, if S is in $\mathbf{B} \circledast \mathbf{N}$, there exists a nilpotent semigroup N of exponent p and a relational morphism τ from S into N such that $\tau^{-1}(0)$ is a band. Pick then $e \in E(S)$ and $s \in S$. Since $\tau(ex) = \tau(e^p x) \supseteq \tau(e)^p \tau(x) = \{0\}$, one gets $ex \in \tau^{-1}(0)$, and therefore ex is idempotent. One would show in the same way that xe is also idempotent. ■

Next, since \mathbf{LI} and \mathbf{B} are in \mathbf{DA} , we shall check that $\mathbf{B} \circledast \mathbf{N} \subseteq \mathbf{DA}$. This fact will be used later to apply the factorization theorem on \mathbf{DS} .

Lemma 3.3. *The pseudovariety $\mathbf{B} \circledast \mathbf{N}$ is included in \mathbf{DA} .*

Proof. Let S be a semigroup of $\mathbf{B} \circledast \mathbf{N}$. We have to show that every regular \mathbf{D} -class of S is an aperiodic semigroup. In fact, S satisfies itself $x^\omega = x^{\omega+1}$: it suffices to take $x = y$ in one of the pseudoidentities defining $\mathbf{B} \circledast \mathbf{N}$.

Next, if z lies in a regular \mathbf{D} -class, then there exists an idempotent e such that $e \mathbf{R} z$, and therefore $ez = z$. Since $E(S)$ is an ideal, z is itself an idempotent. Hence, for any y such that $x \mathbf{D} y$, the \mathbf{H} -class $R_y \cap L_x$ contains an idempotent; therefore, $xy \in R_x \cap L_y$, and xy lies in the same \mathbf{D} -class. Therefore, every regular \mathbf{D} -class is a semigroup. ■

Remark 3.4. The inclusion $\mathbf{B} \circledast \mathbf{N} \subseteq \mathbf{DA}$ is strict, otherwise $\mathbf{B} \circledast \mathbf{N}$ would contain for instance the pseudovariety \mathbf{J} of \mathbf{J} -trivial semigroups, but \mathbf{J} does not satisfy the pseudoidentity $(x^\omega y)^2 = x^\omega y$ (see Almeida [7]). ■

The rest of the proof follows the method developed by Almeida: Reiterman's theorem states that one can write $\mathbf{LI} \vee \mathbf{B} = \llbracket \Sigma \rrbracket_{\mathbf{B} \circledast \mathbf{N}}$, where Σ is a set of pseudoidentities on $\mathbf{B} \circledast \mathbf{N}$. This implies in particular that if $(\pi, \pi') \in \Sigma^2$, then both \mathbf{LI} and \mathbf{B} satisfy $\pi = \pi'$. All we have to show now is that π and π' are equal in $\overline{\mathbf{F}}_n(\mathbf{B} \circledast \mathbf{N})$.

Therefore, we need some information on the implicit operations on $\mathbf{B} \circledast \mathbf{N}$. They take on a very particular form, as shown in the next proposition:

Proposition 3.5. *Let \mathbf{V} be a pseudovariety such that $\mathbf{N} \subseteq \mathbf{V} \subseteq \mathbf{B} \circledast \mathbf{N}$. The three following facts hold in $\overline{\mathbf{F}}_n(\mathbf{V})$:*

1. *Every implicit operation of $\overline{\mathbf{F}}_n(\mathbf{V})$ is either explicit or idempotent.*
2. *An implicit operation of $\overline{\mathbf{F}}_n(\mathbf{V})$ cannot be both explicit and idempotent.*
3. *If $(u_m)_m$ is a sequence of explicit operations such that $\lim_{m \rightarrow \infty} |u_m| = +\infty$, then $(u_m)_m$ cannot converge to an explicit operation.*

Proof. If π is a regular implicit operation, then π is idempotent by aperiodicity. Therefore, as $\mathbf{B} \circledast \mathbf{N} \subseteq \mathbf{DA} \subseteq \mathbf{DS} \cap \mathbf{A}$, every implicit operation of $\overline{\mathbf{F}}_n(\mathbf{B} \circledast \mathbf{N})$ has a factorization in a product of explicit and idempotent operations by Theorem 2.5. But $E(\overline{\mathbf{F}}_n(\mathbf{B} \circledast \mathbf{N}))$ is an ideal. Hence, such a product is idempotent as soon as it contains one idempotent operation. This proves 1.

