

# The Covering Problem: a Unified Approach for Investigating the Expressive Power of Logics

Thomas Place and Marc Zeitoun\*

LaBRI, Bordeaux University, France, [firstname.lastname@labri.fr](mailto:firstname.lastname@labri.fr)

---

## Abstract

An important endeavor in computer science is to precisely understand the expressive power of logical formalisms over discrete structures, such as words. Naturally, “understanding” is not a mathematical notion. Therefore, this investigation requires a concrete objective to capture such a notion. In the literature, the standard choice for this objective is the *membership problem*, whose aim is to find a procedure deciding whether an input regular language can be defined in the logic under study. This approach was cemented as the “right” one by the seminal work of Schützenberger, McNaughton and Papert on first-order logic and has been in use since then.

However, membership questions are hard: for several important fragments, researchers have failed in this endeavor despite decades of investigation. In view of recent results on one of the most famous open questions, namely the quantifier alternation hierarchy of first-order logic, an explanation may be that membership is too restrictive as a setting. These new results were indeed obtained by considering more general problems than membership, taking advantage of the increased flexibility of the enriched mathematical setting. This opens a promising avenue of research and efforts have been devoted at identifying and solving such problems for natural fragments. However, until now, these problems have been *ad hoc*, most fragments relying on a specific one. A unique new problem replacing membership as the right one is still missing.

The main contribution of this paper is a suitable candidate to play this role: the Covering Problem. We motivate this problem with three arguments. First, it admits an elementary set theoretic formulation, similar to membership. Second, we are able to reexplain or generalize all known results with this problem. Third, we develop a mathematical framework as well as a methodology tailored to the investigation of this problem.

**1998 ACM Subject Classification** F.4.3 Formal Languages

**Keywords and phrases** Membership Problem, Separation Problem, Covering Problem, Regular Languages, Logics, Decidable Characterizations

**Digital Object Identifier** 10.4230/LIPIcs.MFCS.2016.78

## 1 Introduction

One of the most successful applications of the notion of regularity in computer science is the investigation of logics on discrete structures such as words or trees. The story began in the 60s when Büchi [4], Elgot [8] and Trakhtenbrot [32] proved that the *regular languages* of finite words are those that can be defined in monadic second order logic (MSO). This result has since been exploited to study the expressive power of important fragments of MSO by relying on a decision problem: the *membership problem*. Given a regular language as input, this problem asks if it can be defined by a sentence of the fragment under investigation.

---

\* This study has been carried out with financial support from the French State, managed by the French National Research Agency (ANR) in the frame of the “Investments for the future” Programme IdEx Bordeaux – CPU (ANR-10-IDEX-03-02).



© Thomas Place and Marc Zeitoun;  
licensed under Creative Commons License CC-BY

41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016).

Editors: Piotr Faliszewski, Anca Muscholl, and Rolf Niedermeier; Article No. 78; pp. 78:1–78:48

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Getting membership algorithms is difficult. In fact, this is still open on finite trees for the most natural fragment of MSO, namely first-order logic (FO). On words however, this question was solved in the 70s by Schützenberger, McNaughton and Papert [27, 12]. This theorem was very influential and has often been revisited [34, 7, 5, 18]. It paved the way to a series of results of the same nature. A famous example is Simon’s Theorem [28], which yields an algorithm for the first level of the quantifier alternation hierarchy of FO. Other examples include [3, 11, 35, 29] which consider fragments of FO where the linear order on positions is replaced by the successor relation or [30] which considers the 2-variable fragment of FO. The relevance of this approach is nowadays validated by a wealth of results.

The reason for this success is twofold. First, these results cemented membership as the “right” question: a solution conveys a deep intuition on the investigated logic. In particular, most results include a *generic method for building a canonical sentence* witnessing membership of an input language in the logic. Second, Schützenberger’s solution established a suitable framework and a methodology to solve membership problems. This methodology is based on a canonical algebraic abstraction of a regular language which is finite and computable, the *syntactic monoid*. The core of the approach is to translate the semantic question (*is the language definable in the fragment?*) into a purely syntactical, easy question to be tested on the syntactic monoid (*does the syntactic monoid satisfy some equation?*).

Unfortunately, this methodology seems to have reached its limits for the hardest questions. An emblematic example is the *quantifier alternation hierarchy of first-order logic* which classifies sentences according to the number of alternations between  $\exists$  and  $\forall$  quantifiers in their prenex normal form. A sentence is  $\Sigma_i$  if its prenex normal form has  $(i-1)$  alternations and starts with a block of existential quantifiers. A sentence is  $\mathcal{B}\Sigma_i$  if it is a boolean combination of  $\Sigma_i$  sentences. Obtaining membership algorithms for all levels in this hierarchy is a major open question and has been given a lot of attention (see [33, 31, 13, 14, 15, 16, 25, 17] for details and a complete bibliography). However, progress on this question has been slow: until recently, only the lowest levels were solved:  $\Sigma_1$  [2, 19],  $\mathcal{B}\Sigma_1$  [28] and  $\Sigma_2$  [2, 19].

It took years to solve higher levels. Recently, membership algorithms were obtained for  $\Sigma_3$  [23],  $\mathcal{B}\Sigma_2$  [23] and  $\Sigma_4$  [20]. This was achieved by introducing new ingredients into Schützenberger’s methodology: problems that are *more general than membership*. For each result, the strategy is the same: first, a well-chosen more general problem is solved for a lower level in the hierarchy, then, this result is transferred into a membership algorithm for the level under investigation. Let us illustrate what we mean by “more general problem” and present the simplest of them: the *separation problem*. It takes *two* regular languages as input and asks whether there exists a third one which is definable in the logic, contains the first, and is disjoint from the second. Being more general, such problems are also more difficult than membership. However, this generality also makes them more rewarding in the insight they give on the investigated logic. This motivated a series of papers on the separation problem [26, 6, 21, 22, 24] which culminated in the three results above [23, 20]. However, while this avenue of research is very promising, it presently suffers three important flaws:

1. The family of problems that have been considered up until now is a jungle: each particular result relies on a specific ad-hoc problem. For example, the results of [23, 20] rely on three different problems. In fact, even if one is only interested in separation, the actual solution often considers an even more general problem (see [26, 23, 20] for example).
2. Among the problems that have been investigated, separation is the only one that admits a simple and generic set-theoretic definition (which is why it is favored as an example). On the other hand, for all other problems, the definition requires to introduce additional concepts such as semigroups and Ehrenfeucht-Fraïssé games.

3. In contrast to membership solutions, the solutions that have been obtained for these more general problems are non-constructive. For example, most of the separation solutions do not include a generic method for building a separator language when it exists (the algorithms are built around the idea of proving that the two inputs are *not* separable).

**Contributions.** Our objective in this paper is to address these three issues. Our first contribution is the presentation of a single general problem, the “*covering problem*”, which admits a purely set-theoretic definition and generalizes all problems that have already been considered. Furthermore, its definition is modular: the covering problem is designed so that it can easily be generalized to accommodate future needs. Its design is based on an analysis of the methods used to solve membership and separation. In both cases, the algorithms almost always exploit the fact that an input regular language  $L$  is *not isolated*: its recognizer defines a *set* of regular languages from which  $L$  is built. This set has a structure upon which the algorithms are based. The covering problem takes this observation into account: an input of the problem is directly *any finite set* of regular languages. Given such a set  $\mathbf{L}$ , the problem asks to compute the “best possible approximation” (called *optimal cover*, hence the name “covering”) of this set of languages by languages belonging to the investigated fragment. In particular, the separation problem is just the special case when the input set is of size 2.

The main advantage of the covering problem is that it comes with a generic framework and a generic methodology designed for solving it. This framework is our second contribution. It generalizes the original framework of Schützenberger for membership in a natural way and lifts all its benefits to a more general setting. In particular, we recover *constructiveness*: a solution to the covering problem associated to a particular fragment yields a generic way for building an actual optimal cover of the input set.

Finally, the relevance of our new framework is supported by the fact that we are able to obtain covering algorithms for the fragments that were already known to enjoy a decidable separation problem. In contrast to the previous algorithms, these more general ones are presented within a single unified framework. This is our third contribution. We present actual covering algorithms for four particular logics: first-order logic (FO), two-variables FO ( $\text{FO}^2$ ) and two logics within the quantifier alternation hierarchy of FO ( $\mathcal{B}\Sigma_1$  and  $\Sigma_2$ ). As explained, the payoff is that we obtain *effective* solutions to the covering problem. Hence, we obtain an effective method for building separators in the weaker separation problem.

**Historical note.** As observed by Almeida [1], separation is tied to a purely algebraic problem of Henckell and Rhodes (see [9, 10]): computing the “pointlike sets of a given finite semigroup with respect to a variety  $\mathbf{V}$ ”. This can probably be lifted to covering. However, there are two main advantages to our approach. First, it is more general: pointlike sets are restricted to classes and inputs that are both more specific than ours. Second, covering admits a simple set theoretic definition that pointlike sets obfuscate with heavy terminology.

**Organization.** We define the covering problem in Section 2 (for arbitrary input sets of languages, *i.e.*, not necessarily made of regular languages). We present our framework for the particular case of regular inputs in Sections 3 and 4. Four examples of covering algorithms are presented in Section 5. Due to lack of space, proofs are deferred to the journal version.

## 2 The Covering Problem

In this section, we define the covering problem. For the whole paper, we fix a finite alphabet  $A$  and work with finite words over  $A$  (*i.e.*, elements of  $A^*$ ). A *language* is a subset of  $A^*$ . Note that we restrict ourselves to words for the sake of simplifying the presentation. However, the covering problem makes sense for *any structure* (such as infinite words or trees).

We focus on two kinds of classes of languages. We say that a class of languages  $\mathcal{C}$  is a *lattice* when it contains the empty and universal languages ( $\emptyset$  and  $A^*$ ) and it is closed under finite union and finite intersection:  $K, L \in \mathcal{C}$  implies  $K \cup L, K \cap L \in \mathcal{C}$ . Furthermore,  $\mathcal{C}$  is a *boolean algebra* when  $\mathcal{C}$  is a lattice that is closed under complement:  $L \in \mathcal{C}$  implies  $\{w \in A^* \mid w \notin L\} \in \mathcal{C}$ . The covering problem then comes into two variants:

- a variant that can be associated to any class of languages that is a lattice. We call this variant the *pointed covering problem*.
- a weaker variant that can be associated to any class of languages that is a boolean algebra. We call it the *covering problem*. While weaker than the first one, this variant enjoys simpler terminology, which makes it our choice when working with boolean algebras.

We now define these two variants. In the definition, we use the separation problem as a foundation to motivate and explain our design choices. As we explained, given a class of languages  $\mathcal{C}$ , solutions to membership and separation exploit the fact that the recognizer of an input regular language  $L$  recognizes a *set* of regular languages from which  $L$  is built. The covering problem is based on this observation: its input is any finite set of languages  $\mathbf{L}$ .

► **Remark.** A “set of languages” is a purely mathematical object. An actual input is a set of recognizing devices for these languages. In particular, it may happen that two such devices recognize the same language. Therefore our inputs are actually finite sets of languages *names* (which may contain “several copies” of the same language). This is harmless: two sets of names for the same underlying set of languages are equivalent for both covering problems.

## 2.1 The Covering Problem for Boolean Algebras

We begin with the simpler covering problem. Let  $\mathcal{C}$  be a boolean algebra<sup>1</sup>. Given a finite set of languages names  $\mathbf{L} = \{L_1, \dots, L_n\}$ , a  $\mathcal{C}$ -*cover* of  $\mathbf{L}$  is a finite set of languages  $\mathbf{K} = \{K_1, \dots, K_m\}$  such that  $K_i \in \mathcal{C}$  for all  $i \leq m$  and:

$$L_1 \cup \dots \cup L_n \subseteq K_1 \cup \dots \cup K_m.$$

Note that since  $\mathcal{C}$  is a boolean algebra, there always exists a  $\mathcal{C}$ -cover of  $\mathbf{L}$ : the singleton  $\{A^*\}$ . When we have a  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  in hand, our main interest will be to know how good  $\mathbf{K}$  is at separating languages in  $\mathbf{L}$ : what languages in  $\mathbf{L}$  are separated by unions of languages in  $\mathbf{K}$ ? What are the “best  $\mathcal{C}$ -covers” of  $\mathbf{L}$  (called optimal  $\mathcal{C}$ -covers)? This information is captured by a new object that we associate to any cover of  $\mathbf{L}$ , its *imprint* on  $\mathbf{L}$ .

**Filterings and Imprints.** Imprints are based on filterings. Given a finite set of names  $\mathbf{L}$  and a language  $K$ , the *filtering of  $\mathbf{L}$  by  $K$* , measures the “interaction” between  $\mathbf{L}$  and  $K$ . More precisely, the filtering of  $\mathbf{L}$  by  $K$ , denoted by  $\langle \mathbf{L} | K \rangle$ , is defined as the following set:

$$\langle \mathbf{L} | K \rangle = \{L \in \mathbf{L} \mid L \cap K \neq \emptyset\} \subseteq \mathbf{L}$$

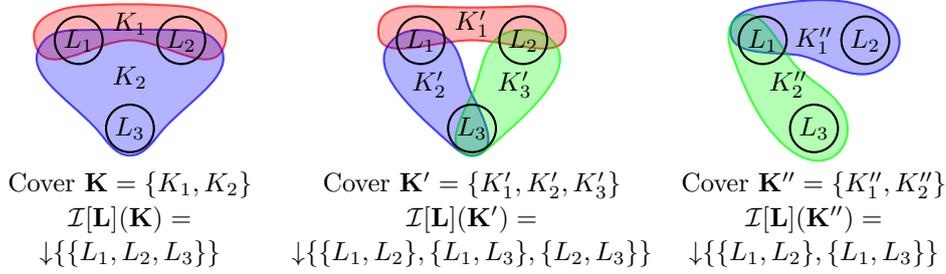
► **Remark.** This notion is what makes the problem modular. It can be strengthened to define harder variants of the problem and accommodate future needs.

We may now define imprints. Given a subset  $E$  of  $2^{\mathbf{L}}$ , we write  $\downarrow E$  to denote the *downset* of  $E$ , i.e., the set  $\downarrow E = \{\mathbf{H} \mid \exists \mathbf{H}' \in E \text{ such that } \mathbf{H} \subseteq \mathbf{H}'\}$ . If  $\mathbf{K}$  is a finite set of languages, the *imprint of  $\mathbf{K}$  on  $\mathbf{L}$*  is the set,

$$\mathcal{I}[\mathbf{L}](\mathbf{K}) = \downarrow \{\langle \mathbf{L} | K \rangle \mid K \in \mathbf{K}\} \subseteq 2^{\mathbf{L}}$$

<sup>1</sup> The problem actually makes sense for any class that contains the universal language and is closed under intersection. However, we need  $\mathcal{C}$  to be a boolean algebra for the connection with separation.

Note that we shall mainly use this definition when  $\mathbf{K}$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$ . However, in some proofs, it will be convenient to have it for an arbitrary set of languages  $\mathbf{K}$ . We present examples of imprints when  $\mathbf{K}$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$  in Figure 1.



■ **Figure 1** Some  $\mathcal{C}$ -covers of  $\mathbf{L} = \{L_1, L_2, L_3\}$  and their imprint on  $\mathbf{L}$

Let us make a few observations about imprints. An imprint on  $\mathbf{L}$  is a subset of  $2^{\mathbf{L}}$ . Therefore, for a fixed *finite* set  $\mathbf{L}$ , there are *finitely many* possible imprints on  $\mathbf{L}$ , even though there are infinitely many finite sets  $\mathbf{K}$  of languages. Another simple observation is that all imprints are closed under downset:  $\mathcal{I}[\mathbf{L}](\mathbf{K}) = \downarrow\mathcal{I}[\mathbf{L}](\mathbf{K})$ . Also notice that if  $\mathbf{K}$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$ , its imprint captures separation-related information: if  $\{L_1, L_2\} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$ , then  $L_1$  (resp.  $L_2$ ) can be separated from  $L_2$  (resp.  $L_1$ ) by a union (in  $\mathcal{C}$ ) of languages in  $\mathbf{K}$ .

► **Remark.** Imprints capture more than just separation-related information. From the separation point of view, the  $\mathcal{C}$ -covers  $\mathbf{K}$  and  $\mathbf{K}'$  of Figure 1 are equivalent: they cannot separate any pair of languages in  $\mathbf{L}$ . However, their imprints on  $\mathbf{L}$  tell us that  $\mathbf{K}'$  is “better” as it covers  $\mathbf{L}$  without containing a language that intersects all languages in  $\mathbf{L}$  at the same time.

Finally, observe that if  $\mathbf{K}$  is a  $\mathcal{C}$ -cover of a finite set  $\mathbf{L}$ , then its imprint on  $\mathbf{L}$  always contains some trivial elements. To any finite set of names  $\mathbf{L}$ , we associate the following set:

$$\mathcal{I}_{triv}[\mathbf{L}] = \downarrow\{\{\mathbf{L}\{w\}\} \mid w \in A^*\} = \{\mathbf{H} \subseteq \mathbf{L} \mid \cap_{H \in \mathbf{H}} H \neq \emptyset\}$$

► **Fact 1.** For any  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ , we have  $\mathcal{I}_{triv}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$ .

**Optimal  $\mathcal{C}$ -covers.** We now use imprints to define our notion of “best”  $\mathcal{C}$ -cover of  $\mathbf{L}$  which we call *optimal  $\mathcal{C}$ -covers*. A necessary (but not sufficient) property for a  $\mathcal{C}$ -cover of  $\mathbf{L}$  to be optimal will be that  $L_1, L_2 \in \mathbf{L}$  are  $\mathcal{C}$ -separable if and only if they can be separated by a union of languages in the  $\mathcal{C}$ -cover. Formally, we say that a  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  is *optimal* when,

$$\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}') \quad \text{for any } \mathcal{C}\text{-cover } \mathbf{K}' \text{ of } \mathbf{L}.$$

In general, there can be infinitely many optimal  $\mathcal{C}$ -covers of a given finite set of names  $\mathbf{L}$ . We now state that for any  $\mathbf{L}$ , there always exists an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Note that the proof only requires  $\mathcal{C}$  to be closed under finite intersection.

► **Lemma 2.** For any finite set of languages names  $\mathbf{L}$ , there exists an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ .

Note that the proof of Lemma 2 is non-constructive. Given a finite set of names  $\mathbf{L}$ , computing an actual optimal  $\mathcal{C}$ -cover is a difficult problem in general. In fact, as seen in Theorem 4 below, this is more general than solving  $\mathcal{C}$ -separability for any pair of languages in  $\mathbf{L}$ . Before we present this theorem, let us make a key observation about optimal  $\mathcal{C}$ -covers.

By definition, given a boolean algebra  $\mathcal{C}$  and a finite set of names  $\mathbf{L}$ , all optimal  $\mathcal{C}$ -covers of  $\mathbf{L}$  have the same imprint on  $\mathbf{L}$ . Hence, this unique imprint on  $\mathbf{L}$  is a *canonical* object for  $\mathcal{C}$  and  $\mathbf{L}$ . We call it the *optimal imprint with respect to  $\mathcal{C}$  on  $\mathbf{L}$*  and we denote it by  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ :

$$\mathcal{I}_{\mathcal{C}}[\mathbf{L}] = \mathcal{I}[\mathbf{L}](\mathbf{K}) \quad \text{for any optimal } \mathcal{C}\text{-cover } \mathbf{K} \text{ of } \mathbf{L}.$$

We can now state the *covering problem*. We parametrize it by two classes of languages, a class  $\mathcal{D}$  constraining the input, and a boolean algebra  $\mathcal{C}$ .

► **Definition 3.** The Covering problem for  $\mathcal{C}$  inside  $\mathcal{D}$  is as follows:

**INPUT:** A finite set of languages  $\mathbf{L} \subseteq \mathcal{D}$ .  
**QUESTION:** Compute  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .

As expected, we only consider the covering problem when the input class  $\mathcal{D}$  is the class of regular languages (in particular we will often simply say “covering problem” for this particular variant). There are two stages when solving the covering problem.

1. *Stage One:* find an algorithm that, given a finite set of regular languages  $\mathbf{L}$  as input, computes  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  (we call such an algorithm a covering algorithm for  $\mathcal{C}$ ). In Theorem 4 below, we prove that this generalizes separation as a *decision problem*.
2. *Stage Two:* find an algorithm that, given a finite set of regular languages  $\mathbf{L}$  as input, computes an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$  (i.e., one whose imprint is  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ ). We prove below that this generalizes separation as a *computational problem*: if one has an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ , one may build a separator in  $\mathcal{C}$  for any two separable languages in  $\mathbf{L}$ .

► **Theorem 4.** Let  $\mathcal{C}$  be a boolean algebra and let  $\mathbf{L}$  be a finite set of languages names. Given any two language name’s  $L_1, L_2 \in \mathbf{L}$ , the following properties are equivalent:

1.  $L_1$  and  $L_2$  are  $\mathcal{C}$ -separable.
2.  $\{L_1, L_2\} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .
3. For any optimal  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ ,  $L_1$  and  $L_2$  are  $\mathcal{C}$ -separable by a union of languages in  $\mathbf{K}$ .

Theorem 4 will be proved in the journal version of this paper. It entails that ‘covering’ is a more general problem than ‘separation’. It is actually *strictly* more general as  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  captures more information than which pairs of languages in  $\mathbf{L}$  are  $\mathcal{C}$ -separable.

## 2.2 The Pointed Covering Problem for Lattices

So far, we connected the separation problem to the more general *covering problem*. Unfortunately, while the definition of the covering problem makes sense for all lattices, the connection with separation stated in Theorem 4 requires the investigated class  $\mathcal{C}$  to be a boolean algebra. When  $\mathcal{C}$  is not closed under complement, the optimal imprint  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  does not capture enough information to decide whether two languages in  $\mathbf{L}$  are  $\mathcal{C}$ -separable.

► **Example 5.** Let  $\mathcal{C}$  be the class of languages which are unions and intersections of languages of the form  $A^*aA^*$  for some  $a \in A$ . Observe that  $L_1 = A^*aA^* \cap A^*bA^*$  is  $\mathcal{C}$ -separable from  $L_2 = a^*$  ( $L_1$  belongs to  $\mathcal{C}$  and  $L_1 \cap L_2 = \emptyset$ ). However, it can be verified that the optimal imprint with respect to  $\mathcal{C}$  on  $\{L_1, L_2\}$  is  $\mathcal{I}_{\mathcal{C}}[\{L_1, L_2\}] = \{\emptyset, \{L_1\}, \{L_2\}, \{L_1, L_2\}\}$ .

We solve this issue with a new problem generalizing separation for any lattice of languages  $\mathcal{C}$ : the *pointed  $\mathcal{C}$ -covering problem*. The main idea behind this new problem is to replace the notion of cover of a finite set of languages names  $\mathbf{L}$  with a more general one: *pointed covers*. When a class of languages  $\mathcal{C}$  is a lattice but not a boolean algebra (i.e.,  $\mathcal{C}$  is *not closed under complement*), the associated separation problem is asymmetric: given  $L_1, L_2 \subseteq A^*$ , the two following problems are non-equivalent:

- finding  $K_1 \in \mathcal{C}$  such that  $L_1 \subseteq K_1$  and  $K_1 \cap L_2 = \emptyset$ .
- finding  $K_2 \in \mathcal{C}$  such that  $L_2 \subseteq K_2$  and  $K_2 \cap L_1 = \emptyset$ .

From the point of view of  $\mathcal{C}$ -covers, this means that we have to define a notion of “ $\mathcal{C}$ -cover of  $\{L_1, L_2\}$ ” making a distinction between the languages used to cover  $L_1$  and those used to cover  $L_2$ . This is what pointed  $\mathcal{C}$ -covers are designed for.

**Pointed  $\mathcal{C}$ -covers.** Let  $\mathbf{L}$  be a finite set of names. An  $\mathbf{L}$ -pointed set of languages is a finite set  $\mathbb{P} \subseteq \mathbf{L} \times 2^{A^*}$  (i.e., elements of  $\mathbb{P}$  are pairs  $(L, K)$  where  $L$  is a name in  $\mathbf{L}$  and  $K$  is an arbitrary language). Furthermore, we call *support* of  $\mathbb{P}$  the set  $\mathbf{K} = \{K \mid (L, K) \in \mathbb{P} \text{ for some } L \in \mathbf{L}\}$ . In other words the support of  $\mathbb{P}$  is the smallest set of languages such that  $\mathbb{P} \subseteq \mathbf{L} \times \mathbf{K}$ . Finally, when we have an  $\mathbf{L}$ -pointed set of languages  $\mathbb{P}$  with support  $\mathbf{K}$  in hand, for all  $L \in \mathbf{L}$ , we will denote by  $\mathbb{P}(L) \subseteq \mathbf{K}$  the set of all  $K \in \mathbf{K}$  such that  $(L, K) \in \mathbb{P}$ .

We may now define *pointed  $\mathcal{C}$ -covers*. Let  $\mathcal{C}$  be a lattice. Given a finite set of languages names  $\mathbf{L}$ , a *pointed  $\mathcal{C}$ -cover* of  $\mathbf{L}$  is an  $\mathbf{L}$ -pointed set of languages  $\mathbb{P}$  such that all  $K$  in the support of  $\mathbb{P}$  belong to  $\mathcal{C}$  and for all  $L \in \mathbf{L}$ ,

$$L \subseteq \bigcup_{K \in \mathbb{P}(L)} K \quad (\text{i.e., } \mathbb{P}(L) \text{ is a cover of } \{L\})$$

Note that since  $\mathcal{C}$  is a lattice, we have  $A^* \in \mathcal{C}$ . Hence, for all finite sets  $\mathbf{L}$ , there always exists a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ : the set  $\{(L, A^*) \mid L \in \mathbf{L}\}$ .

► **Remark.** Pointed  $\mathcal{C}$ -covers are more general than  $\mathcal{C}$ -covers: if  $\mathbb{P}$  is a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ , then the support  $\mathbf{K}$  of  $\mathbb{P}$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Intuitively, pointed  $\mathcal{C}$ -covers capture more information: they record for each  $L \in \mathbf{L}$  which languages in  $\mathbf{K}$  are needed to cover  $L$ . We use this additional information to define a finer notion of optimality.

**Pointed Imprints.** We now generalize imprints to pointed covers with the notion of pointed imprint (also based on the notion of filtering which is unchanged). To define pointed imprints, we first have to generalize the notion of downset to our new setting. If  $\mathbf{L}$  is a finite set of language names and  $E \subseteq \mathbf{L} \times 2^{\mathbf{L}}$ , we denote by  $\downarrow E$  the set,

$$\downarrow E = \{(L, \mathbf{H}) \mid \text{there exists } (L, \mathbf{H}') \in E \text{ such that } \mathbf{H} \subseteq \mathbf{H}'\}$$

We may now define pointed imprints. Let  $\mathbf{L}$  be a finite set of language names and let  $\mathbb{P}$  be an  $\mathbf{L}$ -pointed set of languages. The *pointed imprint of  $\mathbb{P}$  on  $\mathbf{L}$*  is the set,

$$\mathcal{P}[\mathbf{L}](\mathbb{P}) = \downarrow \{(L, \langle \mathbf{L} \mid K \rangle) \mid (L, K) \in \mathbb{P}\} \subseteq \mathbf{L} \times 2^{\mathbf{L}}$$

This new notion of pointed imprint has similar properties to those of the original notion of imprint. For a fixed  $\mathbf{L}$ , any pointed imprint on  $\mathbf{L}$  is a subset of  $\mathbf{L} \times 2^{\mathbf{L}}$ , so there are finitely many pointed imprints on  $\mathbf{L}$ . Furthermore, pointed imprints are closed under downset.

Moreover, as for imprints, pointed imprints contain some trivial elements. If  $\mathbf{L}$  is a finite set of languages, we let

$$\mathcal{P}_{triv}[\mathbf{L}] = \downarrow \{(L, \langle \mathbf{L} \mid \{w\} \rangle) \mid L \in \mathbf{L} \text{ and } w \in L\} = \{(L, \mathbf{H}) \mid (\bigcap_{H \in \mathbf{H}} H) \cap L \neq \emptyset\}$$

► **Fact 6.** For any pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ , we have  $\mathcal{P}_{triv}[\mathbf{L}] \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P})$ .

**Optimal Pointed  $\mathcal{C}$ -Covers.** We can now define optimal pointed  $\mathcal{C}$ -covers. The definition is similar to that of optimal  $\mathcal{C}$ -covers. We say that a pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$  is *optimal* when,

$$\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}') \quad \text{for any pointed } \mathcal{C}\text{-cover } \mathbb{P}' \text{ of } \mathbf{L}$$

► **Lemma 7.** *For any finite set of languages names  $\mathbf{L}$ , there exists an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ .*

As Lemma 2, Lemma 7 is based on closure under intersection. We now generalize the notion of optimal imprint. By definition, all optimal pointed  $\mathcal{C}$ -covers of  $\mathbf{L}$  share the same pointed imprint on  $\mathbf{L}$ . Hence, this unique pointed imprint is a *canonical object* for  $\mathcal{C}$  and  $\mathbf{L}$ . We call it the *optimal pointed imprint with respect to  $\mathcal{C}$  on  $\mathbf{L}$*  denoted by  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ :

$$\mathcal{P}_{\mathcal{C}}[\mathbf{L}] = \mathcal{P}[\mathbf{L}](\mathbf{K}) \quad \text{for any optimal pointed } \mathcal{C}\text{-cover } \mathbf{K} \text{ of } \mathbf{L}$$

We are now ready to state the pointed covering problem. As before, it is parametrized by a class  $\mathcal{D}$  constraining the input, and a lattice  $\mathcal{C}$ .