In order to prove 2, notice that if $u \in \mathbf{F}_n(\mathbf{V})$ is an idempotent, then $\mathbf{N} \subseteq \mathbf{V}$ would satisfy $u = u^2$. But \mathbf{N} does not satisfy any non-trivial identity, so the words u and u^2 are identical, which is impossible.

Finally, for 3, assume that a sequence $(u_m)_m$ of explicit operations converges to an explicit operation u , and let $k = |u| + 1$. Since $\mathbf{N}_k \subseteq \mathbf{V}$ and since $\mathbf{F}_n(\mathbf{N}_k)$ is finite, $\overline{\mathbf{F}}_n(\mathbf{N}_k)$ is a semigroup of \mathbf{V} . Since u is the limit of (u_m) , \mathbf{N}_k satisfies $u_m = u$ for m large enough. This is in contradiction with the fact that \mathbf{N}_k does not satisfy any non-trivial identity $v = u$ where $|u| < k$ and $|v| < k$. So the limit of (u_m) cannot be explicit. ■

Our next lemma deals with the pseudoidentity $x^\omega y x^\omega = x^\omega y^2 x^\omega$. It is immediate that this pseudoidentity is satisfied by \mathbf{LI} and by \mathbf{B} , and hence it is satisfied by $\mathbf{LI} \vee \mathbf{B}$. We show that it holds in $\mathbf{B} \circledast \mathbf{N}$ too.

Lemma 3.6. *The pseudovariety $\mathbf{B} \circledast \mathbf{N}$ satisfies $x^\omega y x^\omega = x^\omega y^2 x^\omega$.*

Proof. Let $\pi = x^\omega y x^\omega$ and $\rho = x^\omega y^2 x^\omega$ in $\overline{\mathbf{F}}_n(\mathbf{B} \circledast \mathbf{N})$. By Lemma 3.2, these implicit operations are both idempotent. Furthermore, $\mathbf{S1} \subseteq \mathbf{B} \circledast \mathbf{N} \subseteq \mathbf{DS}$. As $c(\pi) = c(\rho)$, π and ρ lie in the same \mathbf{J} -class, by Theorem 2.4. By aperiodicity, it suffices therefore to show that they lie in the same \mathbf{H} -class. But

$$x^\omega y x^\omega \cdot x^\omega y^2 x^\omega = (x^\omega y)^2 y x^\omega = x^\omega y^2 x^\omega$$

Similarly,

$$x^\omega y^2 x^\omega \cdot x^\omega y x^\omega = x^\omega y (y x^\omega)^2 = x^\omega y^2 x^\omega$$

Therefore, $\pi \rho = \rho \pi = \rho$, and π and ρ are comparable for the relations \mathbf{R} and \mathbf{L} . Since they lie in the same \mathbf{J} -class, they have to be \mathbf{H} -equivalent. ■

However, the next statement shows that one cannot simplify the intersection in Theorem 3.1. Indeed, $\mathbf{LI} \vee \mathbf{B}$ obviously satisfies the pseudoidentity $x^\omega y z^\omega = x^\omega y^2 z^\omega$ which fails in $\mathbf{B} \circledast \mathbf{N} = \llbracket (x^\omega y) = (x^\omega y)^2, (y x^\omega) = (y x^\omega)^2 \rrbracket$.

Lemma 3.7. *The pseudovariety $\mathbf{B} \circledast \mathbf{N}$ does not satisfy the pseudoidentity $x^\omega y z^\omega = x^\omega y^2 z^\omega$.*

Proof. Consider the free 3-generated semigroup F in the semigroup variety defined by $xy = (xy)^2$. Gerhard proved in [21] that it is finite; therefore, it clearly lies in $\mathbf{B} \circledast \mathbf{N}$. The word problem in F was also solved in [21]. Gerhard's algorithm easily shows that the words $x^2 y z^2$ and $x^2 y^2 z^2$ are different in F . ■

Let now $\mathbf{U} = \mathbf{B} \circledast \mathbf{N} \cap \llbracket x^\omega y z^\omega = x^\omega y^2 z^\omega \rrbracket$ and $(\pi, \pi') \in (\overline{F}_n(\mathbf{U}))^2$ be a non-trivial pseudoidentity satisfied by $\mathbf{LI} \vee \mathbf{B}$: that is, $\pi \neq \pi'$ and $\mathbf{LI} \vee \mathbf{B} \models \pi = \pi'$. By Proposition 3.5, the operations π and π' are either explicit or idempotent. Since in particular $\mathbf{N} \models \pi = \pi'$ (since $\mathbf{N} \subseteq \mathbf{LI}$), and since \mathbf{N} does not satisfy any non-trivial identity, π and π' are not both explicit. For the same reason, if π' is idempotent, then so is π : indeed, if π' was idempotent and π explicit, we would have $\mathbf{N} \models \pi^2 = \pi'^2 = \pi' = \pi$ and therefore $\mathbf{N} \models \pi = \pi^2$, which is impossible if π is explicit. So let us suppose that π and π' are both idempotent. A last lemma is needed before we can conclude.

Lemma 3.8. *Let $(\pi, \pi') \in (\overline{F}_n(\mathbf{U}))^2$ be a pseudoidentity satisfied by \mathbf{LI} . Suppose that π and π' are idempotent. Then, there exist sequences of explicit operations $(r_m)_m, (s_m)_m, (t_m)_m$ and $(t'_m)_m$ such that*

$$\pi = \lim_{m \rightarrow \infty} r_m t_m s_m, \quad \pi' = \lim_{m \rightarrow \infty} r_m t'_m s_m$$

and, for every $m \in \mathbb{N}$

$$|r_m| \geq m, \quad |s_m| \geq m$$

Proof. Let $(u_m)_m$ and $(v_m)_m$ be two sequences of explicit operations that converge to π and π' respectively. For any given k , \mathbf{LI}_k satisfies $\pi = \pi'$. As $F_n(\mathbf{LI}_k)$ is finite, it lies in \mathbf{LI} ; thus, \mathbf{LI}_k satisfies the identity $u_m = v_m$ for all $m \geq m_k$. One can assume for convenience that for these values of m , $|u_m| > 2k$ and $|v_m| > 2k$: otherwise we could first extract a bounded sequence from $(|u_m|)_m$ or $(|v_m|)_m$, and then a constant sequence from $(u_m)_m$ or $(v_m)_m$. This is impossible since these subsequences converge to idempotent operations and since an idempotent cannot be explicit, by Proposition 3.5.

To say that \mathbf{LI}_k satisfies $u_m = v_m$ is equivalent to say that u_m and v_m have the same prefix and same suffix of length k . As $|u_m| > 2k$ and $|v_m| > 2k$, we can write $u_{m_k} = \tilde{r}_{m_k} \tilde{t}_{m_k} \tilde{s}_{m_k}$ and $v_{m_k} = \tilde{r}_{m_k} \tilde{t}'_{m_k} \tilde{s}_{m_k}$ with $|\tilde{r}_{m_k}| = |\tilde{s}_{m_k}| = k$. To conclude, take $r_k = \tilde{r}_{m_k}, s_k = \tilde{s}_{m_k}, t_k = \tilde{t}_{m_k}$. ■

Let us conclude the proof of Theorem 3.1. We may write the implicit operations π and π' as limits of sequences $(r_m t_m s_m)_m$ and $(r_m t'_m s_m)_m$ as in Lemma 3.8. By compactness of $\overline{F}_n(\mathbf{U})$, we may assume that $(r_m)_m, (s_m)_m, (t_m)_m$ and $(t'_m)_m$ also converge, to ρ, σ, τ and τ' respectively. By Proposition 3.5, ρ and