► **Definition 8.** The Pointed covering problem for  $\mathcal{C}$  inside  $\mathcal{D}$  is as follows:

**INPUT:** A finite set of languages  $\mathbf{L} \subseteq \mathcal{D}$ .  
**QUESTION:** Compute  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ .

Similarly to the covering problem, there are two stages when solving the pointed covering problem for a given lattice  $\mathcal{C}$ . The first one is to find an algorithm that computes  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  from  $\mathbf{L}$  and the second one is to find a generic method for constructing optimal pointed  $\mathcal{C}$ -covers. We now make the connection with the  $\mathcal{C}$ -separation problem in the following theorem.

► **Theorem 9.** *Let  $\mathcal{C}$  be a lattice and let  $\mathbf{L}$  be a finite set of languages. Given any two languages  $L_1, L_2 \in \mathbf{L}$ , the following properties are equivalent:*

1.  $L_1$  is  $\mathcal{C}$ -separable from  $L_2$ .
2.  $(L_1, \{L_2\}) \notin \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ .
3. For any optimal pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ , the language  $\bigcup_{K \in \mathbb{P}(L_1)} K$  separates  $L_1$  from  $L_2$ .

Let us make two remarks. The first one is that for any lattice  $\mathcal{C}$ , pointed covering is more general than covering. The second is that while this relation can be strict (see Example 5), this only happens when the class  $\mathcal{C}$  is not closed under complement: if  $\mathcal{C}$  is a boolean algebra, then the two problems are equivalent. In other words, when  $\mathcal{C}$  is a boolean algebra, there is no point in considering pointed covering: the covering problem (which relies on simpler terminology) suffices. We refer to the journal version of this paper for details.

Now that we have defined both covering problems, the remaining sections are devoted to presenting their benefits. In particular, we present a general methodology for regular inputs in Sections 3 and 4 and use it in Section 5 on specific examples. Note that in contrast to this section which was generic to all types of structures and inputs, the remainder of the paper is specific to words and regular languages: we will rely on the fact that our inputs are sets of regular languages of finite words in our methodology.

### 3 Tame Sets of Languages

We now present a special class of input sets for the covering problem that we call the class of *tame* sets of languages names. A tame set contains only regular languages and has a specific algebraic structure (which is connected to language concatenation). While not all finite sets of regular languages are tame, we will be able to restrict our algorithms to such inputs without loss of generality. This restriction is central: we rely heavily on the properties of tame inputs in all our algorithms. The typical example of a tame set is the following.

► **Example 10.** Given a nondeterministic finite automaton (NFA)  $\mathcal{A} = (A, Q, I, F, \delta)$ , the set  $\{L_{q,r} \mid (q, r) \in Q^2\}$  is tame (where  $L_{q,r}$  is a name for the language  $\{w \mid q \xrightarrow{w} r\}$ ).

### 3.1 Definition

A finite set of languages names is said to be *tame* if it can be given a *partial semigroup structure*. Let us first define *partial semigroups*. A partial semigroup is a set  $S$  equipped with a partial multiplication (*i.e.*,  $st$  may not be defined for all  $s, t \in S$ ) such that for all  $r, s, t \in S$ , if  $rs$  and  $st$  are both defined, then  $(rs)t$  and  $r(st)$  are defined and equal.

We may now define tame sets. Let  $\mathbf{L}$  be a finite set of languages names. A *tame multiplication* for  $\mathbf{L}$  is a partial semigroup multiplication “ $\odot$ ” (we use this notation to avoid confusion with language concatenation) that satisfies the following properties:

- (1) for all  $L, L' \in \mathbf{L}$ , if  $L \odot L'$  is defined then  $LL' \subseteq L \odot L'$ .
- (2) for all  $H \in \mathbf{L}$  and all words  $w \in H$ , if  $w$  may be decomposed as  $w = uu'$ , then there exist  $L, L' \in \mathbf{L}$  such that  $u \in L$ ,  $u' \in L'$  and  $H = L \odot L'$ .

We say that a finite set of languages names  $\mathbf{L}$  is *tame* if it can be equipped with a *tame multiplication*. Note that when working with tame sets, we will implicitly assume that we have a tame multiplication “ $\odot$ ” for this set. Furthermore, since  $\mathbf{L}$  is a finite partial semigroup, it is known that there exists an integer  $\omega(\mathbf{L})$  (denoted by  $\omega$  when  $\mathbf{L}$  is understood) such that if  $L \odot L$  is defined, then  $L^\omega$  is defined and idempotent (*i.e.*,  $L^\omega \odot L^\omega = L^\omega$ ).

An important observation is that tame sets of languages names may only contain *regular languages*, as stated in the following lemma (proved in the journal version).

► **Lemma 11.** *Any language in a tame set of languages is regular.*

Unfortunately, the converse of Lemma 11 is not true: there are finite sets of regular languages that are not tame. For example, the set  $\mathbf{L} = \{\{ab\}\}$  fails Condition (2). However, this issue is easily solved with the following proposition.

► **Proposition 12.** *Let  $\mathbf{H} = \{H_1, \dots, H_n\}$  be a finite set of languages given by  $n$  NFAs  $\mathcal{A}_1, \dots, \mathcal{A}_n$ . There exists a tame set of languages names  $\mathbf{L}$  such that for any lattice  $\mathcal{C}$ ,*

- $\mathcal{I}_{\mathcal{C}}[\mathbf{H}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{H}]$ ) can be computed from  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ ).
- any optimal (pointed)  $\mathcal{C}$ -cover of  $\mathbf{L}$  is an optimal (pointed)  $\mathcal{C}$ -cover of  $\mathbf{H}$ .
- $\mathbf{L}$  and its tame multiplication can be computed from  $\mathcal{A}_1, \dots, \mathcal{A}_n$  in polynomial time and has size  $|\mathcal{A}_1|^2 + \dots + |\mathcal{A}_n|^2$  (where  $|\mathcal{A}_i|$  stands for the number of states of  $\mathcal{A}_i$ ).

Proposition 12 is proved in the journal version (the construction is based on Example 10). From now on, we will assume that our inputs are tame. We finish the section by explaining the benefits of considering tame inputs in the covering and pointed covering problems.

### 3.2 Tame Sets of Languages and the Covering Problems

As explained, we will restrict our inputs to tame sets. We now have to explain the benefits of such a restriction. In order to get these benefits, we need the investigated class  $\mathcal{C}$  to satisfy a new property in addition to being a boolean algebra or a lattice. The *left quotient* of a language  $L$  by a word  $w$  is the language  $w^{-1}L = \{u \in A^* \mid wu \in L\}$ . The *right quotient*  $Lw^{-1}$  is defined symmetrically. A class of languages is a *quotienting boolean algebra* if it is a boolean algebra of regular languages closed under left and right quotient. A *quotienting lattice* is a lattice of regular languages closed under left and right quotients.

When  $\mathbf{L}$  is tame, the partial semigroup multiplication  $\odot$  over  $\mathbf{L}$  can be extended as a semigroup multiplication over  $2^{\mathbf{L}}$ :  $\mathbf{S} \odot \mathbf{R} = \{S \odot R \mid S \in \mathbf{S}, R \in \mathbf{R} \text{ and } S \odot R \text{ is defined}\}$ . Hence,  $2^{\mathbf{L}}$  is a semigroup and  $\mathbf{L} \times 2^{\mathbf{L}}$  a partial semigroup. It turns out that when  $\mathcal{C}$  is a quotienting lattice these structures are transferred to  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}] \subseteq 2^{\mathbf{L}}$  and  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] \subseteq \mathbf{L} \times 2^{\mathbf{L}}$ .

► **Lemma 13.** *Let  $\mathcal{C}$  be a quotienting lattice and let  $\mathbf{L}$  be a tame set of languages. Then the two following properties holds:*

- (1)  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  is closed under multiplication: for all  $(L_1, \mathbf{L}_1), (L_2, \mathbf{L}_2)$  in  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ , if  $L_1 \odot L_2$  is defined, then  $(L_1 \odot L_2, \mathbf{L}_1 \odot \mathbf{L}_2) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ .
- (2)  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  is closed under multiplication: for all  $\mathbf{L}_1$  and  $\mathbf{L}_2$  in  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ ,  $\mathbf{L}_1 \odot \mathbf{L}_2 \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .

Lemma 13 will be proved in the full version. Let us explain why it is crucial. We do it in the setting of the covering problem, which is simpler. We start with the following statement.

► **Lemma 14.** *Let  $\mathbf{L}$  be a tame set of languages and let  $K_1, K_2$  be two languages, then  $\langle \mathbf{L}|K_1 \rangle \odot \langle \mathbf{L}|K_2 \rangle = \langle \mathbf{L}|K_1 K_2 \rangle$ .*

Let  $\mathcal{C}$  be a boolean algebra and  $\mathbf{L}$  be a finite set of names. A natural method for building an optimal  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  is to start from  $\mathbf{K} = \mathcal{I}_{triv}[\mathbf{L}]$  and to add new languages  $K$  in  $\mathcal{C}$  to  $\mathbf{K}$  until  $\mathbf{K}$  covers  $\mathbf{L}$ . By definition of imprints, for  $\mathbf{K}$  to be optimal, we need all such candidate languages  $K$  to satisfy  $\langle \mathbf{L}|K \rangle \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ . It follows from Lemma 13 and Lemma 14 that when  $\mathcal{C}$  is a quotienting boolean algebra and  $\mathbf{L}$  is tame, these  $K$  may be built with concatenation: if we already have  $K_1$  and  $K_2$  such that  $\langle \mathbf{L}|K_1 \rangle, \langle \mathbf{L}|K_2 \rangle \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ , then we may add  $K_1 K_2$  as well since by Lemmas 13 and 14,  $\langle \mathbf{L}|K_1 K_2 \rangle = \langle \mathbf{L}|K_1 \rangle \odot \langle \mathbf{L}|K_2 \rangle \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .

This is central for classes of languages defined through logic (such as first-order logic). Indeed, concatenation is a fundamental process for building new languages in such classes.

## 4 General Approach

In this section, we present a natural methodology for attempting to solve the covering or pointed covering problem for a particular input class  $\mathcal{C}$ . This is the methodology that we use for all examples of Section 5.

Let  $\mathcal{C}$  be a quotienting boolean algebra or a quotienting lattice. Recall that since we restrict ourselves to tame sets, the two objectives of the covering (resp. pointed covering) problem are as follows. Given as input a tame set  $\mathbf{L}$ ,

1. we want an algorithm that computes  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ ).
2. we want an algorithm that computes optimal  $\mathcal{C}$ -covers (resp. pointed  $\mathcal{C}$ -covers).

We now detail our methodology for the pointed covering problem (the case of the weaker covering problem is similar, see Section 5). This methodology consists in three steps.

**Step 1: Presentation of the Pointed Covering Algorithm.** The first step presents a solution to stage one: an algorithm that takes as input a tame set  $\mathbf{L}$  and computes  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ . This step only presents the algorithm: the second and third steps are devoted to its proof.

A key point is that pointed covering algorithms are designed as *lowest fixpoint algorithms*. Since  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  is a pointed imprint on  $\mathbf{L}$ , we have  $\mathcal{P}_{triv}[\mathbf{L}] \subseteq \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  (Fact 6). All our algorithms start from  $\mathcal{P}_{triv}[\mathbf{L}]$ , and then add new elements using finitely many operations until a fixpoint is reached. Among these operations, some are specific to the particular quotienting lattice  $\mathcal{C}$  that we consider, and some are *generic* to all quotienting lattices. In particular, the set of operations that we use will always include downset and multiplication (see Lemma 13). To sum up, our algorithms compute  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  as the smallest set  $Sat_{\mathcal{C}}(\mathbf{L}) \subseteq \mathbf{L} \times 2^{\mathbf{L}}$  (*Sat* means ‘saturation’), containing  $\mathcal{P}_{triv}[\mathbf{L}]$  and closed under the following operations:

1. Downset:  $Sat_{\mathcal{C}}(\mathbf{L}) = \downarrow Sat_{\mathcal{C}}(\mathbf{L})$ .
2. Multiplication: if  $(L, \mathbf{H}), (L', \mathbf{H}') \in Sat_{\mathcal{C}}(\mathbf{L})$ , then  $(L \odot L', \mathbf{H} \odot \mathbf{H}') \in Sat_{\mathcal{C}}(\mathbf{L})$  (if defined).
3.  $\dots$  (additional operation(s) specific to  $\mathcal{C}$ ).

**Step 2: Soundness.** The second step is devoted to proving that the covering algorithm of Step 1 is sound, *i.e.*, that  $Sat_{\mathcal{C}}(\mathbf{L}) \subseteq \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ : for any pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ ,  $Sat_{\mathcal{C}}(\mathbf{L}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P})$ . This is the “easy” direction and it involves Ehrenfeucht-Fraïssé arguments.

**Step 3: Completeness.** The third step is devoted to proving that the covering algorithm of Step 1 is complete, *i.e.*, that  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] \subseteq Sat_{\mathcal{C}}(\mathbf{L})$ . While usually difficult, this proof is of particular interest as it yields a solution to second stage of the pointed covering problem as a byproduct: an algorithm that computes optimal pointed  $\mathcal{C}$ -covers.

The proof of this step should be presented as a generic construction for building an actual pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$  whose imprint on  $\mathbf{L}$  is included in  $Sat_{\mathcal{C}}(\mathbf{L})$ . This proves that  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\mathcal{C}}(\mathbf{L})$ , and therefore completeness. However, by combining this with the knowledge that the algorithm is also sound (this is proved in Step 2), we obtain that  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] = \mathcal{P}[\mathbf{L}](\mathbb{P})$ . In other words the proof builds an *optimal pointed  $\mathcal{C}$ -cover*  $\mathbb{P}$  of  $\mathbf{L}$ .

## 5 Examples of Covering Algorithms

We now present examples of covering algorithms for several classical logical fragments, all based on first-order logic on words. Let us first briefly recall the definition of first-order logic over finite words. A word is viewed as logical structure made of a sequence of positions labeled over  $A$ . In first-order logic over words (FO), for each  $a \in A$ , one is allowed to use a unary predicate “ $a(x)$ ” which selects positions  $x$  labeled with an  $a$ , as well as a binary predicate “ $<$ ” for the linear order. A language  $L$  is said to be *first-order definable* if there is an FO sentence  $\varphi$  such that  $L = \{w \mid w \models \varphi\}$ . Also denote by FO the class of all first-order definable languages. We present algorithms for FO itself and its fragments  $\mathcal{B}\Sigma_1$ ,  $\text{FO}^2$ ,  $\Sigma_2$ .

Note that we only present Step 1 of our methodology in the main text, *i.e.*, algorithms without their proofs. An important remark is that these proofs are all difficult: while we have a generic template, proving a covering algorithm always requires arguments specific to the investigated class. We present proofs for  $\mathcal{B}\Sigma_1$ ,  $\text{FO}^2$  and  $\Sigma_2$  in the full version of this paper. The proof for FO is omitted as it is close to proof of [26] (which is based on a prototype of the present framework). On the other hand, the algorithms and proofs for  $\mathcal{B}\Sigma_1$ ,  $\text{FO}^2$  and  $\Sigma_2$  are new.

**First-Order Logic: FO.** The first algorithm that we present is for FO itself, which is among the most famous classes of regular languages in the literature. The decidability of the membership problem for FO was proved by Schützenberger, McNaughton and Papert [27, 12] and the result is among those that started this line of research. Separation was later proved to be decidable as well [9, 10, 26]. As explained the covering algorithm is a generalization of that of [26] (which is based on a prototype of this framework). As FO is known to be a quotienting boolean algebra, we use the covering problem.

► **Theorem 15.** *Let  $\mathbf{L}$  be a tame set of languages. Then  $\mathcal{I}_{\text{FO}}[\mathbf{L}]$  is the smallest subset of  $2^{\mathbf{L}}$  containing  $\mathcal{I}_{\text{triv}}[\mathbf{L}]$ , closed under downset, multiplication and such that for all  $\mathbf{S} \in \mathcal{I}_{\text{FO}}[\mathbf{L}]$ , we have  $\mathbf{S}^\omega \cup \mathbf{S}^{\omega+1} \in \mathcal{I}_{\text{FO}}[\mathbf{L}]$ .*

**Boolean Combinations of  $\Sigma_1$ :  $\mathcal{B}\Sigma_1$ .** The next class that we use as an example is  $\mathcal{B}\Sigma_1$ , which is the restriction of FO to sentences that are boolean combinations of  $\Sigma_1$  sentences. A sentence is  $\Sigma_1$  if its prenex normal form uses only existential quantifiers. The class  $\mathcal{B}\Sigma_1$  is famous in the literature. the decidability of  $\mathcal{B}\Sigma_1$ -membership was proved by Simon [28].  $\mathcal{B}\Sigma_1$ -separation is also known to be decidable [6, 21]. As  $\mathcal{B}\Sigma_1$  is known to be a quotienting boolean algebra, we use the covering problem. Given a word  $w \in A^*$ , we denote by  $\mathbf{alph}(w)$  the set of letters occurring in  $w$ , *i.e.* the smallest subset of  $B$  of  $A$  such that  $w \in B^*$ .

► **Theorem 16.** *Let  $\mathbf{L}$  be a tame set of languages.  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$  is the smallest subset of  $2^{\mathbf{L}}$  containing  $\mathcal{I}_{triv}[\mathbf{L}]$ , closed under downset, multiplication and such that for all  $B \subseteq A$ , if  $\mathbf{H} = \{L \in \mathbf{L} \mid \exists w \in L, \text{s.t. } \mathbf{alph}(w) = B\}$ , then  $\mathbf{H}^\omega \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ .*

**Two-variable First-Order Logic:  $\text{FO}^2$ .** The logic  $\text{FO}^2$  is the restriction of FO to sentences that use at most two distinct variables (which may be reused). That the associated membership problem is decidable is due to Thérien and Wilke [30]. The separation problem was proved to be decidable in [21]. As  $\text{FO}^2$  is known to be a quotienting boolean algebra, we use the covering problem. Our algorithm requires the input to satisfy a new condition in addition to being tame: *alphabet compatibility* (this may be assumed without loss of generality, as will be shown in the full version). A set  $\mathbf{L}$  is said to be *alphabet compatible* if for all languages  $L \in \mathbf{L}$ , there exists a unique  $B \subseteq A$  such that for any  $w \in L$ ,  $\mathbf{alph}(w) = B$ . Note that when  $\mathbf{L}$  is alphabet compatible, then  $\mathbf{alph}(L)$  is well-defined for all  $L \in \mathbf{L}$  as this unique alphabet.

► **Theorem 17.** *Let  $\mathbf{L}$  be a tame and alphabet compatible set of languages.  $\mathcal{I}_{\text{FO}^2}[\mathbf{L}]$  is the smallest subset of  $2^{\mathbf{L}}$  containing  $\mathcal{I}_{triv}[\mathbf{L}]$ , closed under downset, multiplication and such that for all  $B \subseteq A$  and  $\mathbf{S}, \mathbf{T} \in \mathcal{I}_{\text{FO}^2}[\mathbf{L}]$  containing  $S, T$  with  $\mathbf{alph}(S) = \mathbf{alph}(T) = B$ ,*

$$\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \mathcal{I}_{\text{FO}^2}[\mathbf{L}].$$

**One Quantifier Alternation:  $\Sigma_2$ .** Our third example is  $\Sigma_2$ , which is the restriction of FO to sentences whose prenex normal form have a quantifier prefix of the form ‘ $\exists^* \forall^*$ ’. It was proved that  $\Sigma_2$ -membership is decidable in [2, 19] and the same was proved for separation in [23]. As  $\Sigma_2$  is a quotienting lattice but not a boolean algebra, we use the pointed covering problem. Our algorithm requires the input to be tame and alphabet compatible.

► **Theorem 18.** *Let  $\mathbf{L}$  be a tame and alphabet compatible set of languages  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  is the smallest subset of  $\mathbf{L} \times 2^{\mathbf{L}}$  containing  $\mathcal{P}_{triv}[\mathbf{L}]$ , closed under downset, multiplication and such that for any  $B \subseteq A$ , and  $(S, \mathbf{S}) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$  satisfying  $\mathbf{alph}(S) = B$  and  $S \odot S$  is defined,*

$$(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}].$$

## 6 Conclusion

We introduced the covering and pointed covering problems which are designed to investigate quotienting boolean algebras and quotienting lattices respectively. We also presented a methodology outlining how these problems should be approached. Furthermore, we presented four examples of algorithms for the instances associated to FO,  $\mathcal{B}\Sigma_1$ ,  $\text{FO}^2$  and  $\Sigma_2$ .

It is worth noting that while our examples include the most significant logics for which separation is known to be decidable, an important one is missing:  $\Sigma_3$ . This is not surprising as the algorithm of [20] considers an *ad hoc* problem which is associated to two logics at the same time:  $\Sigma_2$  and  $\Sigma_3$ . However, it is possible to generalize this result as well within our framework: this is where the modularity of our problems comes into play. Using a stronger notion of filtering, one can reformulate and generalize the problem of [20] as an instance of the pointed covering problem (we leave the presentation of this instance for further work).

Our results raise several questions. The most natural is to apply our framework to classes for which no membership or separation algorithm is known yet. Another one is related to the classical membership algorithms. These algorithms are usually stated as equations on the syntactic monoid of the language which share similarities with fixpoint operations of our (pointed) covering algorithms. An interesting question would be to find a criterion under which membership equations can be lifted as a fixpoint operation for the covering problem.

## References

- 1 Jorge Almeida. Some algorithmic problems for pseudovarieties. *Publicationes Mathematicae Debrecen*, 54:531–552, 1999.
- 2 Mustapha Arfi. Polynomial operations on rational languages. In *Proceedings of the 4th Annual Symposium on Theoretical Aspects of Computer Science (STACS'87)*, pages 198–206, 1987.
- 3 Janusz A. Brzozowski and Imre Simon. Characterizations of locally testable events. *Discrete Mathematics*, 4(3):243–271, 1973.
- 4 Julius R. Büchi. Weak second-order arithmetic and finite automata. *Mathematical Logic Quarterly*, 6(1-6):66–92, 1960.
- 5 Thomas Colcombet. Green's relations and their use in automata theory. In *Proceedings of Language and Automata Theory and Applications, 5th International Conference (LATA'11)*, pages 1–21, 2011.
- 6 Wojciech Czerwiński, Wim Martens, and Tomáš Masopust. Efficient separability of regular languages by subsequences and suffixes. In *Proceedings of the 40th International Colloquium on Automata, Languages, and Programming (ICALP'13)*, pages 150–161, 2013.
- 7 Volker Diekert and Paul Gastin. First-order definable languages. In Jörg Flum, Erich Grädel, and Thomas Wilke, editors, *Logic and Automata: History and Perspectives*, volume 2 of *Texts in Logic and Games*, pages 261–306. Amsterdam University Press, 2008.
- 8 Calvin C. Elgot. Decision problems of finite automata design and related arithmetics. *Transactions of the AMS*, 98(1):21–51, 1961.
- 9 Karsten Henckell. Pointlike sets: the finest aperiodic cover of a finite semigroup. *J. Pure Appl. Algebra*, 55:85–126, 1988.
- 10 Karsten Henckell, John Rhodes, and Benjamin Steinberg. Aperiodic pointlikes and beyond. *Internat. J. Algebra Comput.*, 20(2):287–305, 2010.
- 11 Robert McNaughton. Algebraic decision procedures for local testability. *Mathematical Systems Theory*, 8(1):60–76, 1974.
- 12 Robert McNaughton and Seymour A. Papert. *Counter-Free Automata*. The MIT Press, 1971.
- 13 Jean-Éric Pin. Finite semigroups and recognizable languages: An introduction. In *Semigroups, Formal Languages and Groups*, pages 1–32. Springer-Verlag, 1995.
- 14 Jean-Éric Pin. Syntactic semigroups. In *Handbook of Formal Languages*, pages 679–746. Springer-Verlag, 1997.
- 15 Jean-Éric Pin. Bridges for concatenation hierarchies. In *Proceedings of the 25th International Colloquium on Automata, Languages and Programming, ICALP'98*, Lecture Notes in Computer Science, pages 431–442, Berlin, Heidelberg, 1998. Springer-Verlag.
- 16 Jean-Éric Pin. Theme and variations on the concatenation product. In *Proceedings of the 4th International Conference on Algebraic Informatics, CAI'11*, Lecture Notes in Computer Science, pages 44–64, Berlin, Heidelberg, 2011. Springer-Verlag.
- 17 Jean-Éric Pin. The dot-depth hierarchy, 45 years later. In *WSPC Proceedings*, 2016. To appear.
- 18 Jean-Éric Pin. Mathematical foundations of automata theory. In preparation, 2016. URL: <https://www.irif.univ-paris-diderot.fr/~jep/MPRI/MPRI.html>.
- 19 Jean-Éric Pin and Pascal Weil. Polynomial closure and unambiguous product. *Theory of Computing Systems*, 30(4):383–422, 1997.
- 20 Thomas Place. Separating regular languages with two quantifiers alternations. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'15)*, pages 202–213, 2015.

- 21 Thomas Place, Lorijn van Rooijen, and Marc Zeitoun. Separating regular languages by piecewise testable and unambiguous languages. In *Proceedings of the 38th International Symposium on Mathematical Foundations of Computer Science*, MFCS'13, pages 729–740, 2013.
- 22 Thomas Place, Lorijn van Rooijen, and Marc Zeitoun. Separating regular languages by locally testable and locally threshold testable languages. *Logical Methods in Computer Science*, 10(3), 2014.
- 23 Thomas Place and Marc Zeitoun. Going higher in the first-order quantifier alternation hierarchy on words. In *Automata, Languages, and Programming - 41st International Colloquium (ICALP'14)*, pages 342–353, 2014.
- 24 Thomas Place and Marc Zeitoun. Separation and the successor relation. In *32nd International Symposium on Theoretical Aspects of Computer Science (STACS'15)*, pages 662–675, 2015.
- 25 Thomas Place and Marc Zeitoun. The tale of the quantifier alternation hierarchy of first-order logic over words. *SIGLOG news*, 2(3):4–17, 2015.
- 26 Thomas Place and Marc Zeitoun. Separating regular languages with first-order logic. *Logical Methods in Computer Science*, 12(1), 2016.
- 27 Marcel Paul Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8(2):190–194, 1965.
- 28 Imre Simon. Piecewise testable events. In *Proc. of the 2nd GI Conf. on Automata Theory and Formal Languages*, pages 214–222, 1975.
- 29 Denis Thérien and Alex Weiss. Graph congruences and wreath products. *J. Pure Appl. Algebra*, 36:205–215, 1985.
- 30 Denis Thérien and Thomas Wilke. Over words, two variables are as powerful as one quantifier alternation. In *Proceedings of the 30th Annual ACM Symposium on Theory of Computing (STOC'98)*, pages 234–240, 1998.
- 31 Wolfgang Thomas. Languages, automata, and logic. In *Handbook of formal languages*. Springer, 1997.
- 32 Boris A. Trakhtenbrot. Finite automata and logic of monadic predicates. *Doklady Akademii Nauk SSSR*, 149:326–329, 1961. In Russian.
- 33 Pascal Weil. Concatenation product: a survey. In *Formal Properties of Finite Automata and Applications*, volume 386 of *Lecture Notes in Computer Science*, pages 120–137. Springer-Verlag, Berlin, Heidelberg, 1989.
- 34 Thomas Wilke. Classifying discrete temporal properties. In *Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science, STACS'99*, Lecture Notes in Computer Science, pages 32–46, Berlin, Heidelberg, 1999. Springer-Verlag.
- 35 Yechezkel Zalcstein. Locally testable languages. *Journal of Computer and System Sciences*, 6(2):151–167, 1972.

## A Appendix to Section 2

In this appendix, we present the missing proofs of Section 2. Furthermore, we connect the covering and pointed covering problems by proving the latter is more general than the former.

### Proofs for the Covering Problem

We prove Fact 1, Lemma 2 and Theorem 4. The three results depend on a boolean algebra  $\mathcal{C}$  and a finite set of language names  $\mathbf{L}$ . We begin with the proof of Fact 1. Let us first restate this fact.

► **Fact 1.** *For any  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ , we have  $\mathcal{I}_{triv}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$ .*

**Proof.** Recall that  $\mathcal{I}_{triv}[\mathbf{L}] = \downarrow\{\langle \mathbf{L}|\{w\} \rangle \mid w \in A^*\}$  and let  $\mathbf{K}$  be a  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Since  $\mathcal{I}[\mathbf{L}](\mathbf{K})$  is closed under downset by definition, we have to prove that for all  $w \in A^*$ ,  $\langle \mathbf{L}|\{w\} \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K})$ . Let  $w \in A^*$ . If  $\langle \mathbf{L}|\{w\} \rangle = \emptyset$ , then  $\langle \mathbf{L}|\{w\} \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K})$  by closure under downset. Otherwise let  $\mathbf{H} = \langle \mathbf{L}|\{w\} \rangle$  and let  $H \in \mathbf{H}$ . By definition  $w \in H$  and since  $\mathbf{K}$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$ , there exists  $K \in \mathbf{K}$  such that  $w \in K$ . It follows that  $\langle \mathbf{L}|\{w\} \rangle \subseteq \langle \mathbf{L}|K \rangle$  and therefore that  $\langle \mathbf{L}|\{w\} \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K})$ , again by closure under downset. ◀

We now prove Lemma 2. We first recall the statement.

► **Lemma 2.** *For any finite set of languages names  $\mathbf{L}$ , there exists an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ .*

**Proof.** We already know that the set of  $\mathcal{C}$ -covers of  $\mathbf{L}$  is not empty, since the singleton containing the universal language only is such a cover. We prove that for any two  $\mathcal{C}$ -covers  $\mathbf{K}'$  and  $\mathbf{K}''$  of  $\mathbf{L}$ , there exists a third  $\mathcal{C}$ -cover  $\mathbf{K}$  such that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$  and  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}'')$ . Since there are only finitely possible imprints on  $\mathbf{L}$ , the statement will follow. Define  $\mathbf{K} = \{K' \cap K'' \mid K' \in \mathbf{K}' \text{ and } K'' \in \mathbf{K}''\}$ . Since  $\mathbf{K}'$  and  $\mathbf{K}''$  are  $\mathcal{C}$ -covers of  $\mathbf{L}$ , the set  $\mathbf{K}$  is also a cover of  $\mathbf{L}$ . Moreover, it is a  $\mathcal{C}$ -cover since  $\mathcal{C}$  is closed under intersection. Finally, it is immediate from the definitions that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$  and  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}'')$ . ◀

We finish with the proof of Theorem 4. We begin by recalling the statement.

► **Theorem 4.** *Let  $\mathcal{C}$  be a boolean algebra and let  $\mathbf{L}$  be a finite set of languages names. Given any two languages  $L_1, L_2 \in \mathbf{L}$ , the following properties are equivalent:*

1.  $L_1$  and  $L_2$  are  $\mathcal{C}$ -separable.
2.  $\{L_1, L_2\} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .
3. For any optimal  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ ,  $L_1$  and  $L_2$  are separable by a union of languages in  $\mathbf{K}$ .

**Proof.** We prove 3)  $\Rightarrow$  1)  $\Rightarrow$  2)  $\Rightarrow$  3). Let us first assume that 3) holds, *i.e.*, that for any optimal  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ ,  $L_1$  and  $L_2$  are separable by a union of languages in  $\mathbf{K}$ . Since there exists at least one  $\mathcal{C}$ -cover of  $\mathbf{L}$  that is optimal (by Lemma 2),  $L_1$  can be separated from  $L_2$  with a union of languages in  $\mathcal{C}$ . Since  $\mathcal{C}$  is closed under union, this separator is in  $\mathcal{C}$  and 1) holds.

We now prove 1)  $\Rightarrow$  2). Assume that 1) holds, *i.e.*, that  $L_1$  is  $\mathcal{C}$ -separable from  $L_2$ . This means that there exists a language  $K \in \mathcal{C}$  such that  $L_1 \subseteq K$  and  $K \cap L_2 = \emptyset$  (*i.e.*,  $L_2 \subseteq A^* \setminus K$ ). Since  $\mathcal{C}$  is closed under complement,  $A^* \setminus K \in \mathcal{C}$  and  $\mathbf{K} = \{K, A^* \setminus K\}$  is a  $\mathcal{C}$ -cover. By construction,  $\{L_1, L_2\} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$ , hence  $\{L_1, L_2\} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  since  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$  by definition.

It remains to prove  $2) \Rightarrow 3)$ . Assume that  $\{L_1, L_2\} \notin \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  and let  $\mathbf{K}$  be an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Since  $\mathbf{K}$  is optimal, we know from our hypothesis that  $\{L_1, L_2\} \notin \mathcal{I}[\mathbf{L}](\mathbf{K})$ . By definition of  $\mathcal{I}[\mathbf{L}](\mathbf{K})$  it follows that no  $K \in \mathbf{K}$  intersects both  $L_1$  and  $L_2$ . Hence the union of all languages  $K \in \mathbf{K}$  that intersect  $L_1$  separates  $L_1$  from  $L_2$ .  $\blacktriangleleft$

### Proofs for the Pointed Covering Problem

We prove Fact 6, Lemma 7 and Theorem 9. The three results depend on a lattice  $\mathcal{C}$  and a finite set of language names  $\mathbf{L}$ . We begin with the proof of Fact 6. Let us first restate this fact.

► **Fact 6.** *For any pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ , we have  $\mathcal{P}_{triv}[\mathbf{L}] \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P})$ .*

**Proof.** Recall that  $\mathcal{P}_{triv}[\mathbf{L}] = \downarrow\{(L, \langle \mathbf{L} \setminus \{w\} \rangle) \mid L \in \mathbf{L} \text{ and } w \in L\}$  and let  $\mathbb{P}$  be a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . By definition, we have to prove that for all  $L \in \mathbf{L}$  and all  $w \in L$ , we have  $(L, \langle \mathbf{L} \setminus \{w\} \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ . Since  $\mathbb{P}$  is a pointed  $\mathcal{C}$ -cover, we have  $L \subseteq \mathbb{P}(L)$ , hence there exists  $K \in \mathbb{P}(L)$  such that  $w \in K$ . Hence  $\langle \mathbf{L} \setminus \{w\} \rangle \subseteq \langle \mathbf{L} \setminus K \rangle$  and  $(L, \langle \mathbf{L} \setminus \{w\} \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$  by closure under downset.  $\blacktriangleleft$

We now prove Lemma 7.

► **Lemma 7.** *For any finite set of languages names  $\mathbf{L}$ , there exists an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ .*

**Proof.** We know that there always exists a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Therefore, in order to prove that there always exists an optimal one, it suffices to prove that for any two pointed  $\mathcal{C}$ -covers  $\mathbb{P}'$  and  $\mathbb{P}''$  of  $\mathbf{L}$ , there exists a third one  $\mathbb{P}$  such that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}')$  and  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}'')$ . Since there are only finitely possible pointed imprints on  $\mathbf{L}$ , the lemma will follow. We define,

$$\mathbb{P} = \{(L, K' \cap K'') \mid (L, K') \in \mathbb{P}' \text{ and } (L, K'') \in \mathbb{P}''\}$$

Since  $\mathbb{P}'$  and  $\mathbb{P}''$  are pointed  $\mathcal{C}$ -covers of  $\mathbf{L}$ ,  $\mathbb{P}$  is also a pointed cover of  $\mathbf{L}$ . Moreover, it is a pointed  $\mathcal{C}$ -cover since  $\mathcal{C}$  is closed under intersection. Finally, it is immediate from the definitions that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}')$  and  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}'')$ .  $\blacktriangleleft$

We finish with Theorem 9.

► **Theorem 9.** *Let  $\mathcal{C}$  be a lattice and let  $\mathbf{L}$  be a finite set of languages. Given any two languages  $L_1, L_2 \in \mathbf{L}$ , the following properties are equivalent:*

1.  $L_1$  is  $\mathcal{C}$ -separable from  $L_2$ .
2.  $(L_1, \{L_2\}) \notin \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ .
3. For any optimal pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ , the language  $\bigcup_{K \in \mathbb{P}(L_1)} K$  separates  $L_1$  from  $L_2$ .

**Proof.** The proof is similar to that of Theorem 4. We prove that  $3) \Rightarrow 1) \Rightarrow 2) \Rightarrow 3)$ . Let us first assume that 3) holds, *i.e.*, that the language  $\bigcup_{K \in \mathbb{P}(L_1)} K$  separates  $L_1$  from  $L_2$ . Since there exists at least one pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$  that is optimal (see Lemma 7),  $L_1$  can be separated from  $L_2$  with a union of languages in  $\mathcal{C}$ . Since  $\mathcal{C}$  is closed under union, this separator is in  $\mathcal{C}$  and 1) holds.

We now prove that  $1) \Rightarrow 2)$ . Assume that 1) holds, *i.e.*, that  $L_1$  is  $\mathcal{C}$ -separable from  $L_2$ . This means that there exists a language  $K \in \mathcal{C}$  such that  $L_1 \subseteq K$  and  $K \cap L_2 = \emptyset$ . We define,

$$\mathbb{P} = \{(L_1, K)\} \cup \{(L, A^*) \mid L \neq L_1\}$$

By definition,  $\mathbb{P}$  is a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$  and  $(L_1, \{L_2\}) \notin \mathcal{P}[\mathbf{L}](\mathbb{P})$ . Therefore,  $(L_1, \{L_2\}) \notin \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  since  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P})$  by definition.

It remains to prove that 2)  $\Rightarrow$  3). Assume that  $(L_1, \{L_2\}) \notin \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  and let  $\mathbb{P}$  be an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Since  $\mathbb{P}$  is optimal, we know from our hypothesis that  $(L_1, \{L_2\}) \notin \mathcal{P}[\mathbf{L}](\mathbb{P})$ . Hence it follows from the definition that no  $K \in \mathbb{P}(L_1)$  intersects  $L_2$ . Hence  $L_1$  can be separated from  $L_2$  by  $\bigcup_{K \in \mathbb{P}(L_1)} K$ .  $\blacktriangleleft$

## Connection with the Covering Problem

In this subsection, we connect the pointed covering problem to the covering problem. First, we formally state that for any class  $\mathcal{C}$  that is a lattice, the pointed covering problem for  $\mathcal{C}$  generalizes the covering problem for  $\mathcal{C}$  (*i.e.*, the latter can be reduced to the former).

► **Proposition 19.** *Let  $\mathbf{L}$  be a finite set of languages names and  $\mathcal{C}$  be a lattice. Then the two following properties hold:*

1. *If  $\mathbb{P}$  is an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ , then its support is an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ .*
2.  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}] = \{\mathbf{L}' \mid (L, \mathbf{L}') \in \mathcal{P}_{\mathcal{C}}[\mathbf{L}] \text{ for some } L \in \mathbf{L}\}$ .

**Proof.** We begin with the proof of the first item. The second one will then be a simple consequence. Both proofs are based on the following simple observation which follows from the definitions of imprints and pointed imprints. If  $\mathbb{Q}$  is a pointed cover of  $\mathbf{L}$  and  $\mathbf{H}$  is its support, then,

$$\mathcal{I}[\mathbf{L}](\mathbf{H}) = \{\mathbf{L}' \mid (L, \mathbf{L}') \in \mathcal{P}[\mathbf{L}](\mathbb{Q}) \text{ for some } L \in \mathbf{L}\} \quad (1)$$

We can now prove the first item in the lemma. Let  $\mathbb{P}$  be an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$  and let  $\mathbf{K}$  be its support. We have to prove that  $\mathbf{K}$  is an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ . By definition, this amounts to proving that for any  $\mathcal{C}$ -cover  $\mathbf{K}'$  of  $\mathbf{L}$ , we have  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$ . Let  $\mathbf{K}'$  be a  $\mathcal{C}$ -cover of  $\mathbf{L}$  and consider the following  $\mathbf{L}$ -pointed set:

$$\mathbb{P}' = \{(L, K) \mid L \in \mathbf{L} \text{ and } K \in \mathbf{K}'\}$$

By definition, for all  $L$ ,  $\mathbb{P}'(L) = \mathbf{K}'$  which is a  $\mathcal{C}$ -cover of  $\mathbf{L}$  by definition. Hence,  $\mathbb{P}'$  is a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Since  $\mathbb{P}$  is optimal by hypothesis, we obtain that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}')$ . It then follows from (1) that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$  which terminates the proof of the first item.

It remains to prove the second item. Let  $\mathbb{P}$  be an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$  and let  $\mathbf{K}$  be its support. By definition,  $\mathcal{P}[\mathbf{L}](\mathbb{P}) = \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  and we just proved that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) = \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ . Hence, the second item is immediate from (1).  $\blacktriangleleft$

As seen in Example 5, the converse of Proposition 19 does not hold: pointed covering is strictly more general than covering. However, as we observe in the proposition below, this only happens when the class  $\mathcal{C}$  is a lattice and not a boolean algebra. When  $\mathcal{C}$  is a boolean algebra, the associated covering and pointed covering problems are equivalent.

► **Proposition 20.** *Let  $\mathbf{L}$  be a finite set of languages and  $\mathcal{C}$  be a boolean algebra. Then the two following properties hold:*

1. *if  $\mathbf{K}$  is an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ , then  $\mathbb{P} = \{(L, K) \mid L \in \mathbf{L}, K \in \mathbf{K} \text{ and } K \cap L \neq \emptyset\}$  is an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ .*
2.  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] = \downarrow\{(L, \mathbf{L}') \mid \mathbf{L}' \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}] \text{ and } L \in \mathbf{L}\}$ .

**Proof.** We begin with the proof of the first item. We then obtain the second item as a corollary. Let  $\mathbf{K}$  be an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$  and let  $\mathbb{P} = \{(L, K) \mid L \in \mathbf{L}, K \in \mathbf{K} \text{ and } K \cap L \neq \emptyset\}$ . We have to prove that  $\mathbb{P}$  is an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ .

The fact that it is a  $\mathcal{C}$ -cover is obvious. By definition, we need to prove that for any pointed  $\mathcal{C}$ -cover  $\mathbb{P}'$  of  $\mathbf{L}$ ,  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{P}')$ . Let  $(L, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ , we prove that  $(L, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P}')$ . Since  $(L, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ , we have  $K \in \mathbb{P}(L)$  such that  $\mathbf{H} \subseteq \langle \mathbf{L} | K \rangle$ . Hence, by closure under downset, we have to prove that  $(L, \langle \mathbf{L} | K \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{P}')$ . Consider the following set of languages,

$$\mathbf{K}' = \mathbb{P}'(L) \cup \{A^* \setminus (\bigcup_{K' \in \mathbb{P}'(L)} K')\}$$

Observe that  $\mathbf{K}'$  is a  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Indeed, by definition  $\mathbf{K}'$  covers  $A^*$  and since  $\mathcal{C}$  is a boolean algebra, each language in  $\mathbf{K}'$  belongs to  $\mathcal{C}$ . Moreover, since  $(L, K) \in \mathbb{P}$  we have  $K \in \mathbf{K}$  and since  $\mathbf{K}$  is optimal, we have  $\langle \mathbf{L} | K \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$ . In other words, there exists  $H \in \mathbf{K}'$  such that,  $\langle \mathbf{L} | K \rangle \subseteq \langle \mathbf{L} | H \rangle$ . We prove that  $H \in \mathbb{P}'(L)$  (i.e.,  $H$  is not the language  $A^* \setminus (\bigcup_{K' \in \mathbb{P}'(L)} K')$ ). This terminates the proof since it means that  $(L, \langle \mathbf{L} | H \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{P}')$  and therefore that  $(L, \langle \mathbf{L} | K \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{P}')$  by closure under downset. Since  $(L, K) \in \mathbb{P}$ , we have  $K \cap L \neq \emptyset$  and  $L \in \langle \mathbf{L} | K \rangle \subseteq \langle \mathbf{L} | H \rangle$ . This suffices to conclude that  $H \in \mathbb{P}'(L)$ , since  $\mathbb{P}'(L)$  is a cover of  $L$  which implies that,

$$L \notin \langle \mathbf{L} | A^* \setminus (\bigcup_{K \in \mathbb{P}'(L)} K) \rangle$$

It remains to prove the second item in the lemma. Let  $\mathbf{K}$  be an optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$ . We just proved that the  $\mathbf{L}$ -pointed set  $\mathbb{P} = \{(L, K) \mid L \in \mathbf{L}, K \in \mathbf{K} \text{ and } K \cap L \neq \emptyset\}$  is an optimal pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . Therefore,

$$\begin{aligned} \mathcal{P}_{\mathcal{C}}[\mathbf{L}] &= \downarrow\{(L, \langle \mathbf{L} | K \rangle) \mid L \in \mathbf{L}, K \in \mathbf{K} \text{ and } K \cap L \neq \emptyset\} \\ &= \downarrow\{(L, \mathbf{L}') \mid \mathbf{L}' \in \mathcal{I}[\mathbf{L}](\mathbf{K}) \text{ and } L \in \mathbf{L}'\} \end{aligned}$$

We conclude that  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}] = \downarrow\{(L, \mathbf{L}') \mid \mathbf{L}' \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}] \text{ and } L \in \mathbf{L}'\}$  since  $\mathbf{K}$  is optimal.  $\blacktriangleleft$

The main consequence of Proposition 20 is that when the investigated class  $\mathcal{C}$  is a boolean algebra, there is no point in considering pointed covering: the covering problem suffices (and relies on simpler terminology).

## B Appendix to Section 3

In this appendix we present the missing proofs in Section 3. We prove Lemma 11, Lemma 13 and Proof of Lemma 14 which state the properties of tame sets. The proof of Proposition 12 is postponed to Appendix C.

### B.1 Proof of Lemma 11

We begin with Lemma 11.

► **Lemma 11.** *Any language in a tame set of languages is regular.*

**Proof.** Let  $\mathbf{L}$  be tame set. the tame multiplication can be lifted as a semigroup multiplication over  $2^{\mathbf{L}}$ . One may verify from the properties of tame sets that the map  $\alpha : A^* \rightarrow 2^{\mathbf{L}}$  defined by  $\alpha(w) = \{L \in \mathbf{L} \mid w \in L\}$  is semigroup morphism. This implies that  $\alpha(A^*) \subseteq 2^{\mathbf{L}}$  is a monoid and this monoid recognizes each language in  $L \in \mathbf{L}$ . Indeed, by definition,  $L = \alpha^{-1}(\{\mathbf{K} \in \alpha(A^*) \mid L \in \mathbf{K}\})$ .  $\blacktriangleleft$

## B.2 Proof of Lemma 13

We now turn to Lemma 13.

► **Lemma 13.** *Let  $\mathcal{C}$  be a quotienting lattice and let  $\mathbf{L}$  be a tame set of languages. Then the two following properties holds:*

- (1)  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  is closed under multiplication: for all  $(L_1, \mathbf{L}_1), (L_2, \mathbf{L}_2)$  in  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ , if  $L_1 \odot L_2$  is defined, then  $(L_1 \odot L_2, \mathbf{L}_1 \odot \mathbf{L}_2) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ .
- (2)  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  is closed under multiplication: for all  $\mathbf{L}_1$  and  $\mathbf{L}_2$  in  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ ,  $\mathbf{L}_1 \odot \mathbf{L}_2 \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ .

**Proof.** We prove the first item. The proof of the second one is similar. Let  $(S, \mathbf{S})$  and  $(T, \mathbf{T})$  be two elements of  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  such that  $S \odot T$  is defined and let  $(H, \mathbf{H}) = (S \odot T, \mathbf{S} \odot \mathbf{T})$ . We prove that  $(H, \mathbf{H}) \in \mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ . By definition, it suffices to prove that for any pointed  $\mathcal{C}$ -cover  $\mathbb{P}$  of  $\mathbf{L}$ , we have  $(H, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ . Let  $\mathbf{S} = \{S_1, \dots, S_m\}$  and  $\mathbf{T} = \{T_1, \dots, T_n\}$ .

Let  $\mathbb{P}$  be such a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$  and let  $\mathbf{K}$  be its support. That  $(H, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$  is a consequence of the following claim.

**Claim** There exist two words  $u_0 \in S$  and  $v_0 \in T$  such that for every  $K \in \mathbf{K}$  and every  $w \in A^*$ ,

1. if  $u_0 \in Kw^{-1}$  then  $(Kw^{-1}) \cap S_i \neq \emptyset$  for all  $1 \leq i \leq m$ .
2. if  $v_0 \in w^{-1}K$ , then  $(w^{-1}K) \cap T_j \neq \emptyset$  for all  $1 \leq j \leq n$ .

Before we prove the claim, let us use it to finish the proof of Lemma 13. By definition,  $u_0v_0 \in ST$ . Moreover,  $ST \subseteq H$  since  $H = S \odot T$ , and therefore  $u_0v_0 \in H$ . Hence, since  $\mathbb{P}$  is a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ ,  $\mathbb{P}(H)$  is a  $\mathcal{C}$ -cover of  $H$  and there exists  $K \in \mathbb{P}(H)$  such that  $u_0v_0 \in K$ . We prove that for all  $S_i \in \mathbf{S}$  and  $T_j \in \mathbf{T}$ ,  $K$  intersects all the languages  $S_i \odot T_j$ . This will exactly mean that  $K$  intersects all languages in  $\mathbf{H} = \mathbf{S} \odot \mathbf{T} = \bigcup_{i,j} \{S_i \odot T_j\}$  and therefore  $(H, \mathbf{H}) \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ , which terminates the proof of Lemma 13.

Set  $S_i \in \mathbf{S}$ ,  $T_j \in \mathbf{T}$  such that  $S_i \odot T_j$  is defined. By definition  $u_0 \in K \cdot (v_0)^{-1}$ . Hence, we know from the first item in the claim that  $K \cdot (v_0)^{-1}$  intersects  $S_i$ . We obtain a word  $u_i \in S_i$  such that  $u_iv_0 \in K$ . It now follows that  $v_0 \in (u_i)^{-1} \cdot K$ . Hence, we know from the second item in the claim that  $(u_i)^{-1} \cdot K$  intersects  $T_j$ . We obtain a word  $v_j \in T_j$  such that  $u_iv_j \in K$ . Finally, since  $S_iT_j \subseteq S_i \odot T_j$ , we have  $u_iv_j \in S_i \odot T_j$ . We conclude that  $K$  intersects  $S_i \odot T_j$ , which terminates the proof.

It now remains to prove the claim. We prove the existence of  $v_0$  (the existence of  $u_0$  is obtained by symmetry). Let  $\mathbf{Q}$  be the set of left quotients of languages in  $\mathbf{K}$ :  $\mathbf{Q} = \{w^{-1} \cdot K \mid w \in A^* \text{ and } K \in \mathbf{K}\}$ . We need to find  $v_0 \in T$  such that for all  $Q \in \mathbf{Q}$ , if  $v_0 \in Q$ , then  $Q$  intersects all languages  $T_1, \dots, T_n$ . Let us begin by stating two properties of the set  $\mathbf{Q}$  that we will use:

1. All  $Q \in \mathbf{Q}$  belong to  $\mathcal{C}$ . This is because  $\mathcal{C}$ , as a quotienting lattice, is closed under quotients and by definition all  $K \in \mathbf{K}$  belong to  $\mathcal{C}$ .
2. The set  $\mathbf{Q}$  is finite. This is because each language in  $\mathbf{K}$  is regular (they belong to  $\mathcal{C}$  which is a quotienting lattice). Hence, we know from Myhill-Nerode Theorem that each language in  $\mathbf{K}$  has finitely many left quotients. Since  $\mathbf{K}$  is finite, so is  $\mathbf{Q}$ .

We can now prove the existence of  $v_0$ . We proceed by contradiction: assume that for all  $v \in T$ , there exists  $Q_v \in \mathbf{Q}$  such that  $v \in Q_v$  and  $Q_v$  does not intersect all languages  $T_1, \dots, T_n$ . We use this hypothesis to construct a pointed  $\mathcal{C}$ -cover  $\mathbb{P}'$  of  $\mathbf{L}$  such that  $(T, \mathbf{T}) \notin \mathcal{P}[\mathbf{L}](\mathbb{P}')$ . The existence of such a pointed cover of  $\mathbf{L}$  is a contradiction since by definition,

$(T, \mathbf{T})$  belongs to  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$  (and therefore to the imprint of any pointed cover of  $\mathbf{L}$ ). Recall that  $\mathbf{Q}$  is finite (and therefore that the set  $\{Q_v \mid v \in T\}$  is finite), we define,

$$\mathbb{P}' = \{(T, Q_v) \mid v \in T\} \cup \{(L, A^*) \mid L \neq T\}$$

By definition,  $\mathbb{P}'$  is a  $\mathbf{L}$ -pointed set that contains only languages in  $\mathcal{C}$ . Moreover, by construction, we know that  $\mathbb{P}'(T)$  is a cover of  $T$ . We conclude that  $\mathbb{P}'$  is a pointed  $\mathcal{C}$ -cover of  $\mathbf{L}$ . By construction, we know that there is no language in  $\mathbb{P}'(T)$  which intersects all the languages  $T_1, \dots, T_n$ . Therefore, it is immediate that  $(T, \mathbf{T}) \notin \mathcal{P}[\mathbb{P}']()$ . We have a contradiction, which terminates the proof.  $\blacktriangleleft$

### B.3 Proof of Lemma 14

We finish with Lemma 14.

► **Lemma 14.** *Let  $\mathbf{L}$  be a tame set of languages and let  $K_1, K_2$  be two languages, then  $\langle \mathbf{L} \mid K_1 \rangle \odot \langle \mathbf{L} \mid K_2 \rangle = \langle \mathbf{L} \mid K_1 K_2 \rangle$ .*

**Proof.** Let  $L \in \langle \mathbf{L} \mid K_1 \rangle \odot \langle \mathbf{L} \mid K_2 \rangle$ . By definition of a tame multiplication, there exists  $L_1 \in \langle \mathbf{L} \mid K_1 \rangle$  and  $L_2 \in \langle \mathbf{L} \mid K_2 \rangle$  such that  $L = L_1 \odot L_2$ . By definition of a tame multiplication again, this implies  $L_1 L_2 \subseteq L$ . Moreover, since  $K_i$  intersects  $L_i$  for  $i = 1, 2$ ,  $K_1 K_2$  intersects  $L_1 L_2$  and therefore  $L$ . Finally,  $L \in \langle \mathbf{L} \mid K_1 K_2 \rangle$

Let  $L \in \langle \mathbf{L} \mid K_1 K_2 \rangle$ . By definition,  $K_1 K_2$  intersects  $L$ . Hence there exists  $w_1 \in K_1$  and  $w_2 \in K_2$  such that  $w_1 w_2 \in L$ . By definition of a tame multiplication, we obtain  $L_1, L_2 \in \mathbf{L}$  such that  $w_1 \in L_1$  and  $w_2 \in L_2$  and  $L = L_1 \odot L_2$ . It is immediate that  $L_1 \in \langle \mathbf{L} \mid K_1 \rangle$  and  $L_2 \in \langle \mathbf{L} \mid K_2 \rangle$ . Therefore,  $L \in \langle \mathbf{L} \mid K_1 \rangle \odot \langle \mathbf{L} \mid K_2 \rangle$  by the first item in the definition of a tame multiplication.  $\blacktriangleleft$

## C From Finite Sets of Regular Languages to Tame Sets

This section, is devoted to the proof of Proposition 12. Let us first recall the statement of proposition.

► **Proposition 12.** *Let  $\mathbf{H} = \{H_1, \dots, H_n\}$  be a finite set of languages given by  $n$  NFAs  $\mathcal{A}_1, \dots, \mathcal{A}_n$ . There exists a tame set of languages names  $\mathbf{L}$  such that for any lattice  $\mathcal{C}$ ,*

- $\mathcal{I}_{\mathcal{C}}[\mathbf{H}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{H}]$ ) can be computed from  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ ).
- any optimal (pointed)  $\mathcal{C}$ -cover of  $\mathbf{L}$  is an optimal (pointed)  $\mathcal{C}$ -cover of  $\mathbf{H}$ .
- $\mathbf{L}$  and its tame multiplication can be computed from  $\mathcal{A}_1, \dots, \mathcal{A}_n$  in polynomial time and has size  $|\mathcal{A}_1|^2 + \dots + |\mathcal{A}_n|^2$  (where  $|\mathcal{A}_i|$  stands for the number of states of  $\mathcal{A}_i$ ).

We proceed in two steps. First, we introduce a new notion called *extension* of input sets. It is designed with the following objective in mind: given two input sets  $\mathbf{L}$  and  $\mathbf{H}$ , if  $\mathbf{L}$  extends  $\mathbf{H}$ , then we want the (pointed) covering problem with input  $\mathbf{H}$  to be easily reducible to that with input  $\mathbf{L}$ . We then prove that for any finite set of regular languages, one can build a tame set that extends it. It will follow that we may restrict inputs to tame ones without loss of generality. Additionally, we use extension to prove that we may also assume that our input sets are *alphabet compatibility*.

## C.1 Extension

Let  $\mathbf{L}$  and  $\mathbf{H}$  be two finite sets of languages. We say that  $\mathbf{L}$  *extends*  $\mathbf{H}$  if and only if any language in  $\mathbf{H}$  is a union of languages in  $\mathbf{L}$ , *i.e.*, for any  $H \in \mathbf{H}$ , there exists  $\mathbf{L}' \subseteq \mathbf{L}$  such that  $H = \bigcup_{L' \in \mathbf{L}'} L'$ .