σ are idempotent since $\lim_{m \rightarrow \infty} |r_m| = \lim_{m \rightarrow \infty} |s_m| = +\infty$. By aperiodicity, \mathbf{U} satisfies therefore

$$\begin{aligned}\pi &= \rho^\omega \tau \sigma^\omega = \rho^\omega (\rho \tau \sigma) \sigma^\omega \\ \pi' &= \rho^\omega \tau' \sigma^\omega = \rho^\omega (\rho \tau' \sigma) \sigma^\omega\end{aligned}$$

By hypothesis, $\overline{\mathbf{F}}_n(\mathbf{B})$ satisfies $\pi = \pi'$. Hence, it satisfies also $\rho \tau \sigma = \rho \tau' \sigma$ and it suffices now to use the pseudoidentity (1) to conclude. \blacksquare

4. Some subpseudovarieties of $\mathbf{LI} \vee \mathbf{B}$

In this section, we compute the joins $\mathbf{V} \vee \mathbf{W}$ where $\mathbf{V} \in \{\mathbf{LI}, \mathbf{K}, \mathbf{D}, \mathbf{N}\}$ and $\mathbf{W} \in \{\mathbf{B}, \mathbf{SI}\}$.

Theorem 4.1. *The pseudovarieties $\mathbf{K} \vee \mathbf{B}$ and $\mathbf{D} \vee \mathbf{B}$ are defined by*

$$\mathbf{K} \vee \mathbf{B} = \llbracket x^\omega y = x^\omega y^2, (x^\omega y) = (x^\omega y)^2, (yx^\omega) = (yx^\omega)^2 \rrbracket = \llbracket x^\omega y = x^\omega y^2 \rrbracket \cap (\mathbf{B} \textcircled{m} \mathbf{N})$$

$$\mathbf{D} \vee \mathbf{B} = \llbracket yx^\omega = y^2 x^\omega, (x^\omega y) = (x^\omega y)^2, (yx^\omega) = (yx^\omega)^2 \rrbracket = \llbracket yx^\omega = y^2 x^\omega \rrbracket \cap (\mathbf{B} \textcircled{m} \mathbf{N})$$

Proof. We just prove the first of these equalities. The second one is dual. Let \mathbf{U} be the pseudovariety defined by the pseudoidentities

$$\begin{aligned}(x^\omega y) &= (x^\omega y)^2, & (yx^\omega) &= (yx^\omega)^2 & \text{and} \\ x^\omega y &= x^\omega y^2 & & & (2)\end{aligned}$$

The containment $\mathbf{K} \vee \mathbf{B} \subseteq \mathbf{U}$ is once again straightforward. We now prove that if $\pi, \pi' \in \overline{\mathbf{F}}_n(\mathbf{U})$ and $\mathbf{K} \vee \mathbf{B} \models \pi = \pi'$, then $\pi = \pi'$. We need to prove analogues of Lemma 3.8. We first have:

Lemma 4.2. *Let π and π' be two implicit operations of $\overline{\mathbf{F}}_n(\mathbf{U})$ that coincide over $\overline{\mathbf{F}}_n(\mathbf{B})$. Then, \mathbf{U} satisfies $x^\omega \pi = x^\omega \pi'$.*

Proof. As before, we just prove this pseudoidentity for explicit operations, and we deduce the general case by taking the limit. It suffices to show here that \mathbf{U} satisfies $x^\omega w w' w'' = x^\omega w w'^2 w''$. But \mathbf{U} satisfies

$$\begin{aligned}x^\omega w w' w'' &= (x^\omega w)^\omega w' w'' & \text{as } E(\overline{\mathbf{F}}_n(\mathbf{U})) \text{ is an ideal} \\ &= (x^\omega w)^\omega w'^2 w'' & \text{by (2)} \\ &= x^\omega w w'^2 w'' & \blacksquare\end{aligned}$$

The natural version of Lemma 3.8 is the following:

Lemma 4.3. *Let π and π' be idempotent elements of $\overline{\mathbf{F}}_n(\mathbf{U})$ such that $\mathbf{K} \models \pi = \pi'$. Then, there exist sequences $(r_m)_m, (t_m)_m$ and $(t'_m)_m$ of explicit operations such that*

$$\pi = \lim_{m \rightarrow \infty} r_m t_m, \quad \pi' = \lim_{m \rightarrow \infty} r_m t'_m$$

with $|r_m| \geq m$ for every $m \in \mathbb{N}$.