For instance,  $\mathbf{L} = \{L_1, L_2, L_3, L_4, L_5\}$  extends  $\mathbf{H} = \{L_1 \cup L_2, L_3 \cup L_5\}$ . Likewise,  $\mathbf{L} = \{L_1, L_2\}$  and  $\mathbf{H} = \{L_1, L_2, L_1 \cup L_2\}$  mutually extend each other. Observe that by definition, extension is a preorder on the set of finite sets of languages. Also note that containment is stronger than extension: given two sets  $\mathbf{L}$  and  $\mathbf{H}$ , if  $\mathbf{L} \supseteq \mathbf{H}$ , then  $\mathbf{L}$  extends  $\mathbf{H}$ . However, containment is not the only case of extension, as shown by the above examples.

► **Lemma 21.** *Let  $\mathcal{C}$  be a lattice and  $\mathbf{L}$  and  $\mathbf{H}$  be two finite sets of languages such that  $\mathbf{L}$  extends  $\mathbf{H}$ . Then, the two following properties hold:*

1.  $\mathcal{I}_{\mathcal{C}}[\mathbf{H}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{H}]$ ) can be computed from  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  (resp.  $\mathcal{P}_{\mathcal{C}}[\mathbf{L}]$ ).
2. Any optimal  $\mathcal{C}$ -cover (resp. pointed  $\mathcal{C}$ -cover) of  $\mathbf{L}$  is an optimal (resp. pointed  $\mathcal{C}$ -cover) of  $\mathbf{H}$ .

**Proof.** We do the proof for the covering problem. The proof for the pointed covering problem is similar. We prove that the three following properties hold:

1. Any  $\mathcal{C}$ -cover of  $\mathbf{L}$  is a  $\mathcal{C}$ -cover of  $\mathbf{H}$ .
2. Any optimal  $\mathcal{C}$ -cover of  $\mathbf{L}$  is an optimal  $\mathcal{C}$ -cover of  $\mathbf{H}$ .
3.  $\mathcal{I}_{\mathcal{C}}[\mathbf{H}]$  can be computed from  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$ :

$$\mathcal{I}_{\mathcal{C}}[\mathbf{H}] = \left\{ \{H_1, \dots, H_m\} \subseteq \mathbf{H} \mid \begin{array}{l} \exists \{L_1, \dots, L_m\} \in \mathcal{I}_{\mathcal{C}}[\mathbf{L}] \\ \forall i, \text{ we have } L_i \subseteq H_i \end{array} \right\}$$

The first item is immediate since by definition,  $\bigcup_{H \in \mathbf{H}} H \subseteq \bigcup_{L \in \mathbf{L}} L$ . For the two other items, it suffices to observe that since  $\mathbf{L}$  extends  $\mathbf{H}$ , for any  $\mathcal{C}$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  (which makes it a  $\mathcal{C}$ -cover of  $\mathbf{H}$  as well),

$$\mathcal{I}[\mathbf{H}](\mathbf{K}) = \left\{ \{H_1, \dots, H_m\} \subseteq \mathbf{H} \mid \begin{array}{l} \exists \{L_1, \dots, L_m\} \in \mathcal{I}[\mathbf{L}](\mathbf{K}) \\ \forall i, \text{ we have } L_i \subseteq H_i \end{array} \right\}$$

It then follows that for any two  $\mathcal{C}$ -covers  $\mathbf{K}, \mathbf{K}'$  of  $\mathbf{L}$  and  $\mathbf{H}$ , if  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}')$ , then  $\mathcal{I}[\mathbf{H}](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{H}](\mathbf{K}')$ , which yields the second item. We can now set  $\mathbf{K}$  as an optimal  $\mathcal{C}$ -cover of both  $\mathbf{L}$  and  $\mathbf{H}$ . The third item is now immediate as  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}] = \mathcal{I}[\mathbf{L}](\mathbf{K})$  and  $\mathcal{I}_{\mathcal{C}}[\mathbf{H}] = \mathcal{I}[\mathbf{H}](\mathbf{K})$ . ◀

## C.2 Constructing Tame Sets

Now that we have extension, that we may restrict ourselves to tame inputs (*i.e.*, Proposition 12) is an immediate consequence of the following proposition: given a finite set of regular languages, it is always possible to construct a tame one that extends it.

► **Proposition 22.** *Let  $\mathbf{H} = \{H_1, \dots, H_n\}$  be a finite set of languages given by  $n$  NFAs  $\mathcal{A}_1, \dots, \mathcal{A}_n$ . There exists a tame set of languages names  $\mathbf{L}$  such that for any lattice  $\mathcal{C}$ ,*

- $\mathbf{L}$  extends  $\mathbf{H}$ .
- $\mathbf{L}$  and its tame multiplication can be computed from  $\mathcal{A}_1, \dots, \mathcal{A}_n$  in polynomial time and has size  $|\mathcal{A}_1|^2 + \dots + |\mathcal{A}_n|^2$  (where  $|\mathcal{A}_i|$  stands for the number of states of  $\mathcal{A}_i$ ).

**Proof.** We first prove that for any NFA  $\mathcal{A} = (A, Q, \delta, I, F)$  (where  $Q$  is the set of states,  $\delta \subseteq Q \times A \times Q$  the set of transitions and  $I, F$  the sets of initial and final states), one may construct a tame set  $\mathbf{L}_{\mathcal{A}}$  of size at most  $Q^2$  that extends the singleton set  $\{L(\mathcal{A})\}$ , which contains only the language recognized by  $\mathcal{A}$ . For all  $q, r \in Q$ , we define  $L_{q,r}$  as a name for the language  $\{w \in A^* \mid q \xrightarrow{w} r\}$ .

► **Remark.** Note that we need names here as it may happen that  $L_{q,r}$  and  $L_{q',r'}$  represent the same language while  $(q, r) \neq (q', r')$ .

We define  $\mathbf{L}_{\mathcal{A}}$  as follows  $\mathbf{L}_{\mathcal{A}} = \{L_{q,r} \mid (q, r) \in Q^2\}$ . Let us explain why  $\mathbf{L}_{\mathcal{A}}$  is tame. We begin by defining our tame multiplication. We define,

$$L_{q,r} \odot L_{s,t} = \begin{cases} L_{q,t} & \text{when } r = s \\ \text{undefined} & \text{otherwise} \end{cases}$$

In particular observe that this multiplication is well-defined because we are working with a set of names and not the underlying set of languages (otherwise the definition above could be ambiguous). One can verify that this is a tame multiplication. Moreover, the following fact is immediate.

► **Fact 23.** *Let  $L$  be the language recognized by  $\mathcal{A}$ . Then  $\mathbf{L}$  extends  $\{L\}$ .*

We may now finish the construction in Proposition 22. The proof is based on the two following facts. In these two facts we speak of the disjoint union of two tame sets. By this we mean that when making the union of a tame set of names  $\mathbf{L}_1$  with a second tame set of names  $\mathbf{L}_2$ , we assume that the two sets are disjoint (which can be assumed without generality using renaming).

► **Fact 24.** *Let  $\mathbf{L}_1, \mathbf{L}_2, \mathbf{H}_1$  and  $\mathbf{H}_2$  be finite sets of languages names such that  $\mathbf{L}_1$  extends  $\mathbf{H}_1$  and  $\mathbf{L}_2$  extends  $\mathbf{H}_2$ . Then the disjoint union  $\mathbf{L}_1 \uplus \mathbf{L}_2$  extends the disjoint union  $\mathbf{H}_1 \uplus \mathbf{H}_2$ .*

► **Fact 25.** *Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be two tame sets of languages names. Then the disjoint union  $\mathbf{L}_1 \uplus \mathbf{L}_2$  is tame and an actual tame multiplication can be constructed from those of  $\mathbf{L}_1$  and  $\mathbf{L}_2$ .*

Before proving the two facts, we finish the proof of Proposition 22. Let  $\mathbf{H} = \{H_1, \dots, H_n\}$  be a finite set of regular languages names and  $\mathcal{A}_1, \dots, \mathcal{A}_n$  the associated NFAs. Then, let  $\mathbf{L}_1, \dots, \mathbf{L}_n$  be the tame sets associated to  $\mathcal{A}_1, \dots, \mathcal{A}_n$  as above. We let  $\mathbf{L} = \mathbf{L}_1 \uplus \dots \uplus \mathbf{L}_n$ . It is immediate from Facts 23 and 24 that  $\mathbf{L}$  extends  $\mathbf{H}$  and from Fact 25 that  $\mathbf{L}$  is tame. Moreover, it can be verified from the construction that  $\mathbf{L}$  satisfies the properties described in the proposition.

It remains to prove the two facts. Fact 24 is immediate from the definition of extension. Let us describe the construction for Fact 25. Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be two tame sets. We have to prove that  $\mathbf{L}_1 \uplus \mathbf{L}_2$  is tame as well. For  $i = 1, 2$ , we let  $\odot_i$  as the tame multiplication on  $\mathbf{L}_i$ . We define a tame multiplication  $\odot$  for  $\mathbf{L}_1 \uplus \mathbf{L}_2$  as follows. For  $i = 1, 2$  and all  $S, T \in \mathbf{L}_i$  such that  $S \odot_i T$  is defined, we define  $S \odot = S \odot_i T$ . All other products are undefined. One may verify that this yields a tame multiplication. ◀

### C.3 Alphabet Compatible Sets of Languages

Recall that *alphabet compatibility* is an additional restriction that we use in our algorithms for  $\text{FO}^2$  and  $\Sigma_2$ . We use extension to prove that it can be assumed without loss of generality as well. Let us first recall the definition.

Recall that  $A$  denotes the alphabet over which our languages are defined. For any word  $w \in A^*$ , we denote by  $\mathbf{alph}(w)$  the set of letters that occur in  $w$  (*i.e.*, the smallest  $B$  such that  $w \in B^*$ ). A finite set of languages  $\mathbf{L}$  is said to be *alphabet compatible* if for all languages  $L \in \mathbf{L}$ , there exists  $B \subseteq A$  such that for any  $w \in L$ ,  $\mathbf{alph}(w) = B$  (*i.e.*, all words in  $L$  have the same alphabet). In particular, note that when  $\mathbf{L}$  is an alphabet compatible set of languages  $\mathbf{L}$ , for each language  $L \in \mathbf{L}$ ,  $\mathbf{alph}(L)$  is well defined as the unique alphabet shared by all words in  $L$ .

It follows from Lemma 26 below that we may assume that our inputs are alphabet compatible without loss of generality.

► **Lemma 26.** *From any finite set of languages  $\mathbf{H}$ , one can construct a finite set of languages  $\mathbf{L}$  of size  $|\mathbf{H}| \times |2^A|$  that extends  $\mathbf{H}$  and is alphabet compatible. Moreover, if  $\mathbf{H}$  is tame,  $\mathbf{L}$  is tame as well.*

**Proof.** For all  $H \in \mathbf{H}$ , we define  $L_{H,B}$  as a name for the language of all words  $w \in H$  such that  $\mathbf{alph}(w) = B$ . We define  $\mathbf{L}$  as the following set of languages:

$$\mathbf{L} = \{L_{H,B} \mid H \in \mathbf{H} \text{ and } B \subseteq A\}.$$

It is immediate from the definition that  $\mathbf{L}$  is alphabet compatible, extends  $\mathbf{H}$ , and has size at most  $|\mathbf{H}| \times |2^A|$ . It remains to prove that if  $\mathbf{H}$  is tame, then so is  $\mathbf{L}$ . Let  $\odot_{\mathbf{H}}$  be the tame multiplication of  $\mathbf{H}$ . We define a tame multiplication  $\odot_{\mathbf{L}}$  on  $\mathbf{L}$ . Given  $L_{H,B}$  and  $L_{G,C}$  in  $\mathbf{L}$ ,  $L_{H,B} \odot_{\mathbf{L}} L_{G,C}$  is defined when  $H \odot_{\mathbf{H}} G$  and,

$$L_{H,B} \odot_{\mathbf{L}} L_{G,C} = L_{H \odot_{\mathbf{H}} G, B \cup C}$$

One may verify that  $\odot_{\mathbf{L}}$  satisfies the properties of tame multiplications. ◀

## D Proof of Theorem 16: $\mathcal{B}\Sigma_1$ -Covering

This appendix is devoted to the proof of Theorem 16, *i.e.*, of our covering algorithm for  $\mathcal{B}\Sigma_1$ . We begin by presenting the algorithm in more details and introduce some terminology that we will need for the proof.

We follow the template introduced in Section 4: our covering algorithm is a lowest fixpoint. Moreover, it is restricted to input sets that are tame, that is sets  $\mathbf{L}$  such that  $\mathbf{L}$  is equipped with a tame multiplication “ $\odot$ ”. Recall that  $A$  denotes our alphabet. Given a finite set of languages names  $\mathbf{L}$  and  $B \subseteq A$ , we write:

$$\mathbf{L}_{=B} = \{L \in \mathbf{L} \mid \text{there exists } w \in L \text{ such that } \mathbf{alph}(w) = B\}$$

► **Definition 27** (Covering Algorithm for  $\mathcal{B}\Sigma_1(<)$ ). Let  $\mathbf{L}$  be a tame set of languages names. We define  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L}) \subseteq 2^{\mathbf{L}}$  as the smallest subset of  $2^{\mathbf{L}}$  containing  $\mathcal{I}_{triv}[\mathbf{L}]$  and closed under the following operations:

1. Downset:  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L}) = \downarrow Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ .
2. Multiplication: for  $\mathbf{S}, \mathbf{T} \in Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ , we have  $\mathbf{S} \odot \mathbf{T} \in Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ .
3. For all  $B \subseteq A$ , we have  $(\mathbf{L}_{=B})^\omega \in Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ .

Observe that  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$  may be computed from  $\mathbf{L}$ . Indeed,  $\mathcal{I}_{triv}[\mathbf{L}]$  may be computed from  $\mathbf{L}$  as well as  $\mathbf{L}_{=B}$  for all  $B \subseteq A$ , hence  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$  is easily computed using a lowest fixpoint algorithm. We now state the correctness of our algorithm.

► **Theorem 28.** *Let  $\mathbf{L}$  be a tame set of languages names. Then  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] = Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ .*

To prove Theorem 28, we begin by proving that the algorithm is sound (*i.e.*, that  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L}) \subseteq \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ ). We then prove the difficult direction which is completeness (*i.e.*, that  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] \subseteq Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ ).

In both proofs, we will rely on the definition of  $\mathcal{B}\Sigma_1$  as the class of *piecewise testable languages*. Let us recall this definition. Let  $v \in A^*$  be a word. A *piece* of  $v$  is a word  $u = a_1 \cdots a_n$  for which there exist words  $v_0, \dots, v_n \in A^*$  such that

$$v = v_0 a_1 v_1 a_2 v_2 \cdots a_n v_n.$$

A language is said to be *piecewise testable* if it is a boolean combination of languages of the form

$$\{v \in A^* \mid u \text{ is a piece of } v\} \quad \text{for some fixed } u \in A^*.$$

It is well-known that a language is piecewise testable if and only if it can be defined by a  $\mathcal{B}\Sigma_1(<)$  sentence.

## D.1 Soundness

In this section, we prove the first half of Theorem 28: given a tame set of languages names  $\mathbf{L}$ , we prove that  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L}) \subseteq \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ . By definition of  $Sat_{\mathcal{B}\Sigma_1}(\mathbf{L})$ , it suffices to prove that  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$  contains the set  $\mathcal{I}_{triv}[\mathbf{L}]$  and that it is closed under the following operations:

1. Downset:  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] = \downarrow \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ .
2. Multiplication: for  $\mathbf{S}, \mathbf{T} \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ , we have  $\mathbf{S} \odot \mathbf{T} \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ .
3. For all  $B \subseteq A$ , we have  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ .

As we saw in the main paper, only one of these properties is specific to  $\mathcal{B}\Sigma_1(<)$ . We know that  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  contains  $\mathcal{I}_{triv}[\mathbf{L}]$  and is closed under downset for any boolean algebra  $\mathcal{C}$  (see Fact 1). Hence, this is true in particular for  $\mathcal{C} = \mathcal{B}\Sigma_1(<)$ . Moreover, since  $\mathcal{B}\Sigma_1(<)$  is known to be a quotienting boolean algebra, we also know from Lemma 13 that  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$  is closed under multiplication. Therefore, we only have to prove that for all  $B \subseteq A$ , we have  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ .

We prove that for  $B \subseteq A$  and any  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  (and therefore in particular for optimal ones),  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}[\mathbf{L}](\mathbf{K})$ . It will then follow that  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$  since  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] = \mathcal{I}[\mathbf{L}](\mathbf{K})$  for any optimal  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ .

So let  $B = \{b_1, \dots, b_n\} \subseteq A$  and let  $\mathbf{K}$  be a  $\mathcal{B}\Sigma_1(<)$ -cover of  $\mathbf{L}$ . Given two words  $w_1, w_2$  and  $k \in \mathbb{N}$ , we write  $w_1 \sim_k w_2$  if and only if  $w_1$  and  $w_2$  have the same pieces of length up to  $k$ . Since all languages in  $\mathbf{K}$  are  $\mathcal{B}\Sigma_1$ -definable, they are piecewise testable and one can verify that there exists  $k \in \mathbb{N}$  such that they are all unions of equivalence classes of  $\sim_k$ . Let  $\mathbf{K}'$  be the set of equivalence classes of  $\sim_k$ . By definition,  $\mathcal{I}[\mathbf{L}](\mathbf{K}') \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$ . Therefore it suffices to prove that  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}[\mathbf{L}](\mathbf{K}')$ . In other words, we have to prove that there exists an equivalence class  $K \in \mathbf{K}'$  of  $\sim_k$  that intersects all languages in  $(\mathbf{L}_{=B})^\omega$ . This is a consequence of the following lemma.

► **Lemma 29.** *For all  $L \in (\mathbf{L}_{=B})^\omega$ , there exists  $w_L \in L$  which satisfies the two following conditions:*

1.  $(b_1 \cdots b_n)^{k\omega}$  is a piece of  $w_L$ .
2.  $\text{alph}(w_L) = B$ .

Before we prove the lemma, let us use it to conclude the main proof. It follows from Lemma 29 that all words  $w_L$  for  $L \in (\mathbf{L}_{=B})^\omega$  admit the same pieces of length less than  $k$ , namely all words whose alphabet is exactly  $B$  and whose length is less than  $k$ . Therefore, all words  $w_L$  are  $\sim_k$ -equivalent and we get a  $\sim_k$ -class  $K \in \mathbf{K}'$  that intersects all languages  $L \in (\mathbf{L}_{=B})^\omega$ , which exactly says that  $(\mathbf{L}_{=B})^\omega \in \mathcal{I}[\mathbf{L}](\mathbf{K}')$ .

It now remains to prove Lemma 29, we finish the section with this proof. Let  $L \in (\mathbf{L}_{=B})^\omega$ . By definition of the idempotent power ' $\omega$ ', we have  $(\mathbf{L}_{=B})^\omega = (\mathbf{L}_{=B})^{nk\omega}$ . Therefore, there exist  $L_1, \dots, L_{nk\omega} \in \mathbf{L}_{=B}$  such that  $L = L_1 \odot \dots \odot L_{nk\omega}$ . Now for all  $i$ , the language  $L_i$  contains some word  $u_i$  of alphabet  $B$  (by definition of  $\mathbf{L}_{=B}$ ). We define  $w_L = u_1 \dots u_{nk\omega}$ . Note that  $w_L \in L$  by definition of a tame multiplication ( $w_L \in L_1 \dots L_{nk\omega} \subseteq L$ ). We prove that  $w_L$  satisfies the conditions of the lemma.

1. Observe that  $(b_1 \dots b_n)^{k\omega}$  has length  $nk\omega$  and let  $c_1 \dots c_{nk\omega}$  be its decomposition as letters. By definition, for all  $i$ ,  $c_i \in B$  and therefore  $c_i$  is a piece of  $u_i$ . It follows that  $(b_1 \dots b_n)^{k\omega}$  is a piece of  $w_L = u_1 \dots u_{nk\omega}$ .
2. Since  $w_L$  is a concatenation of words of alphabet  $B$ , it is immediate that  $\mathbf{alph}(w_L) = B$ .

## D.2 Completeness

In this section, we prove the difficult direction of Theorem 28: for any tame set  $\mathbf{L}$ ,  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$ . As announced in the main paper, we achieve this by presenting a generic method for constructing a  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  such that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$ . Since  $\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$  for any  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  of  $\mathbf{L}$ , this proves the desired result. Moreover, since we already proved that  $\text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L}) \subseteq \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ , our construction actually builds a  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  such that,

$$\mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L}) \subseteq \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$$

In other words,  $\mathcal{I}[\mathbf{L}](\mathbf{K}) = \mathcal{I}_{\mathcal{B}\Sigma_1}[\mathbf{L}]$ : what we present is a generic method for constructing optimal  $\mathcal{B}\Sigma_1(<)$ -covers, that is, a solution to the second stage of the covering problem for  $\mathcal{B}\Sigma_1(<)$ .

We now start the construction. Let  $\mathbf{L}$  be a tame set of languages names. We have to construct a  $\mathcal{B}\Sigma_1(<)$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  such that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$ . For the construction, we fix an arbitrary linear order on the elements of the alphabet  $A$ . To every integers  $p, \ell \geq 1$ , we associate a finite set of  $\mathcal{B}\Sigma_1(<)$ -definable languages  $\mathbf{K}_{p,\ell}$ . We then prove that

- we can find  $p$  large enough so that for any  $\ell$ , we have  $\mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell}) \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$ , and
- that for this  $p$ , we can find  $\ell$  such that  $\mathbf{K}_{p,\ell}$  is a cover of  $\mathbf{L}$  (actually of  $A^*$ ) which terminates the proof.

The languages of  $\mathbf{K}_{p,\ell}$  are defined according to a new notion called *template*: each language in  $\mathbf{K}_{p,\ell}$  corresponds to a template. Let us first define what a template is. Given  $\ell \geq 0$ , a *template of length  $\ell$*  is a sequence  $T = t_1, \dots, t_\ell$  (empty when  $\ell = 0$ ), such that every  $t_i$  is either a letter  $a \in A$  or a triple  $(b, B, b')$  where  $B \subseteq A$  is a subalphabet and  $b, b' \in B$  are two (possibly equal) letters in  $B$ . Moreover, we say that a template is *unambiguous* if all pairs of consecutive elements  $t_i, t_{i+1}$  in the template are either two letters, a letter  $a$  and a triple  $(b, B, b')$  such that  $a \notin B$  or two triples  $(b_i, B_i, b'_i)$  and  $(b_{i+1}, B_{i+1}, b'_{i+1})$  such that  $b'_i \notin B_{i+1}$  and  $b_{i+1} \notin B_i$ . For example,  $T = a, (c, \{b, c\}, b), d, (a, \{a\}, a)$  is an unambiguous template of length 4 while  $T' = \mathbf{b}, (c, \{\mathbf{b}, c\}, c), d, (a, \{a\}, a)$  and  $T'' = a, (c, \{b, \mathbf{c}\}, b), (\mathbf{c}, \{c\}, c), (a, \{a\}, a)$  are ambiguous templates of length 4.

We now define the language associated to a template. Note that while the notion is defined for both unambiguous and ambiguous templates, only the languages associated to unambiguous ones will be definable in  $\mathcal{BS}_1(<)$ . To each  $p \geq 1$  and each triple  $(b, B, b')$  with  $B \subseteq A$  and  $b, b' \in B$ , we associate a language  $\text{IT}_p(b, B, b')$  that we call the  $p$ -iteration of  $(b, B, b')$ . Let  $B = \{b_1, \dots, b_n\}$  (where the order on the elements of  $B$  is the one we fixed on  $A$  at the beginning). We define

$$\text{IT}_p(b, B, b') = B^*b(B^*b_1B^*b_2B^*\dots B^*b_nB^*)^pb'B^*$$

Given a template  $T$  and  $p \geq 1$ , we call  $p$ -implementation of  $T$  the language  $K_{p,T}$  defined as follows. If  $T$  is empty, then  $K_{p,T} = \{\varepsilon\}$ . Otherwise,  $T = t_1, \dots, t_\ell$  for some  $\ell \geq 1$  and we define  $K_{p,T} = K_1 \cdots K_\ell$  such that for all  $i \leq \ell$ ,

- if  $t_i$  is a single letter  $a \in A$ , then  $K_i = \{a\}$ .
- if  $t_i$  is a triple  $(b, B, b')$  with  $B \subseteq A$  and  $b, b' \in B$ , then  $K_i = \text{IT}_p(b, B, b')$ .

Finally, for all  $p, \ell \geq 1$ , we define  $\mathbf{K}_{p,\ell}$  as the set of all languages that are the  $p$ -implementation of some *unambiguous* template of length at most  $\ell$ . Observe that by definition,  $\mathbf{K}_{p,\ell}$  is a finite set as there are finitely many unambiguous templates of length at most  $\ell$ . We may now finish the proof of Theorem 28. We use the two following propositions to prove that we may find  $p$  and  $\ell$  such that  $\mathbf{K}_{p,\ell}$  is a  $\mathcal{BS}_1(<)$ -cover of  $\mathbf{L}$  and  $\mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell}) \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ .

► **Proposition 30.** *Let  $p > |\mathbf{L}|$ . Then for any  $\ell \geq 1$ ,  $\mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell}) \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ .*

► **Proposition 31.** *Let  $p \geq 1$ . Then, for any  $\ell \geq ((p+2)|A|^{|A|} - 1)$ ,  $\mathbf{K}_{p,\ell}$  is a  $\mathcal{BS}_1(<)$ -cover of  $A^*$ .*

Before we prove the propositions, we use them to finish the proof of Theorem 28. Let  $p = |\mathbf{L}| + 1$  and  $\ell = ((p+2)|A|^{|A|} - 1)$ . It is immediate from Proposition 31 that  $\mathbf{K}_{p,\ell}$  is a  $\mathcal{BS}_1(<)$ -cover of  $\mathbf{L}$  and from Proposition 30 that  $\mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell}) \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ . Since by definition,  $\mathcal{I}_{\mathcal{BS}_1}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell})$ , we conclude that  $\mathcal{I}_{\mathcal{BS}_1}[\mathbf{L}] \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$  as desired. It now remains to prove Proposition 30 and Proposition 31. We devote a subsection to each proof.

### Proof of Proposition 30

Let  $p > |\mathbf{L}|$  and  $\ell \geq 1$ , we have to prove that  $\mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell}) \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ . Since  $\text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$  is closed under downset by definition, this amounts to proving that for any language  $K \in \mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell})$ ,  $\langle \mathbf{L} | K \rangle \in \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ .

Let  $K \in \mathcal{I}[\mathbf{L}](\mathbf{K}_{p,\ell})$ , by definition there exists an unambiguous template  $T$  of length  $\ell' \leq \ell$  such that  $K$  is the  $p$ -implementation of  $T$ . If  $\ell' = 0$ , then  $K = \{\varepsilon\}$  is a singleton. Hence,  $\langle \mathbf{L} | K \rangle \in \mathcal{I}_{\text{triv}}[\mathbf{L}] \subseteq \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$  by definition. Otherwise,  $\ell' \geq 1$  and  $T = t_1, \dots, t_{\ell'}$ . For all  $i \leq \ell'$ , we let  $K_i = \text{IT}_p(b, B, b')$  if  $t_i$  is a triple  $(b, B, b')$  and  $K_i = \{a\}$  if  $t_i$  is a single letter  $a$ . By definition,  $K = K_1 \cdots K_{\ell'}$ . We use the following intermediate result.

► **Lemma 32.** *For all  $i \leq \ell'$ ,  $\langle \mathbf{L} | K_i \rangle \in \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$ .*

Before we prove Lemma 32, we use it to finish the proof of Proposition 30. Since  $\mathbf{L}$  is tame, we know from Lemma 14 that,

$$\langle \mathbf{L} | K_1 \cdots K_n \rangle = \langle \mathbf{L} | K_1 \rangle \odot \cdots \odot \langle \mathbf{L} | K_n \rangle$$

Hence, it is immediate from Lemma 32 and the fact that  $\text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$  is closed under multiplication that  $\langle \mathbf{L} | K \rangle \in \text{Sat}_{\mathcal{BS}_1}(\mathbf{L})$  which terminates the proof of Proposition 30. We finish with the proof of Lemma 32.

Let  $i \leq \ell'$ . If  $K_i = \{a\}$ , then  $K_i$  is a singleton and  $\langle \mathbf{L} | K_i \rangle \in \mathcal{I}_{triv}[\mathbf{L}] \subseteq \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$  by definition. Otherwise  $K_i = \text{IT}_p(b, B, b')$  for some  $B \subseteq A$  and some  $b, b' \in B$ . We prove that  $\langle \mathbf{L} | K_i \rangle \subseteq (\mathbf{L}_{=B})^\omega$ . It will then follow from closure under downset and Operation 3 in the definition of  $\text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$  that  $\langle \mathbf{L} | K_i \rangle \in \text{Sat}_{\mathcal{B}\Sigma_1}(\mathbf{L})$ . Let  $L \in \langle \mathbf{L} | K_i \rangle$ , we have to prove that  $L \in (\mathbf{L}_{=B})^\omega$ . By definition,  $L$  contains some word  $w \in \text{IT}_p(b, B, b') = B^*b(B^*b_1B^*b_2B^*\cdots B^*b_nB^*)^pb'B^*$  (with  $B = \{b_1, \dots, b_n\}$ ). This means that  $w$  can be decomposed as  $w = w_1 \cdots w_p$  such that all words  $w_i$  satisfy  $\mathbf{alph}(w_i) = B$ . Since  $\mathbf{L}$  is tame and  $w_1 \cdots w_p \in L$ , we may use the second property in the definition of tame multiplications to obtain  $L_1, \dots, L_p \in \mathbf{L}$  such that,

1. for all  $i$ ,  $w_i \in L_i$ .
2.  $L = L_1 \odot \cdots \odot L_p$ .

Note that since  $p > |\mathbf{L}|$ , a pigeon-hole principle argument yields  $i < j \leq p$  such that,  $L_1 \odot \cdots \odot L_i = L_1 \odot \cdots \odot L_j$ . This means that  $L_1 \odot \cdots \odot L_i = L_1 \odot \cdots \odot L_i \odot (L_{i+1} \odot \cdots \odot L_j)^k$  for all  $k$ , and therefore

$$L = L_1 \odot \cdots \odot L_i \odot (L_{i+1} \odot \cdots \odot L_j)^\omega \odot L_{j+1} \odot \cdots \odot L_p$$

Furthermore, it is immediate from the first item above that for all  $i$ ,  $L_i \in \mathbf{L}_{=B}$  (recall that  $\mathbf{alph}(w_i) = B$ ). It follows that,

$$L \in (\mathbf{L}_{=B})^{(j-i)\omega+i+p-j} = (\mathbf{L}_{=B})^{\omega+i+p-j}$$

It then follows from the next lemma that  $(\mathbf{L}_{=B})^{\omega+i+p-j} \subseteq (\mathbf{L}_{=B})^\omega$  and therefore that  $L \in (\mathbf{L}_{=B})^\omega$ .

► **Lemma 33.**  $\mathbf{L}_{=B} \odot \mathbf{L}_{=B} \subseteq \mathbf{L}_{=B}$ .

**Proof.** By definition, if  $H \in \mathbf{L}_{=B} \odot \mathbf{L}_{=B}$ , there exist  $H_1, H_2 \in \mathbf{L}_{=B}$  such that  $H = H_1 \odot H_2$ . In particular, this means that  $H_1 H_2 \subseteq H$ . Moreover, since  $H_1, H_2 \in \mathbf{L}_{=B}$ , we have  $w_1, w_2 \in H_1, H_2$  such  $\mathbf{alph}(w_1) = \mathbf{alph}(w_2) = B$ . It follows that  $w_1 w_2 \in H$  and since  $\mathbf{alph}(w_1 w_2) = B$ , that  $H \in \mathbf{L}_{=B}$ . ◀

### Proof of Proposition 31

Let  $p \geq 1$  and let  $\ell \geq ((p+2)|A|^{|A|} - 1)$ , we have to prove that  $\mathbf{K}_{p,\ell}$  is a  $\mathcal{B}\Sigma_1(<)$ -cover of  $A^*$ . We prove separately that the union of all languages in  $\mathbf{K}_{p,\ell}$  is  $A^*$  and that each language in  $\mathbf{K}_{p,\ell}$  is definable in  $\mathcal{B}\Sigma_1(<)$ . Note that we only need  $\ell$  to be larger than  $((p+2)|A|^{|A|} - 1)$  for the first result, definability in  $\mathcal{B}\Sigma_1(<)$  is independent from the choice of  $\ell$ . We begin by proving that  $\mathbf{K}_{p,\ell}$  is a cover of  $A^*$ . We state this result in the next lemma.

► **Lemma 34.** *Let  $p \geq 1$  and let  $\ell \geq ((p+2)|A|^{|A|} - 1)$ . Then,*

$$\bigcup_{K \in \mathbf{K}_{p,\ell}} K = A^*$$

**Proof.** Let  $p \geq 1$  and let  $\ell \geq ((p+2)|A|^{|A|} - 1)$ . We have to prove that for any  $w \in A^*$ , there exists a language in  $\mathbf{K}_{p,\ell}$  that contains it. In other words, we want to find an unambiguous template  $T$  of length less than  $\ell$  such that  $w$  belongs to the  $p$ -implementation of  $T$ . First observe that if  $w = \varepsilon$ , then  $w$  belongs to the  $p$ -implementation of the empty template.

We now assume that  $w \neq \varepsilon$ . We begin with an intermediary result. Given  $m \geq 1$ , we say that  $w$  admits a *good decomposition of length  $m$*  if  $w$  can be decomposed as  $w = w_1 \cdots w_m$  such that for all  $i$ ,  $w_i$  is of one the two following types:

1.  $w_i = a_i \in A$  or,
2. there exists  $B_i = \{b_1, \dots, b_n\} \subseteq A$  such that  $w_i \in (B_i^* b_1 B_i^* b_2 \cdots B_i^* b_n B_i^*)^{p+2}$ .

**Claim** There exists  $\ell' \leq \ell$  such that  $w$  admits a good decomposition of length  $\ell'$ .

Before we prove the claim, we use it to construct an unambiguous template  $T$  such that  $w$  belongs to the  $p$ -implementation of  $T$  and conclude the proof of Lemma 34. Using the claim, we obtain a good decomposition of length  $\ell' \leq \ell$  of  $w$ . Observe that we may assume without loss of generality that this decomposition satisfies the following property. For all  $i$ , if  $w_i$  is of type 2, then there exist  $b_i, b'_i \in B_i$  such that,

- if  $w_{i-1}$  (resp.  $w_{i+1}$ ) is of type 1, then  $a_{i-1} \notin B_i$  (resp.  $a_{i+1} \notin B_i$ ).
- if  $w_{i-1}$  (resp.  $w_{i+1}$ ) is of type 2, then  $b_i \notin B_{i-1}$  (resp.  $b'_i \notin B_{i+1}$ ).

Indeed, otherwise we can simply merge pairs of factors  $w_i, w_{i+1}$  to get an even smaller good decomposition of  $w$ . Consider the template  $T = t_1, \dots, t_{\ell'}$  defined as follows. For all  $i$ ,  $t_i = a_i$  if  $w_i$  is of the type 1 and  $t_i = (b_i, B_i, b'_i)$  if  $w_i$  is of type 2. It follows from our hypothesis on the good decomposition that  $T$  is unambiguous and it is immediate from the definition of good decompositions that  $w$  is a  $p$ -implementation of  $T$ . Since  $\ell' \leq \ell$ , this terminates the proof of Lemma 34.

It remains to prove the claim. Let  $B = \mathbf{alph}(w)$ . We proceed by induction on  $|B|$  and prove that  $w$  admits a good decomposition of length smaller than  $((p+2)|B|^{|B|} - 1)$  (which suffices by choice of  $\ell$ ). Observe first that if  $|B| = 1$ , then  $B = \{b\}$  and  $w = b^m$  for some  $m$ . If  $m \leq p+1$ , then we are finished as  $w$  admits a good decomposition of length  $m$  in which each factor is a letter. Otherwise,  $m \geq p+2$  and  $w \in (b^* b b^*)^{p+2}$ . It follows that  $w$  admits a good decomposition of length 1.

Assume now that  $|B| \geq 2$ . We prove that for any  $m \geq ((p+2)|B|^{|B|})$ , if  $w$  admits a good decomposition of length  $m$ , then  $w$  also admits a good decomposition of length strictly smaller than  $m$ . This suffices since  $w$  admits a good decomposition of length  $|w|$  in which each factor is a single letter. Let  $w = w_1 \cdots w_m$  be the good decomposition of  $w$  and let  $m' = ((p+2)|B|^{|B|-1})$ . Observe that by definition  $m \geq (p+2)|B|m'$ . We consider two cases.

In the first case, we assume that there exists  $i$  such that  $\mathbf{alph}(w_{i+1}w_{i+2} \cdots w_{i+m'}) = C \subsetneq B$ . By induction hypothesis,  $w_{i+1}w_{i+2} \cdots w_{i+m'}$  admits a good decomposition of length smaller than  $((p+2)|C|^{|C|} - 1) < m'$ . We may now replace the factor  $w_{i+1}w_{i+2} \cdots w_{i+m'}$  in the good decomposition of  $w$  with this new decomposition to obtain a new good decomposition of  $w$  of length strictly smaller than  $m$ .

Otherwise, we know that for all  $i$ ,  $\mathbf{alph}(w_{i+1}w_{i+2} \cdots w_{i+m'}) = B$ . It follows that  $w$  can be redecomposed into  $(p+2)|B|$  factors  $w = w'_1 \cdots w'_{(p+2)|B|}$  such that for all  $i$ ,  $\mathbf{alph}(w'_i) = B$ . We conclude that  $w \in (B^* b_1 B^* b_2 \cdots B^* b_n B^*)^{p+2}$  with  $\{b_1, \dots, b_n\} = B$ . In other words,  $w$  admits a good decomposition of length 1 and we are finished. ◀

It now remains to prove that each language in  $\mathbf{K}_{p,\ell}$  is definable in  $\mathcal{BS}_1(<)$ . We do this in the next lemma.

► **Lemma 35.** *Let  $p \geq 1$  and let  $T$  be an unambiguous template. Then  $K_{p,T}$ , the  $p$ -implementation of  $T$ , is definable in  $\mathcal{BS}_1(<)$ .*

**Proof.** We adopt a ‘piecewise testable point of view’: we prove that  $K_{p,T}$  is a piecewise testable language which is equivalent to being  $\mathcal{BS}_1(<)$ -definable.

If  $T$  has length 0,  $K_{p,T} = \{\varepsilon\}$  which is the language of words that do not admit any word of length 1 as a piece. Therefore,  $K_{p,T}$  is piecewise testable. Otherwise  $T = t_1, \dots, t_\ell$  for some  $\ell \geq 1$ . We construct a word  $v \in A^*$  and a finite set of words  $U \subseteq A^*$  such that  $K_{p,T}$  is the language of all words  $w \in A^*$  such that  $v$  is a piece of  $w$  while there exists no word in  $U$  which is a piece of  $w$ . It will then be immediate that  $K_{p,T}$  is piecewise testable. We begin by constructing  $v$ . For all  $i \leq \ell$ , we associate a word  $v_i$  to  $t_i$  as follows:

- if  $t_i$  is a letter  $a \in A$ , then  $v_i = a$ .
- if  $t_i$  is a triple  $(b, B, b')$  with  $B \subseteq A$  and  $b, b' \in B$ , then  $v_i = b(b_1 \cdots b_n)^p b'$  (with  $B = \{b_1, \dots, b_n\}$ ).

We define  $v = v_1 \cdots v_\ell$ . Note that by definition of  $K_{p,T}$ ,  $v$  is piece of all words in  $K_{p,T}$ . We now define  $U$  as the following set,

$$U = \{u \in A^* \mid |u| \leq |v| + 2 \text{ and for all } w \in K_{p,T}, u \text{ is not a piece of } w\}$$

Finally, we define  $H$  as the language of all words  $w$  such that  $v$  is a piece of  $w$  and no word in  $U$  is a piece of  $w$ . By definition,  $H$  is a piecewise testable language. We prove that  $H = K_{p,T}$  which terminates the proof of Lemma 35. We start with the simpler  $K_{p,T} \subseteq H$  inclusion. Let  $w \in K_{p,T}$ , by definition of  $K_{p,T}$ ,  $v$  is a piece of  $w$ . Moreover, by definition of  $U$ , no word of  $U$  is a piece of  $w$ . It follows that  $w \in H$  and therefore that  $K_{p,T} \subseteq H$ .

It now remains to prove that  $H \subseteq K_{p,T}$ . Let  $w \in H$ . Proving that  $w \in K_{p,T}$  amounts to proving that  $w$  can be decomposed as  $w = w_1 \cdots w_\ell$  such that for all  $i$ , either  $t_i = a$  and  $w_i = a$  or  $t_i = (b, B, b')$  and  $w_i \in \text{IT}_p(b, B, b')$ . In particular, note that this is where we use the fact that our template  $T$  is unambiguous. The construction is based on the following lemma.

► **Lemma 36.** *For all  $i \leq \ell - 1$ , there exists a decomposition of  $w$  as  $w = w_1 \cdots w_i \cdot w'_i$  such that:*

1. if  $t_i = (b, B, b')$ , then the first letter of  $w'_i$  does not belong to  $B$ .
2.  $v_{i+1} \cdots v_\ell$  is a piece of  $w'_i$ .
3. for all  $j \leq i$ , either  $t_j = a$  and  $w_j = a$  or  $t_j = (b, B, b')$  and  $w_j \in \text{IT}_p(b, B, b')$ .

Before proving the lemma, we use it to finish the proof of Lemma 35. By applying Lemma 36 for  $i = \ell - 1$ , we obtain a decomposition  $w = w_1 \cdots w_{\ell-1} \cdot w'_{\ell-1}$  of  $w$  such that:

1. if  $t_{\ell-1} = (b, B, b')$ , then the first letter of  $w'_i$  does not belong to  $B$ .
2.  $v_\ell$  is a piece of  $w'_{\ell-1}$ .
3. for all  $j \leq \ell - 1$ , either  $t_j = a$  and  $w_j = a$  or  $t_j = (b, B, b')$  and  $w_j \in \text{IT}_p(b, B, b')$ .

One can now verify using the fact that  $T$  is unambiguous that either  $t_\ell = a$  and  $w'_{\ell-1} = a$  or  $t_\ell = (b, B, b')$  and  $w_{\ell-1} \in \text{IT}_p(b, B, b')$  (the argument is similar to the arguments used to prove Lemma 36 and is left to the reader). We conclude that  $w \in K_{p,T}$  which finishes the proof. It now remains to prove Lemma 36.

This is an induction on  $i$ . The proof is quite tedious as there are many cases to consider depending on the nature of consecutive elements  $t_i, t_{i+1}$  in  $T$ . As all these cases are similar, we only consider two cases: one for the induction base and one for the induction step. These two cases are chosen to encompass all arguments that are need to treat the other ones (which are left to the reader).

We begin with the induction base. When  $i = 1$ , there are two cases depending on the nature of  $t_1$ . We treat the case when  $t_1 = a$ . We prove that the first letter of  $w$  is an “a”.

We proceed by contradiction. Observe that  $v_1 = a$ , hence the word  $v$  (which is a piece of  $w \in H$ ) begins with an “ $a$ ”. Therefore, if  $w$  does not begin with an “ $a$ ”, it follows that  $bv$  is a piece of  $w$  for some  $b \neq a$ . One may use the fact that  $T$  is unambiguous to verify that  $bv$  is the piece of no word in  $K_{p,T}$  which means that  $bv \in U$ . This is a contradiction since no word in  $U$  can be a piece of  $w \in H$ .

Assume now that  $i > 1$ . By induction hypothesis, we obtain a decomposition  $w = w_1 \cdots w_{i-1} \cdot w'_{i-1}$  of  $w$  that satisfies the conditions in the lemma. There are four cases depending on the nature of  $t_{i-1}$  and  $t_i$ . We treat the case when  $t_{i-1} = (c, C, c')$  and  $t_i = (b, B, b')$  which is the most involved one. We define  $w_i$  as the largest prefix of  $w'_{i-1}$  such that  $w_i \in B^*$  and we define  $w'_i$  as the corresponding suffix. It is immediate that  $w = w_1 \cdots w_i \cdot w'_i$ . It remains to prove that this choice satisfies the three conditions of the lemma. The first condition is immediate since  $w'_i$  was chosen as the largest prefix in  $B^*$ . For the second condition, we know that  $v_i \cdots v_\ell$  is a piece of  $w'_{i-1} = w_i w'_i$ . Moreover, we know from the fact that  $T$  is unambiguous that the first letter of  $v_{i+1}$  does not belong to  $B$ . Since  $w_i \in B^*$ , it follows that  $v_{i+1} \cdots v_\ell$  is a piece of  $w'_i$ . It now remains to prove the third item. We prove that  $v_i = b(b_1 \cdots b_n)^p b'$  is piece of  $w_i$ . Since  $w_i \in B^*$ , it will follow that  $w_i \in \text{IT}_p(b, B, b') = B^* b (B^* b_1 B^* \cdots B^* b_n B^*)^p b' B^*$ .

We begin by proving that  $w_i$  is nonempty. Let  $b''$  be the first letter in  $w'_{i-1}$ . By hypothesis,  $b'' \notin C$  (since  $w'_{i-1}$  was constructed by induction, it satisfies Item 1 in the lemma). We prove that  $b'' \in B$ , it will then follow that  $b''$  is the first letter of  $w_i$  since it is the largest prefix of  $w'_{i-1}$  that belongs to  $B^*$ . By contradiction assume that  $b'' \notin B$ . By construction of  $w_1, \dots, w_{i-1}$  and  $w'_{i-1}$  (see Items 2 and 3 in the lemma), it follows that  $v' = v_1 \cdots v_{i-1} \cdot b'' \cdot v_i \cdots v_\ell$  is a piece of  $w$ . However, using the fact that  $T$  is unambiguous and  $b'' \notin B \cup C$ , one may verify that  $v'$  is the piece of no word in  $K_{p,T}$ . Therefore  $v' \in U$  which is a contradiction since no word in  $U$  can be a piece of  $w \in H$ . We conclude that  $b'' \in B$  and that  $w_i = b''x$  for some  $x \in B^*$ . Moreover, we know that  $b'' \in B \setminus C$ .

We are now ready to prove that  $v_i$  is a piece of  $w_i$ . By contradiction assume that  $w_i$  is not a piece of  $w_i$ . Let  $d$  be the first letter of  $w'_i$ . By construction of  $w_i$  as the largest prefix in  $B^*$ , we know that  $d \notin B$ . Recall that  $v_i = b(b_1 \cdots b_n)^p b'$  and observe that  $b' \neq d$  since  $d \notin B$ . Since  $v_i \cdots v_\ell$  is a piece of  $w_i w'_i$ , if  $v_i$  is not a piece of  $w_i$ , we obtain that  $b'' d b' \cdot v_{i+1} \cdots v_\ell$  is a piece of  $w'_{i-1} = w_i w'_i$ . It follows that  $v' = v_1 \cdots v_i \cdot b'' d b' \cdot v_{i+1} \cdots v_\ell$  is piece of  $w$ . In particular this means that  $v' \notin U$  since  $w \in H$ . One can verify that this contradicts the fact that  $T$  is unambiguous. We conclude that  $v_i$  is piece of  $w_i$  which terminates the proof. ◀

## **E** Proof of Theorem 17: FO<sup>2</sup>-Covering

This appendix is devoted to the proof of Theorem 17, *i.e.*, of our covering algorithm for FO<sup>2</sup>. We begin by presenting the algorithm in more details and introduce some terminology that we will need for the proof.

► **Definition 37** (Covering Algorithm for FO<sup>2</sup>). Let  $\mathbf{L}$  be a tame set of languages names. We define  $\text{Sat}_{\text{FO}^2}(\mathbf{L}) \subseteq 2^{\mathbf{L}}$  as the smallest subset of  $2^{\mathbf{L}}$  which contains  $\mathcal{I}_{\text{triv}}[\mathbf{L}]$  and is closed under the following operations:

1. Downset:  $\text{Sat}_{\text{FO}^2}(\mathbf{L}) = \downarrow \text{Sat}_{\text{FO}^2}(\mathbf{L})$ .
2. Multiplication: if  $\mathbf{S}, \mathbf{T} \in \text{Sat}_{\text{FO}^2}(\mathbf{L})$ , then  $\mathbf{S} \odot \mathbf{T} \in \text{Sat}_{\text{FO}^2}(\mathbf{L})$ .
3. For any  $B \subseteq A$  and any  $\mathbf{S}, \mathbf{T} \in \text{Sat}_{\text{FO}^2}(\mathbf{L})$  containing  $S, T$  such that  $\mathbf{alph}(S) = \mathbf{alph}(T) = B$ ,  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \text{Sat}_{\text{FO}^2}(\mathbf{L})$ .

It is immediate from the definition that  $Sat_{FO^2}(\mathbf{L})$  can be computed from  $\mathbf{L}$  using a lowest fixpoint algorithm. In the following theorem, we state that  $Sat_{FO^2}$  is a covering algorithm for  $FO^2$ .

► **Theorem 38.** *Let  $\mathbf{L}$  be a finite set of languages names that is both tame and alphabet compatible. Then  $\mathcal{I}_{FO^2}[\mathbf{L}] = Sat_{FO^2}(\mathbf{L})$ .*

It now remains to prove Theorem 38: we follow the methodology presented in Section 4: we first prove soundness:  $\mathcal{I}_{FO^2}[\mathbf{L}] \supseteq Sat_{FO^2}(\mathbf{L})$  and then completeness:  $Sat_{FO^2}(\mathbf{L}) \subseteq \mathcal{I}_{FO^2}[\mathbf{L}]$ .

## E.1 Soundness

In this section that our covering algorithm is sound. In other words, we prove the inclusion  $\mathcal{I}_{FO^2}[\mathbf{L}] \supseteq Sat_{FO^2}(\mathbf{L})$  in Theorem 38. By definition of  $Sat_{FO^2}(\mathbf{L})$ , it suffices to prove that  $\mathcal{I}_{FO^2}[\mathbf{L}]$  contains the set  $\mathcal{I}_{triv}[\mathbf{L}]$  and that it is closed under the following operations:

1. Downset:  $\mathcal{I}_{FO^2}[\mathbf{L}] = \downarrow \mathcal{I}_{FO^2}[\mathbf{L}]$ .
2. Multiplication: for  $\mathbf{S}, \mathbf{T} \in \mathcal{I}_{FO^2}[\mathbf{L}]$ , we have  $\mathbf{S} \odot \mathbf{T} \in \mathcal{I}_{FO^2}[\mathbf{L}]$ .
3. For any  $B \subseteq A$  and any  $\mathbf{S}, \mathbf{T} \in \mathcal{I}_{FO^2}[\mathbf{L}]$  containing  $S, T$  such that  $\mathbf{alph}(S) = \mathbf{alph}(T) = B$ ,  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \mathcal{I}_{FO^2}[\mathbf{L}]$ .

As usual, only one of these properties is specific to  $FO^2$ . We know that  $\mathcal{I}_{\mathcal{C}}[\mathbf{L}]$  contains  $\mathcal{I}_{triv}[\mathbf{L}]$  and is closed under downset for any boolean algebra  $\mathcal{C}$  (see Fact 1). Hence, this is true in particular for  $\mathcal{C} = FO^2$ . Moreover, since  $FO^2$  is a quotienting boolean algebra, we also know from Lemma 13 that  $\mathcal{I}_{FO^2}[\mathbf{L}]$  is closed under multiplication. Therefore, we only need to prove that  $\mathcal{I}_{FO^2}[\mathbf{L}]$  is closed under the third operation.

To prove this, we use the classical equivalence associated to  $FO^2$ . Recall that the *quantifier rank* of a first-order sentence (and therefore in particular of an  $FO^2$  sentence) is the length of the longest sequence of nested quantifiers inside the sentence. Given two words  $w, w' \in A^*$  and  $k \geq 1$ , we write  $w \cong_k w'$  to denote the fact that  $w$  and  $w'$  satisfy the same  $FO^2$ -sentences of quantifier rank  $k$ . One can verify that for all  $k$ ,  $\cong_k$  is an equivalence relation of finite index and that each class can be defined in  $FO^2$ .

For all  $k \geq 1$ , we denote by  $\mathbf{K}_k$  the partition of  $A^*$  into equivalence classes of  $\cong_k$ . By definition  $\mathbf{K}_k$  is an  $FO^2$ -cover of  $\mathbf{L}$  for all  $k$ . We will need the following lemma.

► **Lemma 39.** *For any  $FO^2$ -cover  $\mathbf{H}$  of  $\mathbf{L}$ , there exists  $k$  such that  $\mathcal{I}[\mathbf{L}](\mathbf{K}_k) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{H})$ .*

**Proof.** Since  $\mathbf{H}$  is finite, there exists  $k$  such that all languages in  $\mathbf{H}$  may be defined with  $FO^2$  sentences of rank  $k$ . Since all languages in  $\mathbf{H}$  are unions of classes of  $\cong_k$ , it is immediate that  $\mathcal{I}[\mathbf{L}](\mathbf{K}_k) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{H})$ . ◀

We can now prove that  $\mathcal{I}_{FO^2}[\mathbf{L}]$  is closed under the third operation in the definition of  $Sat_{FO^2}$ . Let  $B \subseteq A$  and  $\mathbf{S}, \mathbf{T} \in \mathcal{I}_{FO^2}[\mathbf{L}]$  that contain  $S, T$  such that  $\mathbf{alph}(S) = \mathbf{alph}(T) = B$ , we have to prove that  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \mathcal{I}_{FO^2}[\mathbf{L}]$ . In other words, we have to prove that for all  $FO^2$ -covers  $\mathbf{H}$  of  $\mathbf{L}$ , we have  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \mathcal{I}[\mathbf{L}](\mathbf{H})$ .

Let  $\mathbf{H}$  be an  $FO^2$ -cover of  $\mathbf{L}$  and let  $k$  be as defined in Lemma 39 for  $\mathbf{H}$ . That is  $\mathcal{I}[\mathbf{L}](\mathbf{K}_k) \subseteq \mathcal{I}[\mathbf{L}](\mathbf{H})$ . We prove that  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{T}^\omega \in \mathcal{I}[\mathbf{L}](\mathbf{K}_k)$ , which terminates the proof. By hypothesis,  $\mathbf{S}, \mathbf{T} \in \mathcal{I}[\mathbf{L}](\mathbf{K}_k)$ . Hence, we have  $K_{\mathbf{S}}, K_{\mathbf{T}} \in \mathbf{K}_k$  such that  $\mathbf{S} \subseteq \langle \mathbf{L} | K_{\mathbf{S}} \rangle$  and  $\mathbf{T} \subseteq \langle \mathbf{L} | K_{\mathbf{T}} \rangle$ . Consider the language,

$$K = (K_{\mathbf{S}})^{(k+1)\omega} B^* (K_{\mathbf{T}})^{(k+1)\omega}$$

with  $\omega$  as the idempotent power of  $2^{\mathbf{L}}$ . Do note that by “ $(K_{\mathbf{S}})^{(k+1)\omega}$ ” in the expression above, we mean the concatenation of  $(k+1)\omega$  copies of the language  $K_{\mathbf{S}}$  and not a tame multiplication. It follows from Lemma 14 that,

$$\langle \mathbf{L}|K \rangle = (\langle \mathbf{L}|K_{\mathbf{S}} \rangle)^{\omega} \odot \langle \mathbf{L}|B^* \rangle \odot (\langle \mathbf{L}|K_{\mathbf{T}} \rangle)^{\omega}$$

Since  $\mathbf{S} \subseteq \langle \mathbf{L}|K_{\mathbf{S}} \rangle$  and  $\mathbf{T} \subseteq \langle \mathbf{L}|K_{\mathbf{T}} \rangle$ , it follows that  $\mathbf{S}^{\omega} \odot \langle \mathbf{L}|B^* \rangle \odot \mathbf{T}^{\omega} \subseteq \langle \mathbf{L}|K \rangle$ . Hence, if we can find  $K' \in \mathbf{K}_k$  such that  $K \subseteq K'$ , we will obtain that

$$\mathbf{S}^{\omega} \odot \langle \mathbf{L}|B^* \rangle \odot \mathbf{T}^{\omega} \subseteq \langle \mathbf{L}|K \rangle \subseteq \langle \mathbf{L}|K' \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K}_k)$$

This will mean that  $\mathbf{S}^{\omega} \odot \langle \mathbf{L}|B^* \rangle \odot \mathbf{T}^{\omega} \in \mathcal{I}[\mathbf{L}](\mathbf{K}_k)$  as desired. Since the languages of  $\mathbf{K}_k$  are the equivalence classes of  $\cong_k$ , proving that there exists  $K' \in \mathbf{K}_k$  such that  $K \subseteq K'$  amounts to proving that all words in  $K$  are  $\cong_k$ -equivalent. To do this, we use two classical properties of the equivalence  $\cong_k$  that we state in the two lemmas below (whose proofs are based on Ehrenfeucht-Fraïssé games and left to the reader).

► **Lemma 40.** *For all  $k \in \mathbb{N}$ , the  $\cong_k$  is a congruence for concatenation: if  $w_1 \cong_k w'_1$  and  $w_2 \cong_k w'_2$ , then  $w_1w_2 \cong_k w'_1w'_2$ .*

► **Lemma 41.** *Let  $B \subseteq A$  and  $u, v \in A^*$  such that  $\mathbf{alph}(u) = \mathbf{alph}(v) = B$ . Finally, let  $k, \ell \in \mathbb{N}$  such that  $\ell \geq k$ . Then for any  $w, w' \in B^*$ ,*

$$u^{\ell}wv^{\ell} \cong_k u^{\ell}w'v^{\ell}$$

Since  $\mathbf{S}, \mathbf{T}$  contain languages names  $S, T$  of alphabet  $B$  which are intersected by  $K_{\mathbf{S}}$  and  $K_{\mathbf{T}}$  respectively, there exist  $u \in K_{\mathbf{S}}$  and  $v \in K_{\mathbf{T}}$  such that  $\mathbf{alph}(u) = \mathbf{alph}(v) = B$ . Let  $w$  be any word in  $B^*$  and let  $\ell = (k+1)\omega$ . We prove that all words in  $K = (K_{\mathbf{S}})^{\ell}B^*(K_{\mathbf{T}})^{\ell}$  are  $\cong_k$ -equivalent to the word  $u^{\ell}wv^{\ell}$ . By definition, a word in  $K = (K_{\mathbf{S}})^{\ell}B^*(K_{\mathbf{T}})^{\ell}$  is of the form  $u'w'v'$  with  $u' \in (K_{\mathbf{S}})^{\ell}$ ,  $v' \in (K_{\mathbf{T}})^{\ell}$  and  $w' \in B^*$ . Observe that it follows from Lemma 41 that,

$$u^{\ell}wv^{\ell} \cong_k u^{\ell}w'v^{\ell}$$

Moreover, since all words in  $K_{\mathbf{S}}$  (resp. in  $K_{\mathbf{T}}$ ) are  $\cong_k$ -equivalent, it follows from Lemma 40 that  $u' \cong_k u^{\ell}$  and  $v' \cong_k v^{\ell}$ . Using Lemma 40 a second time, we obtain that,

$$u'w'v' \cong_k u^{\ell}w'v^{\ell} \cong_k u^{\ell}wv^{\ell}$$

We conclude that all words in  $K = (K_{\mathbf{S}})^{\ell}B^*(K_{\mathbf{T}})^{\ell}$  are  $\cong_k$ -equivalent, which terminates the proof.