Proof. It is the same as that of Lemma 3.8, using \mathbf{K}_k instead of \mathbf{LI}_k . \blacksquare

In order to complete the proof of Theorem 4.1, we apply the above lemma, and assume furthermore that $(r_m)_m$, $(t_m)_m$, and $(t'_m)_m$ converge to ρ , τ and τ' respectively: we have $\pi = \rho^\omega \tau$ and $\pi' = \rho^\omega \tau'$. By Proposition 3.5, ρ is idempotent since $\lim_{m \rightarrow \infty} |r_m| = +\infty$. Therefore, by aperiodicity, \mathbf{U} satisfies

$$\begin{aligned}\pi &= \rho^\omega \tau = \rho^\omega(\rho\tau) \\ \pi' &= \rho^\omega \tau' = \rho^\omega(\rho\tau')\end{aligned}$$

But by hypothesis, $\overline{\mathbf{F}}_n(\mathbf{B})$ satisfies $\pi = \pi'$, and therefore, it satisfies also $\rho\tau = \rho\tau'$. The conclusion follows from Lemma 4.2. \blacksquare

The next computation is easier:

Theorem 4.4. *The pseudovariety $\mathbf{N} \vee \mathbf{B}$ is defined by*

$$\mathbf{N} \vee \mathbf{B} = \llbracket x^\omega y = (xy)^\omega = xy^\omega \rrbracket$$

Proof. Let $\mathbf{U} = \llbracket x^\omega y = (xy)^\omega = xy^\omega \rrbracket$. It is clear that, if $S \in \mathbf{U}$, then $E(S)$ is an ideal of S , that is, $\mathbf{U} \subseteq \mathbf{B} \circledast \mathbf{N}$. Moreover, \mathbf{N} and \mathbf{B} clearly satisfy $x^\omega y = (xy)^\omega = xy^\omega$, so $\mathbf{N} \vee \mathbf{B} \subseteq \mathbf{U}$. Let now π and π' be two implicit operations of $\overline{\mathbf{F}}_n(\mathbf{U})$ such that $\mathbf{N} \vee \mathbf{B} \models \pi = \pi'$. We need to show that $\pi = \pi'$. As in the proof of Theorem 3.1, we verify that if one of π and π' is explicit, then \mathbf{N} satisfies a non trivial identity. Since this is not the case, both π and π' are idempotent. Let $(u_m)_m$ and $(v_m)_m$ be sequences of explicit operations on \mathbf{U} that converge to π and π' respectively. Since $\overline{\mathbf{F}}_n(\mathbf{B})$ is finite, $u_m \equiv_n v_m$ holds for m large enough. Now, \mathbf{U} satisfies

$$\begin{aligned}(ww'^2w'')^\omega &= (ww'^2)^\omega w'' \\ &= w(w'^2)^\omega w'' \\ &= ww'^\omega w''\end{aligned}$$

and in the same way, $(ww'w'')^\omega = ww'^\omega w''$. Therefore, \mathbf{U} satisfies $(ww'^2w'')^\omega = (ww'w'')^\omega$. Hence, the explicit operations u_m^ω and v_m^ω are equal for m large enough. Therefore, $\pi = \pi'$. \blacksquare

The next result was first proved by Almeida [7]. Recall that \mathbf{Perm} is the pseudovariety of all finite semigroups that satisfy a non-trivial permutation identity. It is defined by $x^\omega yzx^\omega = x^\omega zyx^\omega$. It is the join of two other pseudovarieties, $\mathbf{Perm}_{(+\infty, 2, 0)} = \llbracket x^\omega yz = x^\omega zy \rrbracket$ and $\mathbf{Perm}_{(0, 2, +\infty)} = \llbracket yzx^\omega = zyx^\omega \rrbracket$.