## E.2 Completeness

In this section, we prove the difficult direction of Theorem 38 which corresponds to completeness of our pointed covering algorithm for  $\text{FO}^2$ . In other words, we prove that given a tame and alphabet compatible set of languages names  $\mathbf{L}$ ,  $\mathcal{I}_{\text{FO}^2}[\mathbf{L}] \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L})$ . More precisely, we present a generic construction for building a  $\text{FO}^2$ -cover  $\mathbf{K}$  of  $\mathbf{L}$  such that  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L})$ . Since by definition,  $\mathcal{I}_{\text{FO}^2}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K})$ , the result will follow.

In particular, recall that this generic construction is also a solution to the second stage of the covering problem: it is a generic method for building an optimal  $\text{FO}^2$ -cover of an input

set of languages. Indeed, since we already know that  $\mathcal{I}_{\text{FO}^2}[\mathbf{L}] \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L})$ , the  $\text{FO}^2$ -cover  $\mathbf{K}$  that we construct satisfies,

$$\mathcal{I}_{\text{FO}^2}[\mathbf{L}] \subseteq \mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}) \subseteq \mathcal{I}_{\text{FO}^2}[\mathbf{L}]$$

This means that the  $\text{FO}^2$ -cover that we construct is actually optimal. We start with a few facts that we will use in the construction.

Recall that since we work with an alphabet compatible set of languages names, all words belonging to the same  $L \in \mathbf{L}$  have the same alphabet that we denote  $\mathbf{alph}(L)$ . An important process in our construction will be to use induction and reduce the problem of covering  $\mathbf{L}$  to covering only  $\langle \mathbf{L}|B^* \rangle = \{L \in \mathbf{L} \mid \mathbf{alph}(L) \subseteq B\}$  for increasingly smaller subsets  $B$  of the alphabet  $A$ . In particular, to simplify notations, we will simply write  $\mathbf{L}_B$  for  $\langle \mathbf{L}|B^* \rangle$ . Let us begin with an important observation,

► **Fact 42.** *Let  $B \subseteq A$ , then the restriction of the tame multiplication of  $\mathbf{L}$  to  $\mathbf{L}_B$  remains a tame multiplication. In particular,  $\mathbf{L}_B$  is tame.*

**Proof.** Immediate from alphabet compatibility. ◀

Furthermore the two following facts will prove useful.

► **Fact 43.** *Let  $B, C \subseteq A$  such that  $C \subseteq B$ . Then,*

$$\text{Sat}_{\text{FO}^2}(\mathbf{L}_C) \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$$

**Proof.** Immediate by definition since  $\mathcal{I}_{\text{triv}}[\mathbf{L}_C] \subseteq \mathcal{I}_{\text{triv}}[\mathbf{L}_B]$ . ◀

► **Fact 44.** *Let  $B, C \subseteq A$  such that  $C \subseteq B$  and let  $\mathbf{K}$  be a finite set of languages such that all  $K \in \mathbf{K}$  satisfy  $K \subseteq C^*$ . Then,*

$$\mathcal{I}[\mathbf{L}_B](\mathbf{K}) = \mathcal{I}[\mathbf{L}_C](\mathbf{K})$$

**Proof.** Since  $C \subseteq B$ , we have  $\mathbf{L}_B \subseteq \mathbf{L}_C$ . Therefore it is immediate that  $\mathcal{I}[\mathbf{L}_C](\mathbf{K}) \subseteq \mathcal{I}[\mathbf{L}_B](\mathbf{K})$  for any  $\mathbf{K}$ . We prove the converse inclusion. Let  $\mathbf{H} \in \mathcal{I}[\mathbf{L}_B](\mathbf{K})$ . By definition,  $\mathbf{H} \subseteq \langle \mathbf{L}_B|K \rangle$  for some  $K \in \mathbf{K}$ . In particular, since  $K \subseteq C^*$ , this means that all languages in  $H \in \mathbf{H}$  satisfy  $H \in \mathbf{L}_C$ . Therefore,  $\langle \mathbf{L}_B|K \rangle = \langle \mathbf{L}_C|K \rangle$  which means that  $\mathbf{H} \in \mathcal{I}[\mathbf{L}_C](\mathbf{K})$ . ◀

We are now ready to start the construction. We state it in Proposition 45 below. Observe that for all  $B \subseteq A$ ,  $\text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$  is a subsemigroup of  $2^{\mathbf{L}}$  by definition. In particular, we use the notation  $(\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  to denote the following monoid:

$$(\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1 = \begin{cases} \text{Sat}_{\text{FO}^2}(\mathbf{L}_B) & \text{if } \text{Sat}_{\text{FO}^2}(\mathbf{L}_B) \text{ is a monoid} \\ \text{Sat}_{\text{FO}^2}(\mathbf{L}_B) \cup \{\mathbf{1}\} & \text{where } \mathbf{1} \text{ is an artificial neutral element, otherwise} \end{cases}$$

► **Proposition 45.** *Let  $B \subseteq A$  and  $\mathbf{T}_\ell, \mathbf{T}_r \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$ . Then there exists a  $\text{FO}^2$ -cover  $\mathbf{K}$  of  $\mathbf{L}_B$  such that,*

$$\mathbf{T}_\ell \odot \mathcal{I}[\mathbf{L}_B](\mathbf{K}) \odot \mathbf{T}_r \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$$

Before we prove Proposition 45, we use it to finish the proof of Theorem 38. We apply Proposition 45 in the case when  $B = A$  and  $\mathbf{T}_\ell, \mathbf{T}_r$  are both equal to the neutral element of  $(\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$ . We obtain a  $\text{FO}^2$ -cover  $\mathbf{K}$  of  $\mathbf{L}_A = \mathbf{L}$  such that,  $\mathcal{I}[\mathbf{L}](\mathbf{K}) \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L})$  which terminates the proof.

It now remains to prove Proposition 45. We let  $B, \mathbf{T}_\ell$  and  $\mathbf{T}_r$  be as in the statement of the proposition. The proof is by induction on three parameters that we define now. The first and most important parameter is the size of  $B$ . The other two also depend on  $\mathbf{T}_\ell$  and  $\mathbf{T}_r$  respectively. Given  $\mathbf{R}_1, \mathbf{R}_2 \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$ , we say that,

- $\mathbf{R}_2$  is *left reachable* from  $\mathbf{R}_1$  if there exists  $\mathbf{Q} \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  such that  $\mathbf{R}_2 \subseteq \mathbf{Q} \odot \mathbf{R}_1$ .
- $\mathbf{R}_2$  is *right reachable* from  $\mathbf{R}_1$  if there exists  $\mathbf{Q} \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  such that  $\mathbf{R}_2 \subseteq \mathbf{R}_1 \odot \mathbf{Q}$ .

One may verify that left and right reachability are both preorder relations on the set  $(\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  (this is because tame multiplications are compatible with inclusion). For each  $\mathbf{R} \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$ , we call *left index of  $\mathbf{R}$*  (resp. *right index of  $\mathbf{R}$* ) the number of elements that are left reachable (resp. right reachable) from  $\mathbf{R}$ . We proceed by induction on the following parameters listed by order of importance:

1.  $|B|$
2. The right index of  $\mathbf{T}_\ell$ .
3. The left index of  $\mathbf{T}_r$ .

We consider three cases depending on a property of  $\mathbf{T}_\ell$  and a property of  $\mathbf{T}_r$ . These properties are based on the following fact.

► **Fact 46.** For all  $b \in B$ ,  $\langle \mathbf{L}_B | \{b\} \rangle \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ .

**Proof.** Let  $\mathbf{H} = \langle \mathbf{L}_B | \{b\} \rangle$ . Since  $\{b\}$  is a singleton, all language in  $\mathbf{H}$  contain the word  $b$ . Therefore  $\bigcap_{H \in \mathbf{H}} H \neq \emptyset$  and  $\mathbf{H} \in \mathcal{I}_{\text{triv}}[\mathbf{L}_B] \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$  by definition. ◀

We may now define the two properties that specify our three cases.

- we say that  $\mathbf{T}_\ell$  is *right saturated* by  $B$  if for all  $b \in B$ , there exists  $\mathbf{Q} \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  such that  $\mathbf{T}_\ell$  is right reachable from  $\mathbf{T}_\ell \odot \mathbf{Q} \odot \langle \mathbf{L}_B | \{b\} \rangle$ .
- we say that  $\mathbf{T}_r$  is *left saturated* by  $B$  if for all  $b \in B$ , there exists  $\mathbf{Q} \in (\text{Sat}_{\text{FO}^2}(\mathbf{L}_B))^1$  such that  $\mathbf{T}_r$  is left reachable from  $\langle \mathbf{L}_B | \{b\} \rangle \odot \mathbf{Q} \odot \mathbf{T}_r$ .

The base case happens when  $\mathbf{T}_\ell$  is right saturated and  $\mathbf{T}_r$  is left saturated. Otherwise we use induction. We begin with the base case.

### Base Case: $\mathbf{T}_\ell$ is right saturated and $\mathbf{T}_r$ is left saturated

In this case, we simply define  $\mathbf{K} = \{B^*\}$  (which is clearly a  $\text{FO}^2$ -cover of  $\mathbf{L}_B$ ). We use our hypothesis to prove that it satisfies  $\mathbf{T}_\ell \odot \mathcal{I}[\mathbf{L}_B](\mathbf{K}) \odot \mathbf{T}_r \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . Since by definition,  $\mathcal{I}[\mathbf{L}_B](\mathbf{K}) = \downarrow\{\mathbf{L}_B\}$ , it suffices to prove that  $\mathbf{T}_\ell \odot \mathbf{L}_B \odot \mathbf{T}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . There is a special case when  $\mathbf{T}_\ell = \emptyset$  or  $\mathbf{T}_r = \emptyset$ . In that case,  $\mathbf{T}_\ell \odot \mathbf{L}_B \odot \mathbf{T}_r = \emptyset \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . Assume now that  $\mathbf{T}_\ell \neq \emptyset$  and  $\mathbf{T}_r \neq \emptyset$ . We use the following lemma.

► **Lemma 47.** There exist  $\mathbf{Q}_\ell, \mathbf{Q}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$  containing  $Q_\ell, Q_r$  such that  $\text{alph}(Q_\ell) = \text{alph}(Q_r) = B$ ,  $\mathbf{T}_\ell \subseteq \mathbf{T}_\ell \odot \mathbf{Q}_\ell$  and  $\mathbf{T}_r \subseteq \mathbf{Q}_r \odot \mathbf{T}_r$ .

Before we prove the lemma we use it to conclude this case. Using a little algebra and the fact that “ $\odot$ ” is compatible with inclusion, we obtain  $\mathbf{T}_\ell \subseteq \mathbf{T}_\ell \odot (\mathbf{Q}_\ell)^\omega$  and  $\mathbf{T}_r \subseteq (\mathbf{Q}_r)^\omega \odot \mathbf{T}_r$ . In particular, observe that we obtain,

$$\mathbf{T}_\ell \odot \mathbf{L}_B \odot \mathbf{T}_r \subseteq \mathbf{T}_\ell (\mathbf{Q}_\ell)^\omega \odot \mathbf{L}_B \odot (\mathbf{Q}_r)^\omega \mathbf{T}_r$$

Moreover, since  $\mathbf{alph}(Q_\ell) = \mathbf{alph}(Q_r) = B$ , we know from the definition of  $Sat_{FO^2}$  (see Operation 3) that,

$$(\mathbf{Q}_\ell)^\omega \odot \mathbf{L}_B \odot (\mathbf{Q}_r)^\omega \in Sat_{FO^2}(\mathbf{L}_B)$$

Using closure under multiplication, we then obtain that,  $\mathbf{T}_\ell \odot (\mathbf{Q}_\ell)^\omega \odot \mathbf{L}_B \odot (\mathbf{Q}_r)^\omega \odot \mathbf{T}_r \in Sat_{FO^2}(\mathbf{L}_B)$ . Since  $\mathbf{T}_\ell \odot \mathbf{L}_B \odot \mathbf{T}_r \subseteq \mathbf{T}_\ell (\mathbf{Q}_\ell)^\omega \odot \mathbf{L}_B \odot (\mathbf{Q}_r)^\omega \mathbf{T}_r$ , we use closure under downset to conclude that  $\mathbf{T}_\ell \odot \mathbf{L}_B \odot \mathbf{T}_r \in Sat_{FO^2}(\mathbf{L}_B)$  which terminates the proof of this case. We finish with the proof of Lemma 47.

**Proof.** We only prove the existence of  $\mathbf{Q}_\ell$  using the fact that  $\mathbf{T}_\ell$  is right saturated (the existence of  $\mathbf{Q}_r$  is proved symmetrically using the fact that  $\mathbf{T}_r$  is left saturated). By definition of right saturation, we know that for each  $b \in B$ , there exists  $\mathbf{U}_b, \mathbf{V}_b \in (Sat_{FO^2}(\mathbf{L}_B))^1$  such that,

$$\mathbf{T}_\ell \subseteq \mathbf{T}_\ell \odot \mathbf{U}_b \odot \langle \mathbf{L}_B | \{b\} \rangle \odot \mathbf{V}_b \in Sat_{FO^2}(\mathbf{L}_B)$$

We define  $\mathbf{Q}_\ell = \prod_{b \in B} (\mathbf{U}_b \odot \langle \mathbf{L}_B | \{b\} \rangle \odot \mathbf{V}_b)$  (where the product is made in an arbitrarily chosen order). By definition, we have  $\mathbf{T}_\ell \subseteq \mathbf{T}_\ell \odot \mathbf{Q}_\ell$ . It remains to prove that  $\mathbf{Q}_\ell$  contains  $Q_\ell$  of alphabet  $B$ . We prove an even stronger property: any  $Q_\ell \in \mathbf{Q}_\ell$  has alphabet  $B$ . This suffices since there exists at least one  $Q_\ell \in \mathbf{Q}_\ell$  (by hypothesis  $\mathbf{T}_\ell \neq \emptyset$  and  $\mathbf{T}_\ell \subseteq \mathbf{T}_\ell \odot \mathbf{Q}_\ell$ ).

Observe that for any  $b \in B$ , all languages names  $Q_b \in \mathbf{U}_b \odot \langle \mathbf{L}_B | \{b\} \rangle \odot \mathbf{V}_b$  satisfy  $b \in \mathbf{alph}(Q_b) \subseteq B$ . By definition, any  $Q_\ell \in \mathbf{Q}_\ell$  includes the concatenation of a sequence of languages  $Q_b \in \mathbf{U}_b \odot \langle \mathbf{L}_B | \{b\} \rangle \odot \mathbf{V}_b$  for each  $b \in B$ . Therefore,  $\mathbf{alph}(Q_\ell) = B$ . ◀

We are now finished with the base case. It remains to treat the case when either  $\mathbf{T}_\ell$  is not right saturated or  $\mathbf{T}_r$  is not left saturated. These two case are symmetrical. We consider the case when  $\mathbf{T}_\ell$  is not right saturated. The second case is left to the reader.

### Induction Case: $\mathbf{T}_\ell$ is not right saturated

Since  $\mathbf{T}_\ell$  is not right saturated, this means that there exists some  $b \in B$  such that for all  $\mathbf{Q} \in Sat_{FO^2}(\mathbf{L}_B)$ ,  $\mathbf{T}_\ell$  is **not** right reachable from  $\mathbf{T}_\ell \odot \mathbf{Q} \odot \langle \mathbf{L}_B | \{b\} \rangle$ . We use  $b$  to construct our  $FO^2$ -cover  $\mathbf{K}$  of  $\mathbf{L}_B$ .

Let  $C = B \setminus \{b\}$ . Using induction on the size of the alphabet, we obtain a  $FO^2$ -cover  $\mathbf{H}$  of  $\mathbf{L}_C$  such that  $\mathcal{I}[\mathbf{L}_C](\mathbf{H}) \subseteq Sat_{FO^2}(\mathbf{L}_C)$ . Moreover, we may assume without loss of generality that for all  $H \in \mathbf{H}$ , we have  $H \subseteq C^*$ . Indeed, we can always restrict the languages  $H$  to words belonging to  $C^*$  with  $FO^2$ . In particular, this means that  $\mathcal{I}[\mathbf{L}_C](\mathbf{H}) = \mathcal{I}[\mathbf{L}_B](\mathbf{H})$  (this is Fact 44). Moreover, since  $Sat_{FO^2}(\mathbf{L}_C) \subseteq Sat_{FO^2}(\mathbf{L}_B)$  (this is Fact 43), we obtain that,

$$\mathcal{I}[\mathbf{L}_B](\mathbf{H}) \subseteq Sat_{FO^2}(\mathbf{L}_B) \tag{2}$$

For each  $H \in \mathbf{H}$  let  $\mathbf{T}_H = \mathbf{T}_\ell \odot \langle \mathbf{L}_B | H \rangle \odot \langle \mathbf{L}_B | \{b\} \rangle$ . Since  $\mathcal{I}[\mathbf{L}_B](\mathbf{H}) \subseteq Sat_{FO^2}(\mathbf{L}_B)$ , we know that  $\langle \mathbf{L}_B | H \rangle \in Sat_{FO^2}(\mathbf{L}_B)$  and therefore that  $\mathbf{T}_H \in Sat_{FO^2}(\mathbf{L}_B)$ . By definition of  $b$ , we know that  $\mathbf{T}_\ell$  is not right reachable from  $\mathbf{T}_H$ . In particular, it follows that  $\mathbf{T}_H$  has a right index that is strictly smaller than the right index of  $\mathbf{T}_\ell$ . Hence, we may use induction to construct a  $FO^2$ -cover  $\mathbf{K}_H$  of  $\mathbf{L}_B$  such that,

$$\mathbf{T}_H \odot \mathcal{I}[\mathbf{L}_B](\mathbf{K}_H) \odot \mathbf{T}_r \subseteq Sat_{FO^2}(\mathbf{L}_B) \tag{3}$$

We are now ready to construct our  $FO^2$ -cover  $\mathbf{K}$  of  $\mathbf{L}_B$ . We build the languages of  $\mathbf{K}$  from the languages of  $\mathbf{H}$  and the languages in the sets  $\mathbf{K}_H$ . We define,

$$\mathbf{K} = \mathbf{H} \cup \bigcup_{H \in \mathbf{H}} \{HbK \mid K \in \mathbf{K}_H\}$$

It remains to prove that  $\mathbf{K}$  is an  $\text{FO}^2$ -cover of  $\mathbf{L}_B$  and that  $\mathbf{T}_\ell \odot \mathcal{I}[\mathbf{L}_B](\mathbf{K}) \odot \mathbf{T}_r \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . We begin by proving that all languages  $K \in \mathbf{K}$  are definable in  $\text{FO}^2$ . This is immediate by construction of  $\mathbf{H}$  if  $K \in \mathbf{H}$ . Otherwise,  $K = HbK'$  with  $H \in \mathbf{H}$  and  $K' \in \mathbf{K}_H$ . Observe that by definition,  $H \subseteq C^*$ ,  $b \in B \setminus C$ ,  $H$  is definable in  $\text{FO}^2$  and  $K'$  is definable in  $\text{FO}^2$ . Hence, since  $H \subseteq C^*$  and  $b \notin C$ , a word  $w$  belongs to  $HbK'$  if and only if it satisfies the following three properties:

1.  $w$  contains a  $b$ .
2. the prefix of  $w$  obtained by keeping all positions that are strictly smaller than the position with the leftmost  $b$  belongs to  $H$ .
3. the suffix of  $w$  obtained by keeping all positions that are strictly larger than the position with the leftmost  $b$  belongs to  $K'$ .

Therefore, it suffices to prove that these three properties can be expressed in  $\text{FO}^2(<)$ . This is clear for the first property. For the second and third properties, observe that one can select the position with the leftmost  $b$  using the following  $\text{FO}^2$  formula:

$$\varphi(x) := P_b(x) \wedge \neg \exists y (P_b(y) \wedge y < x)$$

Therefore a sentence for the second property above can be obtained from a sentence  $\Psi_H$  that defines  $H$  by restricting all quantifications to positions  $x$  that satisfy the formula  $\exists y x < y \wedge \varphi(y)$ . Similarly, a sentence for the third property can be obtained from a sentence  $\Psi_L$  that defines  $L$  by restricting all quantifications to positions  $x$  that satisfy the formula  $\exists y y < x \wedge \varphi(y)$ .

We now prove that  $\mathbf{K}$  covers  $\mathbf{L}_B$ . Let  $L \in \mathbf{L}_B$  and  $w \in L$ , we need to find  $K \in \mathbf{K}$  such that  $w \in K$ . If  $L \in \mathbf{L}_c$ ,  $w$  belongs to some  $H \in \mathbf{H} \subseteq \mathbf{K}$  since  $\mathbf{H}$  was constructed as a cover of  $\mathbf{L}_C$ . Otherwise,  $b \in \mathbf{alph}(L) = \mathbf{alph}(w)$  and  $w$  can be decomposed as  $w = ubv$  with  $\mathbf{alph}(u) \subseteq C$  and  $\mathbf{alph}(v) \subseteq B$ . Since  $\mathbf{L}$  is tame, there exists  $L_u \in \mathbf{L}_C$  that contains  $u$ ,  $L_b \in \mathbf{L}_B$  that contains  $b$  and  $L_v \in \mathbf{L}_B$  that contains  $v$  such that  $L = L_u \odot L_b \odot L_v$ . It follows that  $u$  belongs to some  $H \in \mathbf{H}$  ( $\mathbf{H}$  is a cover of  $\mathbf{L}_C$ ) and that  $v$  belongs to some  $K' \in \mathbf{K}_H$  ( $\mathbf{K}_H$  is a cover of  $\mathbf{L}_B$ ). Hence  $w \in HbK'$  which is a language of  $\mathbf{K}$  and we are finished.

Finally, we have to prove that  $\mathbf{T}_\ell \odot \mathcal{I}[\mathbf{L}_B](\mathbf{K}) \odot \mathbf{T}_r \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . By definition of  $\mathcal{I}[\mathbf{L}_B](\mathbf{K})$ , it suffices to prove that for all  $K \in \mathbf{K}$ ,  $\mathbf{T}_\ell \odot \langle \mathbf{L}_B | K \rangle \odot \mathbf{T}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . We have two cases depending on  $K$ . If  $K \in \mathbf{H}$ , then  $\langle \mathbf{L}_B | K \rangle \in \mathcal{I}[\mathbf{L}_B](\mathbf{H}) \subseteq \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$  (see (2)). Moreover, since  $\mathbf{T}_\ell, \mathbf{T}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$  it follows from closure under multiplication that  $\mathbf{T}_\ell \odot \langle \mathbf{L}_B | K \rangle \odot \mathbf{T}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . Assume now that  $K = HbK'$  for  $H \in \mathbf{H}$  and  $K' \in \mathbf{K}_H$ . Using Lemma 14, we obtain that,

$$\langle \mathbf{L}_B | K \rangle = \langle \mathbf{L}_B | H \rangle \odot \langle \mathbf{L}_B | \{b\} \rangle \odot \langle \mathbf{L}_B | K' \rangle$$

Recall that by definition,  $\mathbf{T}_H = \mathbf{T}_\ell \odot \langle \mathbf{L}_B | H \rangle \odot \langle \mathbf{L}_B | \{b\} \rangle$ . Therefore,  $\mathbf{T}_\ell \odot \langle \mathbf{L}_B | K \rangle \odot \mathbf{T}_r = \mathbf{T}_H \odot \langle \mathbf{L}_B | K' \rangle \odot \mathbf{T}_r$ . Finally, since  $K' \in \mathbf{K}_H$ , we have  $\langle \mathbf{L}_B | K' \rangle \in \mathcal{I}[\mathbf{L}](\mathbf{K}_H)$ . It follows from (3) that  $\mathbf{T}_\ell \odot \langle \mathbf{L}_B | K \rangle \odot \mathbf{T}_r \in \text{Sat}_{\text{FO}^2}(\mathbf{L}_B)$ . This concludes the proof of Proposition 45.

## **F** Proof of Theorem 18: Pointed $\Sigma_2$ -Covering

This appendix is devoted to the proof of Theorem 18, *i.e.*, of our pointed covering algorithm for  $\Sigma_2$ . We begin by presenting the algorithm in more details and introduce some terminology that we will need for the proof.

Consider a finite set of languages names  $\mathbf{L}$  that is both tame and alphabet compatible. Recall that since  $\mathbf{L}$  is alphabet compatible,  $\mathbf{alph}(L)$  is well-defined for all  $L \in \mathbf{L}$  as the

alphabet  $B$  shared by all words in  $L$ . Moreover, recall that for any  $B \subseteq A$ , we denote by  $\mathbf{L}_B$  the set  $\{L \in \mathbf{L} \mid \mathbf{alph}(L) \subseteq B\}$ .

► **Definition 48** (Pointed Covering Algorithm for  $\Sigma_2$ ). Let  $\mathbf{L}$  be a tame set of language names such that  $\mathbf{L}$  is both tame and alphabet compatible. We define  $Sat_{\Sigma_2}(\mathbf{L}) \subseteq \mathbf{L} \times 2^{\mathbf{L}}$  as the smallest subset of  $\mathbf{L} \times 2^{\mathbf{L}}$  containing  $\mathcal{P}_{triv}[\mathbf{L}]$  and closed under the following operations:

1. Downset:  $Sat_{\Sigma_2}(\mathbf{L}) = \downarrow Sat_{\Sigma_2}(\mathbf{L})$ .
2. Multiplication: if  $(S, \mathbf{S}), (T, \mathbf{T}) \in Sat_{\Sigma_2}(\mathbf{L})$  and  $S \odot T$  is defined, then,  $(S \odot T, \mathbf{S} \odot \mathbf{T}) \in Sat_{\Sigma_2}(\mathbf{L})$ .
3. For any  $B \subseteq A$  and  $(S, \mathbf{S}) \in Sat_{\Sigma_2}(\mathbf{L})$  such that  $\mathbf{alph}(S) = B$  and  $S \odot S$  is defined,  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} \mid B^* \rangle \odot \mathbf{S}^\omega) \in Sat_{\Sigma_2}(\mathbf{L})$ .

As usual it is immediate that  $Sat_{\Sigma_2}(\mathbf{L})$  may be computed from  $\mathbf{L}$  using a lowest fixpoint algorithm. This terminates the presentation of our pointed covering algorithm for  $\Sigma_2$ . We state its correctness below.

► **Theorem 49.** *Let  $\mathbf{L}$  be a set of language names such that  $\mathbf{L}$  is both tame and alphabet compatible. Then  $\mathcal{P}_{\Sigma_2}[\mathbf{L}] = Sat_{\Sigma_2}(\mathbf{L})$ .*

It now remains to prove Theorem 49. We follow the methodology presented in Section 4: we first prove soundness ( $\mathcal{P}_{\Sigma_2}[\mathbf{L}] \supseteq Sat_{\Sigma_2}(\mathbf{L})$ ) and then completeness ( $\mathcal{P}_{\Sigma_2}[\mathbf{L}] \subseteq Sat_{\Sigma_2}(\mathbf{L})$ ).

## F.1 Soundness

In this section we prove the soundness of our algorithm, that is the inclusion  $\mathcal{P}_{\Sigma_2}[\mathbf{L}] \supseteq Sat_{\Sigma_2}(\mathbf{L})$  in Theorem 49. By definition of  $Sat_{\Sigma_2}(\mathbf{L})$ , we need to prove that  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  contains  $\mathcal{P}_{triv}[\mathbf{L}]$  and that it is closed under the following operations:

1. Downset:  $\mathcal{P}_{\Sigma_2}[\mathbf{L}] = \downarrow \mathcal{P}_{\Sigma_2}[\mathbf{L}]$ .
2. Multiplication: if  $(S, \mathbf{S}), (T, \mathbf{T}) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$  and  $S \odot T$  is defined, then,  $(S \odot T, \mathbf{S} \odot \mathbf{T}) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$ .
3. For any  $B \subseteq A$  and  $(S, \mathbf{S}) \in Sat_{\Sigma_2}(\mathbf{L})$  such that  $\mathbf{alph}(S) = B$  and  $S \odot S$  is defined,  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} \mid B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$ .

As usual, only one of these properties is specific to  $\Sigma_2$ . Since  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  is a pointed imprint, we already know that it contains  $\mathcal{P}_{triv}[\mathbf{L}]$  (this is Fact 6) and that it is closed under downset. Moreover,  $\mathbf{L}$  is tame and  $\Sigma_2$  is a known quotienting lattice, hence we also know that  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  is closed under multiplication (this is Lemma 13). It remains to prove that  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  is closed under the third operation. This is what we do now.

Recall that the *quantifier rank* of a first-order sentence (and therefore in particular of a  $\Sigma_2$  sentence) is the length of the longest sequence of nested quantifiers inside the sentence. Given two words  $w, w' \in A^*$  and  $k \geq 1$ , we write  $w \lesssim_k w'$  to denote the fact that all  $\Sigma_2$ -sentences of rank  $k$  that are satisfied by  $w$  are satisfied by  $w'$  as well. One may verify that for all  $k$ ,  $\lesssim_k$  is a preorder. Moreover, the following set of languages is a finite set of  $\Sigma_2$ -definable languages:

$$\{\{u \in A^* \mid w \lesssim_k u\} \mid w \in A^*\}.$$

For all  $k \in \mathbb{N}$ , we denote by  $\mathbb{P}_k$  the following  $\mathbf{L}$ -pointed set:

$$\mathbb{P}_k = \{(L, \{w \mid u \lesssim_k w\}) \mid L \in \mathbf{L} \text{ and } u \in L\}$$

► **Lemma 50.** *For any pointed  $\Sigma_2$ -cover  $\mathbb{Q}$  of  $\mathbf{L}$ , there exists  $k \in \mathbb{N}$  such that  $\mathcal{P}[\mathbf{L}](\mathbb{P}_k) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{Q})$ .*

**Proof.** Since the support  $\mathbf{H}$  of  $\mathbb{Q}$  is finite, there exists  $\ell$  such that all languages in  $\mathbf{H}$  may be defined with  $\Sigma_2$  sentences of rank  $\ell$ . It suffices to choose  $k = \ell$ . ◀

We can now prove that  $\mathcal{P}_{\Sigma_2}[\mathbf{L}]$  is closed under the third operation in the definition of  $\text{Sat}_{\Sigma_2}(\mathbf{L})$  and conclude the soundness proof. Let  $B \subseteq A$ ,  $(S, \mathbf{S}) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$  such that  $\mathbf{alph}(S) = B$ , and  $S \odot S$  is defined. We have to prove that  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$ . In other words, we have to prove that for all pointed  $\Sigma_2$ -covers  $\mathbb{Q}$  of  $\mathbf{L}$ , we have  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}[\mathbf{L}](\mathbb{Q})$ .

Let  $\mathbb{Q}$  be a pointed  $\Sigma_2$ -cover of  $\mathbf{L}$  and let  $k$  be as defined in Lemma 50 for  $\mathbb{Q}$ . We prove that  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}[\mathbf{L}](\mathbb{P}_k)$ , which terminates the proof since  $\mathcal{P}[\mathbf{L}](\mathbb{P}_k) \subseteq \mathcal{P}[\mathbf{L}](\mathbb{Q})$ .

By hypothesis  $(S, \mathbf{S}) \in \mathcal{P}_{\Sigma_2}[\mathbf{L}]$ . Hence, we have  $K \in \mathbb{P}_k(S)$  such that  $\mathbf{S} \subseteq \langle \mathbf{L} | K \rangle$ . Consider the language,

$$H = (K)^{2^{k+1}\omega} B^* (K)^{2^{k+1}\omega}$$

with  $\omega$  as the idempotent power of  $2^{\mathbf{L}}$ . Observe that  $\langle \mathbf{L} | H \rangle = (\langle \mathbf{L} | K \rangle)^\omega \odot \langle \mathbf{L} | B^* \rangle \odot (\langle \mathbf{L} | K \rangle)^\omega \supseteq \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega$  (see Lemma 14). Hence, if we can find  $H' \in \mathbb{P}_k(S^\omega)$  such that  $H \subseteq H'$ , we will obtain that  $\mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega \in \langle \mathbf{L} | H' \rangle$  and therefore that  $(S^\omega, \mathbf{S}^\omega \odot \langle \mathbf{L} | B^* \rangle \odot \mathbf{S}^\omega) \in \mathcal{P}[\mathbf{L}](\mathbb{P}_k)$  as desired.

We begin by defining  $H'$  and then prove that it satisfies  $H \subseteq H'$ . Recall that by definition, since  $(S, K) \in \mathbb{P}_k$ , there exists  $u \in S$  such that  $K = \{w \in A^* \mid u \lesssim_k w\}$ . In particular, we know from the definition of tame multiplications that,

$$v = u^{2^{k+2}\omega} \in S^{2^{k+2}\omega} = S^\omega$$

By definition of  $\mathbb{P}_k$  we know that  $H' = \{w \in A^* \mid v \lesssim_k w\}$  belongs to  $\mathbb{P}_k(S^\omega)$ . It now remains to prove that  $H \subseteq H'$ . By definition of  $H'$ , this amounts to proving that all  $w \in H$  satisfy  $v \lesssim_k w$ . We use the two following lemmas (whose proof is based on Ehrenfeucht-Fraïssé games and left to the reader).

► **Lemma 51.** *For all  $i \geq 1$ , the preorder  $\lesssim_k i$  is a precongruence for concatenation: if  $u \lesssim_k iu'$  and  $v \lesssim_k iv'$ , then  $uv \lesssim_k iu'v'$ .*

► **Lemma 52.** *Let  $i \geq 1$ , and let  $k, \ell, r, \ell', r' \in \mathbb{N}$  be such that  $\ell, r, \ell', r' \geq 2^k$  and let  $u, y \in A^*$  such that  $\mathbf{alph}(y) \subseteq \mathbf{alph}(u)$ . Then we have:*

$$u^\ell u^r \lesssim_k u^{\ell'} y u^{r'}.$$

Let  $w \in H = (K)^{2^{k+1}\omega} B^* (K)^{2^{k+1}\omega}$ . Since  $K$  is the language  $\{w' \in A^* \mid u \lesssim_k w'\}$ , using Lemma 51 one may verify that there exists  $y \in B^*$  such that,

$$u^{2^{k+1}\omega} y u^{2^{k+1}\omega} \lesssim_k w$$

Furthermore, by definition,  $v = u^{2^{k+2}\omega}$  and since  $u \in S$  and  $B = \mathbf{alph}(S)$ , we have  $\mathbf{alph}(u) = B$ . Therefore, it follows from Lemma 52 that we have

$$v \lesssim_k u^{2^{k+1}\omega} y u^{2^{k+1}\omega}$$

We can now combine the two inequalities and obtain  $v \lesssim_k w$  which terminates the proof.

## F.2 Completeness

In this section we prove the difficult direction in Theorem 49 which corresponds to completeness of our pointed covering algorithm for  $\Sigma_2$ . More precisely, we prove that given a tame and alphabet compatible set of languages names  $\mathbf{L}$ ,  $\mathcal{P}_{\Sigma_2}[\mathbf{L}] \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$ . As usual, this proof is also a solution to the second stage of the pointed covering problem: following it yields a generic method for building an optimal pointed  $\Sigma_2$ -cover of an input tame set of languages. We present a generic construction for building a pointed  $\Sigma_2$ -cover  $\mathbb{P}$  of  $\mathbf{L}$  such that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$ .

We begin with an overview of the construction. Consider the  $\mathbf{L}$ -pointed set  $\mathbb{T}$  that contains all pairs  $(L, \{w\})$  where  $L \in \mathbf{L}$  and  $w$  belongs to  $L \cap (A \cup \{\varepsilon\})$  (i.e.  $w$  belongs to  $L$  and is either a single letter or the empty word). One can verify that  $\mathbb{T}$  satisfies the two following properties:

1. all languages in the support of  $\mathbb{T}$  are definable in  $\Sigma_2$  (they are finite languages).
2.  $\mathcal{P}[\mathbf{L}](\mathbb{T}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$  (by definition,  $\mathcal{P}[\mathbf{L}](\mathbb{T}) \subseteq \mathcal{P}_{\text{triv}}[\mathbf{L}] \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$ ).

However, observe as soon as some  $L \in \mathbf{L}$  contains a word with more than two letters,  $\mathbb{T}$  is clearly **not** a pointed  $\Sigma_2$ -cover of  $\mathbf{L}$ . While  $\mathbb{T}$  is not a pointed  $\Sigma_2$ -cover of  $\mathbf{L}$  itself, we use it to build one by relying on a new notion called “ $\mathbb{Q}$ -decompositions”.

**$\mathbb{Q}$ -Decompositions.** Let  $\mathbb{Q}$  be a  $\mathbf{L}$ -pointed finite set of languages. Given  $L \in \mathbf{L}$  and  $w \in A^*$ , we say that  $w$  admits a  $\mathbb{Q}$ -decomposition for  $L$  if  $w$  may be decomposed as  $w = w_1 \cdots w_n$  such that there exist pairs  $(L_1, H_1), \dots, (L_n, H_n) \in \mathbb{Q}$  such that,

- $L = L_1 \odot \cdots \odot L_n$ .
- for all  $i \leq n$ ,  $w_i \in H_i$ .

We write  $D_L(\mathbb{Q})$  the language of all words  $w$  which admit a  $\mathbb{Q}$ -decomposition for  $L$ . Observe that the  $\mathbf{L}$ -pointed set  $\mathbb{T}$  above satisfies the following fact.

► **Fact 53.** For all  $L \in \mathbf{L}$ ,  $L \subseteq D_L(\mathbb{T})$ .

**Proof.** Let  $L \in \mathbf{L}$  and  $w \in L$ . If  $w = \varepsilon$ , then  $(L, \{w\}) \in \mathbb{T}$  and therefore  $w$  admits a  $\mathbb{T}$ -decomposition of length 1 for  $L$ . We conclude that  $w \in D_L(\mathbb{T})$ . Otherwise let  $w = a_1 \cdots a_n$  be the decomposition of  $w$  as a concatenation of letters. Since  $w \in L$ , by definition of tame multiplications, we obtain  $L_1, \dots, L_n \in \mathbf{L}$  such that  $a_i \in L_i$  for all  $i$  and  $L = L_1 \odot \cdots \odot L_n$ . By definition of  $\mathbb{T}$ , it follows that  $(L_i, \{a_i\}) \in \mathbb{T}$  for all  $i$  and therefore that  $w = a_1 \cdots a_n$  is a  $\mathbb{T}$ -decomposition of length  $n$  for  $L$ , i.e.  $w \in D_L(\mathbb{T})$ . ◀

We use induction to prove that if we have a  $\mathbf{L}$ -pointed set  $\mathbb{Q}$  of  $\Sigma_2$ -definable languages such that  $\mathcal{P}[\mathbf{L}](\mathbb{Q}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$  in hand (for example  $\mathbb{T}$ ), we can use it to build a new  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}$  which still satisfies  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$  and such that,

$$\text{for all } L \in \mathbf{L}, \mathbb{P}(L) \text{ covers } D_L(\mathbb{Q}) \text{ (i.e. } D_L(\mathbb{Q}) \subseteq \cup_{K \in \mathbb{P}(L)} K)$$

In view of Fact 53 applying this result to  $\mathbb{T}$  will yield the desired  $\Sigma_2$ -cover  $\mathbb{P}$  of  $\mathbf{L}$  such that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$ . Before we formally present this result, we introduce some terminology that we will need.

## Preliminaries

**Safe  $\mathbf{L}$ -pointed sets.** When constructing  $\mathbf{L}$ -pointed sets of languages, it will be convenient choose them so that they satisfy a specific property that we call “safe”. Given a  $\mathbf{L}$ -pointed finite set of languages  $\mathbb{Q}$ , we say that  $\mathbb{Q}$  is *safe* if and only if all pairs  $(L, H) \in \mathbb{Q}$  satisfy  $H \neq \emptyset$  and  $H \subseteq (\mathbf{alph}(L))^*$ .

**Parameters of  $\mathbf{L}$ -pointed sets.** In the proof, we rely on three parameters that can be associated to the  $\mathbf{L}$ -pointed set  $\mathbb{Q}$  from which we start. Let  $\mathbf{H}$  be the support of  $\mathbb{Q}$  (i.e.  $\mathbf{H} = \{H \mid (L, H) \in \mathbb{Q} \text{ for some } L \in \mathbf{L}\}$ ).

1. The *alphabet* of  $\mathbb{Q}$  is the smallest set  $A_{\mathbb{Q}} \subseteq A$  such that  $\mathbf{alph}(L) \subseteq A_{\mathbb{Q}}$  for all  $L \in \mathbf{L}$  satisfying  $\mathbb{Q}(L) \neq \emptyset$ . Note that if  $\mathbb{Q}$  is safe, we also have  $H \subseteq (A_{\mathbb{Q}})^*$  for all  $H \in \mathbf{H}$ .
2. Furthermore, we denote by  $\mathbf{Dec}(\mathbb{Q})$  the set of all  $L \in \mathbf{L}$  for which  $D_L(\mathbb{Q}) \neq \emptyset$ .
3. Finally, we denote by  $Cov(\mathbb{Q})$  the language,

$$Cov(\mathbb{Q}) = \left( \bigcup_{H \in \mathbf{H}} H \right) \cup \left( \bigcup_{H \in \mathbf{H}} H \right) (A_{\mathbb{Q}})^* \left( \bigcup_{H \in \mathbf{H}} H \right)$$

We finish with a few facts about these parameters that can be verified from the definitions and will prove useful.

► **Fact 54.** *Let  $\mathbb{Q}$  be a safe  $\mathbf{L}$ -pointed set.  $\mathbf{Dec}(\mathbb{Q})$  is the smallest sub-partial semigroup of  $\mathbf{L}$  which contains all  $L \in \mathbf{L}$  such that  $\mathbb{Q}(L) \neq \emptyset$ . Moreover, for all  $L \in \mathbf{Dec}(\mathbb{Q})$ ,  $\mathbf{alph}(L) \subseteq A_{\mathbb{Q}}$ . In particular,  $A_{\mathbb{Q}} = \cup_{L \in \mathbf{Dec}(\mathbb{Q})} \mathbf{alph}(L)$ .*

► **Fact 55.** *Let  $\mathbb{Q}$  be a safe  $\mathbf{L}$ -pointed set, then  $Cov(\mathbb{Q}) \subseteq (A_{\mathbb{Q}})^*$  and  $Cov(\mathbb{Q})Cov(\mathbb{Q}) \subseteq Cov(\mathbb{Q})$ .*

## Main proof

We are now ready to start the proof. We start our induction in the following proposition.

► **Proposition 56.** *Let  $\mathbb{Q}$  be a safe  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages such that  $\mathcal{P}[\mathbf{L}](\mathbb{Q}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ . There exists a safe  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}$  such that:*

1. for all  $(L, K) \in \mathbb{P}$ ,  $L \in \mathbf{Dec}(\mathbb{Q})$  and  $K \subseteq Cov(\mathbb{Q})$ .
2. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}(L)$  is a cover of  $D_L(\mathbb{Q})$ .
3.  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

Before we prove the proposition, we use it to finish the proof of Theorem 49. As we observed, the  $\mathbf{L}$ -pointed set  $\mathbb{T}$  above is a  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages such that  $\mathcal{P}[\mathbf{L}](\mathbb{T}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ . Moreover, one can verify from the definition that  $\mathbb{T}$  is safe. Altogether, we can apply Proposition 56 to obtain a new  $\mathbf{L}$ -pointed finite set of  $\Sigma_2$ -definable languages  $\mathbb{P}$  such that,

1. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}(L)$  is a cover of  $D_L(\mathbb{T})$ .
2.  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

We may now combine these two items with Fact 53 to obtain that  $\mathbb{P}$  is a pointed  $\Sigma_2$ -cover of  $\mathbf{L}$  such that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$  which terminates the proof of Theorem 49.

We now turn to the proof of Proposition 56. Let  $\mathbb{Q}$  be as in the statement of the proposition. The proof is by induction on the three following parameters of  $\mathbb{Q}$ .

1.  $|\mathbf{Dec}(\mathbb{Q})|$ .
2.  $|\langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle|$ .
3.  $|\mathbb{Q}|$

We separate the proof in two cases which depend on a property of  $\mathbb{Q}$  that we define now. We say that  $\mathbb{Q}$  is *saturated* if and only if the two following properties hold for all  $(L, H) \in \mathbb{Q}$ :

$$\left\{ \begin{array}{l} \{L\} \odot \mathbf{Dec}(\mathbb{Q}) = \mathbf{Dec}(\mathbb{Q}) \\ \langle \mathbf{L} | H \rangle \odot \langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle = \langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \mathbf{Dec}(\mathbb{Q}) \odot \{L\} = \mathbf{Dec}(\mathbb{Q}) \\ \langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle \odot \langle \mathbf{L} | H \rangle = \langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle \end{array} \right.$$

The base case happens when  $\mathbb{Q}$  is saturated. In that case, we construct  $\mathbb{Q}$  directly (i.e. we do not use induction). Otherwise, we refine the  $\mathbb{Q}$ -decompositions and conclude by induction.

### Base Case: $\mathbb{Q}$ is saturated.

In this case we construct  $\mathbb{P}$  directly. The construction is based on the following lemma (which is where we use our hypothesis that  $\mathbb{Q}$  is saturated).

► **Lemma 57.** *For all  $L \in \mathbf{Dec}(\mathbb{Q})$ ,  $(L, \langle \mathbf{L} | \mathbf{Cov}(\mathbb{Q}) \rangle) \in \mathbf{Sat}_{\Sigma_2}(\mathbf{L})$ .*

Before we prove the lemma, we first use it to construct  $\mathbb{P}$ . We let  $\mathbb{P} = \{(L, \mathbf{Cov}(\mathbb{Q})) \mid L \in \mathbf{Dec}(\mathbb{Q})\}$ . Note that by definition, the support  $\mathbf{K}$  of  $\mathbb{P}$  is the set  $\mathbf{K} = \{\mathbf{Cov}(\mathbb{Q})\}$ . Let  $\mathbf{H}$  be the support of  $\mathbb{Q}$  and recall that,

$$\mathbf{Cov}(\mathbb{Q}) = \left( \bigcup_{H \in \mathbf{H}} H \right) \cup \left( \bigcup_{H \in \mathbf{H}} H \right) (A_{\mathbb{Q}})^* \left( \bigcup_{H \in \mathbf{H}} H \right)$$

Observe that  $(A_{\mathbb{Q}})^*$  and  $\bigcup_{H \in \mathbf{H}} H$  are definable in  $\Sigma_2$  (by hypothesis in Proposition 56, all  $H \in \mathbf{H}$  are definable in  $\Sigma_2$ ). Therefore,  $\mathbf{Cov}(\mathbb{Q})$  is a concatenation of  $\Sigma_2$ -definable languages and is itself a  $\Sigma_2$ -definable language since  $\Sigma_2$  is closed under concatenation. We now use Lemma 57 to verify that  $\mathbb{Q}$  is safe and satisfies the three conditions in Proposition 56. We begin by proving that  $\mathbb{P}$  is safe. Since by definition  $\mathbf{Cov}(\mathbb{Q}) \subseteq (A_{\mathbb{Q}})^*$  (see Fact 55), this follows from the next fact.

► **Fact 58.** *For all  $L \in \mathbf{Dec}(\mathbb{Q})$ ,  $\mathbf{alph}(L) = A_{\mathbb{Q}}$ .*

**Proof.** We prove that all  $L \in \mathbf{Dec}(\mathbb{Q})$  have the same alphabet (which must then be  $A_{\mathbb{Q}}$  by Fact 54). Let  $L, L' \in \mathbf{Dec}(\mathbb{Q})$ . Since  $\mathbb{Q}$  is saturated, we have  $(L_1, H_1), (L_2, H_2) \in \mathbb{Q}$  such that  $L \odot L_1 = L'$  and  $L' \odot L_2 = L$ . We obtain that  $\mathbf{alph}(L) \subseteq \mathbf{alph}(L) \cup \mathbf{alph}(L_1) = \mathbf{alph}(L')$  and  $\mathbf{alph}(L') \subseteq \mathbf{alph}(L') \cup \mathbf{alph}(L_2) = \mathbf{alph}(L)$ . We conclude that  $\mathbf{alph}(L) = \mathbf{alph}(L')$ . ◀

The first condition in Proposition 56 is immediate from the definition of  $\mathbb{P}$ . We prove the second and third conditions. We start by proving that for all  $L \in \mathbf{L}$ ,  $\mathbb{P}(L)$  is a cover of  $\mathbf{D}_L(\mathbb{Q})$ . If  $\mathbf{D}_L(\mathbb{Q}) = \emptyset$ , this is immediate. Otherwise let  $w \in \mathbf{D}_L(\mathbb{Q})$ . By definition, we have to prove that  $w \in \mathbf{Cov}(\mathbb{Q})$ . By definition of  $\mathbf{D}_L(\mathbb{Q})$ ,  $w$  can be decomposed as  $w = w_1 \cdots w_n$  and for all  $i$  there exists  $(L_i, H_i) \in \mathbb{Q}$  such that  $w_i \in H_i$ . Hence, in particular, for all  $i$ ,  $w_i \in \bigcup_{H \in \mathbf{H}} H$ . If  $n = 1$ , then  $w \in \bigcup_{H \in \mathbf{H}} H \subseteq \mathbf{Cov}(\mathbb{Q})$ . Otherwise, one can verify from the fact that  $\mathbb{Q}$  is safe that for all  $i$ ,  $H_i \subseteq (A_{\mathbb{Q}})^*$ . Therefore, we obtain,

$$w \in \left( \bigcup_{H \in \mathbf{H}} H \right) (A_{\mathbb{Q}})^* \left( \bigcup_{H \in \mathbf{H}} H \right) \subseteq \mathbf{Cov}(\mathbb{Q})$$

We conclude that  $\mathbb{P}(L)$  is a cover of  $D_L(\mathbb{Q})$ .

Finally, we prove that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ . Consider  $(L, \mathbf{L}') \in \mathcal{P}[\mathbf{L}](\mathbb{P})$ . By definition of pointed imprints, we get  $K \in \mathbf{K}$  such that  $(L, K) \in \mathbb{P}$  and  $\mathbf{L}' \subseteq \langle \mathbf{L} | K \rangle$ . By closure under downset in the definition of  $Sat_{\Sigma_2}$ , it suffices to prove that  $(L, \langle \mathbf{L} | K \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$ . This is immediate since by definition of  $\mathbf{K}$ ,  $K = Cov(\mathbb{Q})$  and  $(L, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$  by Lemma 57.

It remains to prove Lemma 57. This is where we use the fact that  $\mathbb{Q}$  is saturated. The proof is based on the following fact which treats a special case.

► **Fact 59.** *Let  $(S, H) \in \mathbb{Q}$  such that  $S \odot S$  is defined. Then,*

$$(S^\omega, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$$

**Proof.** By definition of pointed imprints, we know that  $(S, \langle \mathbf{L} | H \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{Q})$  and therefore  $(S, \langle \mathbf{L} | H \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$  since  $\mathcal{P}[\mathbf{L}](\mathbb{Q}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$  by hypothesis in Proposition 56. Also note that by Fact 54,  $S \in \mathbf{Dec}(\mathbb{Q})$  and therefore we know from Fact 58 that  $\mathbf{alph}(S) = A_{\mathbb{Q}}$ . Hence, we may use Operation 3 in the definition of  $Sat_{\Sigma_2}$  to conclude that,

$$(S^\omega, \langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | (A_{\mathbb{Q}})^* \rangle \odot \langle \mathbf{L} | H \rangle^\omega) \in Sat_{\Sigma_2}(\mathbf{L})$$

By closure under downset, it now suffices to prove that  $\langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \subseteq \langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | (A_{\mathbb{Q}})^* \rangle \odot \langle \mathbf{L} | H \rangle^\omega$ . We know from Fact 55 that  $\langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \subseteq \langle \mathbf{L} | (A_{\mathbb{Q}})^* \rangle$ . Therefore, we obtain,

$$\langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \odot \langle \mathbf{L} | H \rangle^\omega \subseteq \langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | (A_{\mathbb{Q}})^* \rangle \odot \langle \mathbf{L} | H \rangle^\omega$$

Moreover, using the fact that  $\mathbb{Q}$  is saturated (this is our hypothesis), we have  $\langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \odot \langle \mathbf{L} | H \rangle^\omega = \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle$ . We conclude that,  $\langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \subseteq \langle \mathbf{L} | H \rangle^\omega \odot \langle \mathbf{L} | (A_{\mathbb{Q}})^* \rangle \odot \langle \mathbf{L} | H \rangle^\omega$  which terminates the proof. ◀

We may now use Fact 59 to finish the proof of Lemma 57. Let  $L \in \mathbf{Dec}(\mathbb{Q})$ , we have to prove that  $(L, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$ . Let  $(S, H) \in \mathbb{Q}$ . Since  $\mathbb{Q}$  is saturated, we know that,

$$\mathbf{Dec}(\mathbb{Q}) = \{S\} \odot \mathbf{Dec}(\mathbb{Q}) = \{S\}^\omega \odot \mathbf{Dec}(\mathbb{Q}) \quad (4)$$

In particular, note that since  $\mathbf{Dec}(\mathbb{Q}) \neq \emptyset$  (it contains  $L$ ), this means that  $\{S\}^\omega \neq \emptyset$ , i.e.  $S \odot S$  is defined. In particular, we know from Fact 59 that  $(S^\omega, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$ . Moreover, we obtain from (4) above that there exists  $L' \in \mathbf{Dec}(\mathbb{Q})$  such that  $L = S^\omega \odot L'$ . By definition, of  $\mathbf{Dec}(\mathbb{Q})$  there exists  $(L_1, H_1), \dots, (L_n, H_n) \in \mathbb{Q}$  such that  $L_1 \odot \dots \odot L_n = L'$ . It follows that  $L = S^\omega \odot L_1 \odot \dots \odot L_n$ . Furthermore, by definition of pointed imprints, we have,  $(L_i, \langle \mathbf{L} | H_i \rangle) \in \mathcal{P}[\mathbf{L}](\mathbb{Q})$  for all  $i$ , and since  $\mathcal{P}[\mathbf{L}](\mathbb{Q}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$  by hypothesis in Proposition 56, we obtain,

$$(L_i, \langle \mathbf{L} | H_i \rangle) \in Sat_{\Sigma_2}(\mathbf{L}) \quad \text{for all } i$$

Using closure under multiplication in the definition of  $Sat_{\Sigma_2}$  we obtain that,

$$(S^\omega \odot L_1 \odot \dots \odot L_n, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle \odot \langle \mathbf{L} | H_1 \rangle \odot \dots \odot \langle \mathbf{L} | H_n \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$$

By definition, the first component of the pair above is equal to  $L$  and by saturation of  $\mathbb{Q}$ , the second component is equal to  $\langle \mathbf{L} | Cov(\mathbb{Q}) \rangle$ . We conclude that  $(L, \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$  which terminates the proof of the base case.

**Induction Case:  $\mathbb{Q}$  is not saturated.**

By hypothesis there exists  $(L, H) \in \mathbb{Q}$  such that one of the four following properties hold:

- $\{L\} \odot \mathbf{Dec}(\mathbb{Q}) \neq \mathbf{Dec}(\mathbb{Q})$  or,
- $\langle \mathbf{L}|H \rangle \odot \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle = \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle$  or,
- $\mathbf{Dec}(\mathbb{Q}) \odot \{L\} \neq \mathbf{Dec}(\mathbb{Q})$  or,
- $\langle \mathbf{L}|Cov(\mathbb{Q}) \rangle \odot \langle \mathbf{L}|H \rangle = \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle$ .

In the remainder of the proof we denote by  $\mathfrak{b} = (L_{\mathfrak{b}}, H_{\mathfrak{b}})$  this pair and by symmetry we suppose that one of the *two* following properties holds:

$$\{L_{\mathfrak{b}}\} \odot \mathbf{Dec}(\mathbb{Q}) \neq \mathbf{Dec}(\mathbb{Q}) \quad (5)$$

$$\langle \mathbf{L}|H_{\mathfrak{b}} \rangle \odot \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle \neq \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle \quad (6)$$

Note that this means that we have two cases to treat depending on the property that holds. However, both cases are similar. In particular, the construction of  $\mathbb{P}$  is the same in both cases. The only difference is the induction parameter that we use to conclude at the end of the proof (it depends on the property that holds).