Theorem 4.5. Almeida [7] *The pseudovariety $\mathbf{LI} \vee \mathbf{SI}$ is defined by:*

$$\mathbf{LI} \vee \mathbf{SI} = \llbracket x^\omega yzx^\omega = x^\omega zyx^\omega \rrbracket_{\mathbf{B} \circledast \mathbf{N}} = (\mathbf{B} \circledast \mathbf{N}) \cap \mathbf{Perm}$$

which can be written as

$$\mathbf{LI} \vee (\mathbf{B} \cap \mathbf{Com}) = (\mathbf{LI} \vee \mathbf{B}) \cap (\mathbf{LI} \vee \mathbf{Com})$$

Proof. The second statement follows from the first and from the fact that $\mathbf{LI} \vee \mathbf{Com} = \mathbf{Perm}$ (Almeida, [2]). There remains to prove the first statement. Let \mathbf{U} be the pseudovariety $(\mathbf{B} \overset{m}{\circ} \mathbf{N}) \cap \mathbf{Perm}$. It is necessary to know a little more about the structure of $\overline{\mathbf{F}}_n(\mathbf{LI} \vee \mathbf{SI})$ to conclude the proof:

Lemma 4.6. *Let ρ, τ, τ' and σ be implicit operations of $\overline{\mathbf{F}}_n(\mathbf{U})$ such that $c(\rho\tau\sigma) = c(\rho\tau'\sigma)$. Then the pseudovariety \mathbf{U} satisfies $\rho^\omega\tau\sigma^\omega = \rho^\omega\tau'\sigma^\omega$.*

Proof. It suffices to show the lemma for explicit operations u, v, v' , and w instead of ρ, τ, τ' and σ respectively. The result follows by taking the limit, since the content function is continuous. One can now show that \mathbf{U} satisfies the pseudoidentity (1) and also $x^\omega yzt^\omega = x^\omega zyt^\omega$. Now, if two words of A_n^+ have the same content, one can pass from one to the other by using a finite number of times the rewriting rules $xyt \rightarrow xy^2t$, $xy^2t \rightarrow xyt$ and $xyzt \rightarrow xzyt$. Therefore, if $r, s \in A_n^+$ have the same content, \mathbf{U} satisfies $x^\omega rt^\omega = x^\omega st^\omega$. So, if $c(uvw) = c(uv'w)$, \mathbf{U} finally satisfies

$$\begin{aligned} u^\omega v w^\omega &= u^\omega (uvw) w^\omega && \text{by aperiodicity} \\ &= u^\omega (uv'w) w^\omega && \text{since } c(uvw) = c(uv'w) \\ &= u^\omega v' w^\omega && \text{by aperiodicity} \end{aligned}$$

■

Now, Theorem 4.5 is a consequence of this lemma: let $(\pi, \pi') \in \overline{\mathbf{F}}_n(\mathbf{U})^2$ such that $\mathbf{LI} \vee \mathbf{SI} \models \pi = \pi'$. We want to show that $\pi = \pi'$. As in the proof of Theorem 3.1, we can assume that both π and π' are idempotent. By Lemma 3.8, π and π' can be written as limits of sequences $(r_m t_m s_m)_m$ and $(r'_m t'_m s'_m)_m$ with $|r_k| = |s_k| = k$. By compactness, we can suppose that $(r_m)_m, (s_m)_m, (t_m)_m$ and $(t'_m)_m$ converge to ρ, σ, τ and τ' respectively. Furthermore, by fact 3 of Proposition 3.5, ρ and σ have to be idempotent. Therefore, $\pi = \rho^\omega \tau \sigma^\omega$ and $\pi' = \rho^\omega \tau' \sigma^\omega$. Now, \mathbf{SI} satisfies $\pi = \pi'$, that is $c(\rho\tau\sigma) = c(\rho\tau'\sigma)$. We then conclude using Lemma 4.6. ■

With the same techniques, one can compute the pseudovarieties $\mathbf{K} \vee \mathbf{SI}$, $\mathbf{D} \vee \mathbf{SI}$ and $\mathbf{N} \vee \mathbf{SI}$.

Theorem 4.7. *The pseudovarieties $\mathbf{K} \vee \mathbf{SI}$ and $\mathbf{D} \vee \mathbf{SI}$ are defined by*

$$\mathbf{K} \vee \mathbf{SI} = (\mathbf{K} \vee \mathbf{B}) \cap \llbracket x^\omega yz = x^\omega zy \rrbracket = (\mathbf{K} \vee \mathbf{B}) \cap \mathbf{Perm}_{(+\infty, 2, 0)}$$

$$\mathbf{D} \vee \mathbf{SI} = (\mathbf{D} \vee \mathbf{B}) \cap \llbracket yzx^\omega = zyx^\omega \rrbracket = (\mathbf{D} \vee \mathbf{B}) \cap \mathbf{Perm}_{(0, 2, +\infty)}$$

which can be written as

$$\mathbf{K} \vee (\mathbf{B} \cap \mathbf{Com}) = (\mathbf{K} \vee \mathbf{B}) \cap (\mathbf{K} \vee \mathbf{Com})$$

$$\mathbf{D} \vee (\mathbf{B} \cap \mathbf{Com}) = (\mathbf{D} \vee \mathbf{B}) \cap (\mathbf{D} \vee \mathbf{Com})$$

Proof. The statements regarding \mathbf{K} and \mathbf{D} are dual, and we prove only that regarding \mathbf{K} . The second part of the statement is a consequence of its first part and from the fact that $\mathbf{K} \vee \mathbf{Com} = \mathbf{Perm}_{(+\infty, 2, 0)}$ (Almeida, [2]). Let again \mathbf{U} be the pseudovariety $(\mathbf{K} \vee \mathbf{B}) \cap \mathbf{Perm}_{(+\infty, 2, 0)}$. The following lemma is analogous to Lemma 4.6:

Lemma 4.8. *Let ρ , τ , and τ' be implicit operations of $\overline{F}_n(\mathbf{U})$ such that $c(\rho\tau) = c(\rho\tau')$. Then the pseudovariety \mathbf{U} satisfies $\rho^\omega\tau = \rho^\omega\tau'$.*

Proof. As usual, it suffices to show this lemma for explicit operations u , v , and v' instead of ρ , τ and τ' respectively, and the result follows by taking the limit. With the same proof as in Lemma 4.2, one shows that this pseudoidentity holds in \mathbf{U} and that $x^\omega yz = x^\omega zy$ also holds. It is not difficult to check then, as in Lemma 4.6, that \mathbf{U} satisfies $x^\omega u = x^\omega u'$ if u and u' have the same content. ■

As before, one can then write $\pi = \rho^\omega\tau$ and $\pi' = \rho^\omega\tau'$, and it suffices then to use the same technique as for $\mathbf{LI} \vee \mathbf{SI}$. ■

The last result goes back to Almeida again. Recall that \mathbf{ZE} is the pseudovariety of semigroups in which idempotents are central.

Theorem 4.9. Almeida [7, 3] *The pseudovariety $\mathbf{N} \vee \mathbf{SI}$ is defined by*

$$\mathbf{N} \vee \mathbf{SI} = (\mathbf{N} \vee \mathbf{B}) \cap \mathbf{ZE}$$

One can also write

$$\mathbf{N} \vee (\mathbf{B} \cap \mathbf{Com}) = (\mathbf{N} \vee \mathbf{B}) \cap (\mathbf{N} \vee \mathbf{Com})$$

Proof. The inclusion from left to right of each of the equalities is once again clear. Let \mathbf{U} be the pseudovariety $(\mathbf{N} \vee \mathbf{B}) \cap \mathbf{ZE}$. The proof that \mathbf{U} is contained in $\mathbf{N} \vee \mathbf{SI}$ follows the same pattern as the preceding proofs: it suffices to show that \mathbf{U} satisfies $\pi = \pi'$ if π and π' are two idempotents which have the same content.

The proof of the second equality is easy: the inclusion from left to right is straightforward, and the one from right to left comes from the inclusion $\mathbf{N} \vee \mathbf{Com} \subseteq \mathbf{ZE}$. Notice however that \mathbf{ZE} is not permutative, so $\mathbf{ZE} \neq \mathbf{N} \vee \mathbf{Com}$. ■

5. Implicit operations of $\mathbf{LI} \vee \mathbf{B}$

For comparison, recall that an implicit operation on \mathbf{LI} that is not explicit can be viewed as a pair of a left and a right infinite word [7, 34]. Indeed, as \mathbf{N} , which does not satisfy any non-trivial identity, is contained in \mathbf{LI} , the semigroup $F_n(\mathbf{LI})$ is free over $\{x_1, \dots, x_n\}$. If a sequence $(u_p)_p$ of $F_n(\mathbf{LI})$ converges in $\overline{F}_n(\mathbf{LI})$, then, this sequence converges in every (finite) semigroup $F_n(\mathbf{LI}_k) \in \mathbf{LI}$. If it is not a constant sequence, the prefixes (resp. the suffixes) of length k of u_p and u_q coincide if p and q are large enough.

Conversely, since the semigroups $F_n(\mathbf{LI}_k)$ generate \mathbf{LI} , every sequence satisfying this condition converges in $\overline{F}_n(\mathbf{LI})$. One can therefore view the non explicit operations as pairs of $A_n^\omega \times {}^\omega A_n$ where ${}^\omega A_n$ is the set of left infinite words and A_n^ω the set of right infinite words over the alphabet A_n . The product over $\overline{F}_n(\mathbf{LI})$ is defined by

$$\begin{aligned} u \cdot (\pi, \rho) &= (u\pi, \rho) && \text{if } \pi \in A_n^\omega, \rho \in {}^\omega A_n \text{ and } u \in A_n^+ \\ (\pi, \rho) \cdot u &= (\pi, \rho u) && \text{if } \pi \in A_n^\omega, \rho \in {}^\omega A_n \text{ and } u \in A_n^+ \\ (\pi, \rho) \cdot (\pi', \rho') &= (\pi, \rho') && \text{if } \pi, \pi' \in A_n^\omega \text{ and } \rho, \rho' \in {}^\omega A_n \end{aligned}$$

In other words, $\overline{F}_n(\mathbf{LI}) \setminus F_n(\mathbf{LI})$ is a rectangular band and $\overline{F}_n(\mathbf{LI})$ is a nilpotent extension of a rectangular band.

The semigroup $\overline{F}_n(\mathbf{LI} \vee \mathbf{B})$ has quite a similar structure: since $\mathbf{LI} \vee \mathbf{B}$ contains \mathbf{N} , the semigroup $F_n(\mathbf{LI} \vee \mathbf{B})$ is the free semigroup over $\{x_1, \dots, x_n\}$ (equipped with the discrete topology). Furthermore, as an implicit operation $\pi \in \overline{F}_n(\mathbf{B} \circledast \mathbf{N})$ is determined by its value over \mathbf{LI} on one hand and by its value over $\overline{F}_n(\mathbf{B})$ on the other hand, a non explicit operation over $\mathbf{LI} \vee \mathbf{B}$ can be viewed as a triple of $A_n^\omega \times F_n(\mathbf{B}) \times {}^\omega A_n$. The product over $\overline{F}_n(\mathbf{LI})$ is now defined by

$$\begin{aligned} v \cdot (\pi, u, \rho) &= (v\pi, vu, \rho) && \text{if } \pi \in A_n^\omega, \rho \in {}^\omega A_n \text{ and } v \in A_n^+ \text{ and } u \in F_n(\mathbf{B}) \\ (\pi, u, \rho) \cdot v &= (\pi, uv, \rho v) && \text{if } \pi \in A_n^\omega, \rho \in {}^\omega A_n \text{ and } v \in A_n^+ \text{ and } u \in F_n(\mathbf{B}) \\ (\pi, u, \rho) \cdot (\pi', u', \rho') &= (\pi, uu', \rho') && \text{if } \pi, \pi' \in A_n^\omega, \rho, \rho' \in {}^\omega A_n, u, u' \in F_n(\mathbf{B}) \end{aligned}$$

where the product of u by v is evaluated in free band.

In other words, non explicit operations on $\mathbf{LI} \vee \mathbf{B}$ form a band, more precisely the direct product of the rectangular band $A_n^\omega \times {}^\omega A_n$ by the free band. Therefore, the semigroup $\overline{F}_n(\mathbf{LI} \vee \mathbf{B})$ is a nilpotent extension of a band, which is not surprising since $\mathbf{LI} \vee \mathbf{B} = \mathbf{B} \circledast \mathbf{N}$.

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