We begin by constructing two new  $\mathbf{L}$ -pointed sets  $\mathbb{Q}_{pr}$  and  $\mathbb{Q}_{su}$ . We define,  $\mathbb{Q}_{pr} = \mathbb{Q} \setminus \{\mathfrak{b}\}$  and  $\mathbb{Q}_{su} = \{\mathfrak{b}\}$ . Observe that by definition,  $\mathbb{Q}_{pr}, \mathbb{Q}_{su} \subseteq \mathbb{Q}$  and therefore,  $A_{\mathbb{Q}_{pr}}, A_{\mathbb{Q}_{su}} \subseteq A_{\mathbb{Q}}$ ,  $\mathbf{Dec}(\mathbb{Q}_{pr}), \mathbf{Dec}(\mathbb{Q}_{su}) \subseteq \mathbf{Dec}(\mathbb{Q})$  and  $\langle \mathbf{L}|Cov(\mathbb{Q}_{pr}) \rangle, \langle \mathbf{L}|Cov(\mathbb{Q}_{su}) \rangle \subseteq \langle \mathbf{L}|Cov(\mathbb{Q}) \rangle$ . For each  $L \in \mathbf{L}$ , consider the three following languages,

1.  $D_L(\mathbb{Q}_{pr})$ , the language of all words which admit a  $\mathbb{Q}_{pr}$ -decomposition for  $L$ .
2.  $D_L(\mathbb{Q}_{su})$ , the language of all words which admit a  $\mathbb{Q}_{su}$ -decomposition for  $L$ .
3.  $\text{INF}_L$  is the language of all words  $w \in A^*$  that admit a decomposition of the form  $u_1 v_1 \cdots u_n v_n$  such that for all  $i$  there exist  $U_i, V_i \in \mathbf{L}$  satisfying,
  - for all  $i$ ,  $u_i \in D_{U_i}(\mathbb{Q}_{su})$ .
  - for all  $i$ ,  $v_i \in D_{V_i}(\mathbb{Q}_{pr})$ .
  - $L = U_1 \odot V_1 \odot \cdots \odot U_n \odot V_n$ .

Recall that given  $L \in \mathbf{L}$ ,  $w \in D_L(\mathbb{Q})$  if  $w$  can be decomposed as  $w = w_1 \dots w_n$  such for all  $i$  there exists  $(L_i, H_i) \in \mathbb{Q}$  such that  $w_i \in H_i$  and  $L = L_1 \odot \cdots \odot L_n$ . Observe that we may sort the factors  $w_i$  into two kinds: those such that  $(L_i, H_i) = \mathfrak{b}$  and those such that  $(L_i, H_i) \neq \mathfrak{b}$ . We may then regroup the factors into maximal blocks such that all factors in a block have the same kind. This means that each block of factors admits either a  $\mathbb{Q}_{pr}$ -decomposition or a  $\mathbb{Q}_{su}$ -decomposition. Altogether, we have the following fact.

► **Fact 60.** *Let  $L \in \mathbf{L}$ , then any word  $w \in D_L(\mathbb{Q})$  can be decomposed in one of the following ways:*

- $w = w_1 w_2 w_3$  such that there exist  $L_1, L_2, L_3 \in \mathbf{L}$  satisfying  $w_1 \in D_{L_1}(\mathbb{Q}_{pr})$ ,  $w_2 \in \text{INF}_{L_2}$ ,  $w_3 \in D_{L_3}(\mathbb{Q}_{su})$  and  $L = L_1 \odot L_2 \odot L_3$ .
- $w = w_1 w_2$  such that there exist  $L_1, L_2 \in \mathbf{L}$  satisfying  $w_1 \in D_{L_1}(\mathbb{Q}_{pr})$ ,  $w_2 \in \text{INF}_{L_2}$ , and  $L = L_1 \odot L_2$ .
- $w = w_2 w_3$  such that there exist  $L_2, L_3 \in \mathbf{L}$  satisfying  $w_2 \in \text{INF}_{L_2}$ ,  $w_3 \in D_{L_3}(\mathbb{Q}_{su})$  and  $L = L_2 \odot L_3$ .
- $w = w_1 w_3$  such that there exist  $L_1, L_3 \in \mathbf{L}$  satisfying  $w_1 \in D_{L_1}(\mathbb{Q}_{pr})$ ,  $w_3 \in D_{L_3}(\mathbb{Q}_{su})$  and  $L = L_1 \odot L_3$ .
- $w \in D_L(\mathbb{Q}_{pr})$ ,  $w \in \text{INF}_L$  or  $w \in D_L(\mathbb{Q}_{su})$ .

The construction of  $\mathbb{P}$  is based on Fact 60 and the three following lemmas.

► **Lemma 61** (Pointed Covering of the Prefixes). *There exists a safe  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}_{pr}$  such that:*

1. for all  $(L, K) \in \mathbb{P}_{pr}$ ,  $L \in \mathbf{Dec}(\mathbb{Q}_{pr})$  and  $K \subseteq Cov(\mathbb{Q}_{pr})$ .
2. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}_{pr}(L)$  is a cover of  $D_L(\mathbb{Q}_{pr})$ .
3.  $\mathcal{P}[\mathbf{L}](\mathbb{P}_{pr}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

► **Lemma 62** (Pointed Covering of the Suffixes). *There exists a safe  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}_{su}$  such that:*

1. for all  $(L, K) \in \mathbb{P}_{su}$ ,  $L \in \mathbf{Dec}(\mathbb{Q}_{su})$  and  $K \subseteq H_b \cup H_b Cov(\mathbb{Q}_{su})$ .
2. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}_{su}(L)$  is a cover of  $D_L(\mathbb{Q}_{su})$ .
3.  $\mathcal{P}[\mathbf{L}](\mathbb{P}_{su}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

► **Lemma 63** (Pointed Covering of the Infixes). *There exists a safe  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}_{in}$  such that:*

1. for all  $(L, K) \in \mathbb{P}_{in}$ ,  $L \in \mathbf{Dec}(\mathbb{Q})$  and  $K \subseteq Cov(\mathbb{Q})$ .
2. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}_{in}(L)$  is a cover of  $INF_L$ .
3.  $\mathcal{P}[\mathbf{L}](\mathbb{P}_{in}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

Before, we prove the three lemmas, we use them to conclude the proof of Proposition 56 and construct  $\mathbb{P}$ . We define,

$$\begin{aligned} \mathbb{P} = & \{(L_1 \odot L_2 \odot L_3, K_1 K_2 K_3) \mid (L_1, K_1) \in \mathbb{P}_{pr}, (L_2, K_2) \in \mathbb{P}_{in} \text{ and } (L_3, K_3) \in \mathbb{P}_{su}\} \\ & \cup \{(L_2 \odot L_3, K_2 K_3) \mid (L_2, K_2) \in \mathbb{P}_{in} \text{ and } (L_3, K_3) \in \mathbb{P}_{su}\} \\ & \cup \{(L_1 \odot L_3, K_1 K_3) \mid (L_1, K_1) \in \mathbb{P}_{pr} \text{ and } (L_3, K_3) \in \mathbb{P}_{su}\} \\ & \cup \{(L_1 \odot L_2, K_1 K_2) \mid (L_1, K_1) \in \mathbb{P}_{pr} \text{ and } (L_2, K_2) \in \mathbb{P}_{in}\} \\ & \cup \mathbb{P}_{pr} \cup \mathbb{P}_{in} \cup \mathbb{P}_{su} \end{aligned}$$

Observe that all languages in the support  $\mathbf{K}$  of  $\mathbb{P}$  are  $\Sigma_2$ -definable: by definition they are concatenations of  $\Sigma_2$ -definable languages and  $\Sigma_2$  is closed under concatenation. Moreover, the fact that  $\mathbb{P}$  is safe can be verified from the fact that  $\mathbb{P}_{pr}$ ,  $\mathbb{P}_{in}$  and  $\mathbb{P}_{su}$  were safe. We now explain why  $\mathbb{P}$  satisfies the three conditions in Proposition 56.

We begin with the first condition. Let  $(L, K) \in \mathbb{P}$ , we have to prove that  $L \in \mathbf{Dec}(\mathbb{Q})$  and  $K \subseteq Cov(\mathbb{Q})$ . We treat the case when  $(L, K) = (L_1 \odot L_2 \odot L_3, K_1 K_2 K_3)$  with  $(L_1, K_1) \in \mathbb{P}_{pr}$ ,  $(L_2, K_2) \in \mathbb{P}_{in}$  and  $(L_3, K_3) \in \mathbb{P}_{su}$  (other cases are similar). By hypothesis in the lemmas, we have  $L_1, L_2, L_3 \in \mathbf{Dec}(\mathbb{Q})$ . Hence  $L = L_1 \odot L_2 \odot L_3 \in \mathbf{Dec}(\mathbb{Q})$  (see Fact 54). Moreover, we know that  $K_1, K_2, K_3 \in Cov(\mathbb{Q})$ . Hence  $K_1 K_2 K_3 \subseteq (Cov(\mathbb{Q}))^3 \subseteq Cov(\mathbb{Q})$  (see Fact 55) which terminates the proof.

The second condition (that  $\mathbb{P}(L)$  is a cover of  $D_L(\mathbb{Q})$  for all  $L \in \mathbf{L}$ ) can be verified from Fact 60 and Item 2 in Lemmas 61, 62 and 63. Let us now prove that  $\mathcal{P}[\mathbf{L}](\mathbb{P}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ . By closure under downset, we have to prove that for all  $(L, K) \in \mathbb{P}$ ,  $(L, \langle \mathbf{L} | K \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$ . We treat the case when  $(L, K) = (L_1 \odot L_2 \odot L_3, K_1 K_2 K_3)$  with  $(L_1, K_1) \in \mathbb{P}_{pr}$ ,  $(L_2, K_2) \in \mathbb{P}_{in}$  and  $(L_3, K_3) \in \mathbb{P}_{su}$  (other cases are similar). We know from Lemma 14 that,  $\langle \mathbf{L} | K \rangle = \langle \mathbf{L} | K_1 \rangle \odot \langle \mathbf{L} | K_2 \rangle \odot \langle \mathbf{L} | K_3 \rangle$ . moreover, by the third item in the three lemmas, we have,

$$\begin{aligned} (L_1, \langle \mathbf{L} | K_1 \rangle) & \in \mathcal{P}[\mathbf{L}](\mathbb{P}_{pr}) \subseteq Sat_{\Sigma_2}(\mathbf{L}) \\ (L_2, \langle \mathbf{L} | K_2 \rangle) & \in \mathcal{P}[\mathbf{L}](\mathbb{P}_{in}) \subseteq Sat_{\Sigma_2}(\mathbf{L}) \\ (L_3, \langle \mathbf{L} | K_3 \rangle) & \in \mathcal{P}[\mathbf{L}](\mathbb{P}_{su}) \subseteq Sat_{\Sigma_2}(\mathbf{L}) \end{aligned}$$

Since  $L = L_1 \odot L_2 \odot L_3$  we conclude using closure under downset and multiplication that  $\langle \mathbf{L}|K \rangle \in \text{Sat}_{\Sigma_2}(\mathbf{L})$  which terminates the proof.

It now remains to prove the three lemmas. By definition of  $\mathbb{Q}_{pr}$ , Lemma 61 is immediate using induction on the third parameter in Proposition 56 (the size of  $\mathbb{Q}$ ). It remains to prove Lemma 62 and Lemma 63. We devote a subsection to each proof.

## Proof of Lemma 62

First observe that we cannot simply use induction on the size of  $\mathbb{Q}$ . There are two reasons for this. First, it may happen that  $\mathbb{Q} = \mathbb{P}_{su}$  (i.e.  $\mathfrak{b}$  was already the only element in  $\mathbb{Q}$ ). In that case, the size of  $\mathbb{Q}$  has not decreased. Second, the first condition in Lemma 62 is stronger than the corresponding one in Proposition 56.

Recall that  $\mathbb{Q}_{su} = \{(L_{\mathfrak{b}}, H_{\mathfrak{b}})\}$ . There are two cases depending on whether  $L_{\mathfrak{b}} \odot L_{\mathfrak{b}}$  is defined. If  $L_{\mathfrak{b}} \odot L_{\mathfrak{b}}$  is not defined then  $D_{L_{\mathfrak{b}}}(\mathbb{Q}_{su}) = H_{\mathfrak{b}}$  and  $D_L(\mathbb{Q}_{su}) = \emptyset$  for  $L \neq L_{\mathfrak{b}}$ . In that case, we simply define  $\mathbb{P}_{su} = \mathbb{Q}_{su}$ . One can verify that this choice satisfies the conditions of the lemma.

Otherwise,  $L_{\mathfrak{b}} \odot L_{\mathfrak{b}}$  is defined and therefore  $(L_{\mathfrak{b}})^n$  is defined for all  $n \geq 1$ . It follows from a standard semigroup theory argument that there exists an integer  $m \leq |2^{\mathbf{L}}|$  such that  $(L_{\mathfrak{b}})^m = (L_{\mathfrak{b}})^\omega$  and  $\langle \mathbf{L}|H_{\mathfrak{b}} \rangle^m = \langle \mathbf{L}|H_{\mathfrak{b}} \rangle^\omega$ . Moreover, we may assume without loss of generality that  $m \geq 2$  (we will need this property to ensure that Item 1 in the lemma holds). We define,

$$\mathbb{P}_{su} = \{(L_{\mathfrak{b}}^k, H_{\mathfrak{b}}^k) \mid k \leq 2m - 1\} \cup \{(L_{\mathfrak{b}}^k, H_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m) \mid k \geq 2m\}$$

Note that in the expression above,  $L_{\mathfrak{b}}^k$  denotes the tame multiplication of  $k$  copies of  $L_{\mathfrak{b}}$  whereas  $H_{\mathfrak{b}}^k$  denotes the concatenation of  $k$  copies of  $H_{\mathfrak{b}}$ .

It remains to prove that  $\mathbb{P}_{su}$  satisfies the conditions in the lemma.  $\mathbb{P}_{su}$  is safe by definition since  $\mathbb{Q}_{su}$  was safe. Moreover, all languages in the support of  $\mathbb{P}_{su}$  are definable in  $\Sigma_2$  as they are concatenations of  $\Sigma_2$ -definable languages. We now verify the three conditions in the lemma.

We begin with the first condition. Let  $(L, K) \in \mathbb{P}_{in}$ , we have to prove that  $L \in \mathbf{Dec}(\mathbb{Q})$  and  $K \subseteq H_{\mathfrak{b}} \cup H_{\mathfrak{b}} \text{Cov}(\mathbb{Q}_{su})$ . We treat the case when  $(L, K) = (L_{\mathfrak{b}}^k, H_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m)$  for  $k \geq m$ . The other case is similar. That  $L_{\mathfrak{b}}^k \in \mathbf{Dec}(\mathbb{Q}_{su})$  follows from Fact 54. Moreover, observe that by definition,  $\mathbf{alph}(L_{\mathfrak{b}}) = A_{\mathbb{Q}_{su}}$  and by Fact 55,  $H_{\mathfrak{b}} \subseteq (\mathbf{alph}(L_{\mathfrak{b}}))^*$ . Therefore, since we chose  $m \geq 2$ , we have,

$$H_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m \subseteq H_{\mathfrak{b}}(H_{\mathfrak{b}}^{m-1}(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m) \subseteq H_{\mathfrak{b}} \text{Cov}(\mathbb{Q}_{su})$$

We now turn to the second condition. Let  $L \in \mathbf{L}$ , we have to prove that  $\mathbb{P}_{su}(L)$  is a cover of  $D_L(\mathbb{Q}_{su})$ . Let  $w \in D_L(\mathbb{Q}_{su})$ , we have to find  $K \in \mathbb{P}_{su}(L)$  such that  $w \in K$ . By definition, since  $(L_{\mathfrak{b}}, H_{\mathfrak{b}})$  is the only element in  $\mathbb{Q}_{\mathfrak{b}}$ , there exists  $k \geq 1$  such that  $L = (L_{\mathfrak{b}})^k$  and  $w \in H_{\mathfrak{b}}^k$ . If  $k \leq 2m - 1$ , then  $H_{\mathfrak{b}}^k \in \mathbb{P}_{su}(L)$  and we are finished. Otherwise,  $k \geq 2m$  and  $H_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m \in \mathbb{P}_{su}(L)$ . Since  $H_{\mathfrak{b}} \subseteq (\mathbf{alph}(L_{\mathfrak{b}}))^*$  ( $\mathbb{Q}_{su}$  is safe), we obtain that  $w \in H_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* H_{\mathfrak{b}}^m$ .

We finish with the third condition. This amounts to proving that for all  $(L, K) \in \mathbb{P}_{su}$ ,  $(L, \langle \mathbf{L}|K \rangle) \in \text{Sat}_{\Sigma_2}(\mathbf{L})$ . If  $(L, K) = (L_{\mathfrak{b}}^k, K_{\mathfrak{b}}^k)$ , then by Lemma 14, we obtain,

$$(L, \langle \mathbf{L}|K \rangle) = (L_{\mathfrak{b}}^k, \langle \mathbf{L}|K_{\mathfrak{b}} \rangle^k)$$

By hypothesis in Proposition 56  $(L_{\mathfrak{b}}, \langle \mathbf{L}|K_{\mathfrak{b}} \rangle) \in \text{Sat}_{\Sigma_2}(\mathbf{L})$ . Hence, we obtain by closure under multiplication that  $(L, \langle \mathbf{L}|K \rangle) \in \text{Sat}_{\Sigma_2}(\mathbf{L})$ . Otherwise,  $(L, K) = (L_{\mathfrak{b}}^k, K_{\mathfrak{b}}^m(\mathbf{alph}(L_{\mathfrak{b}}))^* K_{\mathfrak{b}}^m)$

for some  $k \geq m$ . Let  $k' \geq 0$  such that  $k = m + k'$ . By choice of  $m$ , we have  $L = L_{\mathfrak{b}}^{m+k'} = L_{\mathfrak{b}}^{\omega} \odot L_{\mathfrak{b}}^{k'}$ . Furthermore, using Lemma 14 again, we obtain,

$$\begin{aligned} \langle \mathbf{L} | K \rangle &= \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \\ \langle \mathbf{L} | K \rangle &= \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{k'\omega} \\ \langle \mathbf{L} | K \rangle &= \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{k'(\omega-1)} \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{k'} \\ \langle \mathbf{L} | K \rangle &\subseteq \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{k'} \text{ (since } H_{\mathfrak{b}} \subseteq (\mathbf{alph}(L_{\mathfrak{b}}))^* \text{)} \end{aligned}$$

Finally, using Operation (3) in the definition of  $Sat_{\Sigma_2}$ , we know that,

$$(L_{\mathfrak{b}}^{\omega}, \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega}) \in Sat_{\Sigma_2}(\mathbf{L})$$

Therefore using closure under multiplication, we obtain that,

$$(L_{\mathfrak{b}}^{\omega} \odot L_{\mathfrak{b}}^{k'}, \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | (\mathbf{alph}(L_{\mathfrak{b}}))^* \rangle \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{\omega} \odot \langle \mathbf{L} | H_{\mathfrak{b}} \rangle^{k'}) \in Sat_{\Sigma_2}(\mathbf{L})$$

By closure under downset we conclude that  $(L, \langle \mathbf{L} | K \rangle) \in Sat_{\Sigma_2}(\mathbf{L})$  which terminates the proof of Lemma 62.

### Proof of Lemma 63

The proof is by induction on either the first or the second parameter in Proposition 56 (that is the size of  $\mathbf{Dec}(\mathbb{Q})$  or the size of  $\langle \mathbf{L} | Cov(\mathbb{Q}) \rangle$ ). The induction parameter that we use depends on whether (5) holds or (6) holds for  $\mathfrak{b} = (L_{\mathfrak{b}}, H_{\mathfrak{b}})$ .

We use the  $\mathbf{L}$ -pointed sets  $\mathbb{P}_{su}$  and  $\mathbb{P}_{pr}$  (obtained from Lemmas 62 and 61) to define a new  $\mathbf{L}$ -pointed finite set of  $\Sigma_2$ -definable languages  $\mathbb{S}$  to which we apply induction. We define,

$$\mathbb{S} = \{(L_1 \odot L_2, K_1 K_2) \mid (L_1, K_1) \in \mathbb{P}_{su}, (L_2, K_2) \in \mathbb{P}_{pr} \text{ and } L_1 \odot L_2 \text{ is defined}\}$$

We now state two simple facts about  $\mathbb{S}$ . We use these two facts to connect  $\mathbb{S}$  to the  $\mathbf{L}$ -pointed set  $\mathbb{P}_{in}$  that we have to construct.

► **Fact 64.**  $Dec(\mathbb{S}) \subseteq Dec(\mathbb{Q})$  and  $Cov(\mathbb{S}) \subseteq Cov(\mathbb{Q})$

**Proof.** Immediate using the definition, Fact 54 and Fact 55. ◀

► **Fact 65.** For all  $L \in \mathbf{L}$ ,  $INF_L \subseteq D_L(\mathbb{S})$ .

**Proof.** Let  $L \in \mathbf{L}$  and  $w \in INF_L$ . By definition,  $w = u_1 v_1 \cdots u_n v_n$  such that for all  $i$  there exist  $U_i, V_i \in \mathbf{L}$  satisfying,  $u_i \in D_{U_i}(\mathbb{Q}_{su})$ ,  $v_i \in D_{V_i}(\mathbb{Q}_{pr})$  and  $L = U_1 \odot V_1 \odot \cdots \odot U_n \odot V_n$ . For all  $i$ , let  $w_i = u_i v_i$ , we prove that  $w = w_1 \cdots w_n$  is a  $\mathbb{S}$ -decomposition of  $w$  for  $L$  which exactly says that  $w \in D_L(\mathbb{S})$ . By definition of  $\mathbb{P}_{pr}$  and  $\mathbb{P}_{su}$  in Lemmas 61 and 62 (see Item 2), for all  $i$ , there exists  $K_i, K'_i$  such that  $(U_i, K_i) \in \mathbb{P}_{su}$ ,  $(V_i, K'_i) \in \mathbb{P}_{pr}$ ,  $u_i \in K_i$  and  $v_i \in K'_i$ . It follows that  $w_i = u_i v_i \in K_i K'_i$ . Moreover, by definition of  $\mathbb{S}$ ,  $(U_i \odot V_i, K_i K'_i) \in \mathbb{S}$ . We conclude that  $w \in D_L(\mathbb{S})$ . ◀

An immediate consequence of Facts 64 and 65 is that it suffices to construct our  $\mathbf{L}$ -pointed set of  $\Sigma_2$ -definable languages  $\mathbb{P}_{in}$  in Lemma 63 so that:

1. for all  $(L, K) \in \mathbb{P}_{in}$ ,  $L \in \mathbf{Dec}(\mathbb{S})$  and  $K \subseteq Cov(\mathbb{S})$ .
2. for all  $L \in \mathbf{L}$ ,  $\mathbb{P}_{in}(L)$  is a cover of  $D_L(\mathbb{S})$ .
3.  $\mathcal{P}[\mathbf{L}](\mathbb{P}_{in}) \subseteq Sat_{\Sigma_2}(\mathbf{L})$ .

This is what we do now. As expected we are going to apply induction in Proposition 56 to  $\mathbb{S}$ . To that end we first need to verify that  $\mathbb{S}$  satisfies the conditions required to be an input in Proposition 56. That  $\mathbb{S}$  is safe can be verified from the definition (this is because this was the case for  $\mathbb{P}_{pr}$  and  $\mathbb{P}_{su}$ ). Moreover, the support of  $\mathbb{S}$  contains only  $\Sigma_2$ -definable languages (they are all the concatenation of two  $\Sigma_2$ -definable languages). Finally, we prove that  $\mathcal{P}[\mathbf{L}](\mathbb{S}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$  in the following fact.

► **Fact 66.**  $\mathcal{P}[\mathbf{L}](\mathbb{S}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L})$

**Proof.** Let  $(L, L') \in \mathcal{P}[\mathbf{L}](\mathbb{S})$ . By definition,  $L' \subseteq \langle \mathbf{L} | G \rangle$  for some  $G \in \mathbb{S}(L)$ . By closure under downset, it suffices to prove that  $(L, \langle \mathbf{L} | G \rangle) \in \text{Sat}_{\Sigma_2}(\mathbf{L})$ . By definition of  $\mathbb{S}$ , we have  $(L_1, K_1) \in \mathbb{P}_{su}$  and  $(L_2, K_2) \in \mathbb{P}_{pr}$  such  $L = L_1 \odot L_2$  and  $G = K_1 K_2$ . In particular, using the second item in Lemmas 61 and 62, we obtain that,

$$\begin{aligned} (L_1, \langle \mathbf{L} | K_1 \rangle) &\in \mathcal{P}[\mathbf{L}](\mathbb{P}_{su}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L}) \\ (L_2, \langle \mathbf{L} | K_2 \rangle) &\in \mathcal{P}[\mathbf{L}](\mathbb{P}_{pr}) \subseteq \text{Sat}_{\Sigma_2}(\mathbf{L}) \end{aligned}$$

Using closure under multiplication in  $\text{Sat}_{\Sigma_2}$  and Lemma 14, we conclude that,  $(L, \langle \mathbf{L} | G \rangle) \in \text{Sat}_{\Sigma_2}(\mathbf{L})$ . ◀

It now remains to prove that  $\mathbb{S}$  has strictly smaller induction parameters than  $\mathbb{Q}$  and to use this fact to apply induction in Proposition 56 to construct the desired  $\mathbf{L}$ -pointed set  $\mathbb{P}_{in}$ . We use the two following lemmas.

► **Lemma 67.**  $\text{Dec}(\mathbb{S}) \subseteq \{L_b\} \odot \text{Dec}(\mathbb{Q})$ .

► **Lemma 68.**  $\langle \mathbf{L} | \text{Cov}(\mathbb{S}) \rangle \subseteq \langle \mathbf{L} | H_b \rangle \odot \langle \mathbf{L} | \text{Cov}(\mathbb{Q}) \rangle$ .

Before proving Lemma 67 and Lemma 68, we use them to conclude the proof of Lemma 63. Recall that we know from our choice of  $b$  that either (5) holds (i.e.  $\{L_b\} \odot \text{Dec}(\mathbb{Q}) \subsetneq \text{Dec}(\mathbb{Q})$ ) or (6) holds (i.e.  $\langle \mathbf{L} | H_b \rangle \odot \langle \mathbf{L} | \text{Cov}(\mathbb{Q}) \rangle \subsetneq \langle \mathbf{L} | \text{Cov}(\mathbb{Q}) \rangle$ ). If (5) holds, it follows from Lemma 67 that we may apply induction on the first parameter (the size of  $\text{Dec}(\mathbb{Q})$ ) in Proposition 56. Otherwise, if (6) holds, it follows from Lemma 68 that we may apply induction on the second parameter (the size of  $\langle \mathbf{L} | \text{Cov}(\mathbb{Q}) \rangle$ ) in Proposition 56. In both cases we get a  $\mathbf{L}$ -pointed set  $\mathbb{P}_{in}$  that satisfies the conditions of Lemma 63 as desired.

We finish with the proofs of Lemma 67 and Lemma 68. We begin with Lemma 67. Let  $L \in \text{Dec}(\mathbb{S})$ . By definition, there exists  $(L_1, G_1), \dots, (L_n, G_n) \in \mathbb{S}$  such that  $L = L_1 \odot \dots \odot L_n$ . We prove that for all  $i$ ,  $L_i \in \{L_b\} \odot \text{Dec}(\mathbb{Q})$  which suffices to conclude that  $L \in \{L_b\} \odot \text{Dec}(\mathbb{Q})$ . Let  $i \leq n$ . By definition of  $\mathbb{S}$ , there exists  $(L'_i, K'_i) \in \mathbb{P}_{su}$  and  $(L''_i, K''_i) \in \mathbb{P}_{pr}$  such that  $L_i = L'_i \odot L''_i$ . Moreover, we know from the first item in Lemmas 61 and 62 that  $L'_i \in \text{Dec}(\mathbb{Q}_{su}) \subseteq \text{Dec}(\mathbb{Q})$  and  $L''_i \in \text{Dec}(\mathbb{Q}_{pr}) \subseteq \text{Dec}(\mathbb{Q})$ . By definition of  $\mathbb{Q}_{su}$ , since  $L'_i \in \text{Dec}(\mathbb{Q}_{su})$ , we know that  $L'_i = (L_b)^m$  for some  $m \geq 1$  and we conclude that  $L_i \in \{L_b\} \odot \text{Dec}(\mathbb{Q})$  which terminates the proof of Lemma 67.

We now prove Lemma 68. Let  $\mathbf{G}$  be the support of  $\mathbb{S}$ . Recall that by definition, we have,

$$\text{Cov}(\mathbb{S}) = \left( \bigcup_{G \in \mathbf{G}} G \right) \cup \left( \bigcup_{G \in \mathbf{G}} G(A_{\mathbb{S}})^* \left( \bigcup_{G \in \mathbf{G}} G \right) \right)$$

Moreover, we know that for all  $G \in \mathbf{G}$ , there exists  $(L, K) \in \mathbb{P}_{su}$  and  $(L', K') \in \mathbb{P}_{pr}$  such that  $G = KK'$ . We may combine this fact with the first item in Lemmas 61 and 62 to conclude that,

$$\bigcup_{G \in \mathbf{G}} G \subseteq H_b \text{Cov}(\mathbb{Q}_{su}) \text{Cov}(\mathbb{Q}_{pr}) \subseteq H_b \text{Cov}(\mathbb{Q}) \text{Cov}(\mathbb{Q})$$

**78:48 The Covering Problem**

Moreover, we know from Fact 55 that  $Cov(\mathbb{Q})Cov(\mathbb{Q}) \subseteq Cov(\mathbb{Q})$ . We then obtain,

$$Cov(\mathbb{S}) \subseteq H_b Cov(\mathbb{Q}) \cup H_b Cov(\mathbb{Q})(A_{\mathbb{Q}})^* H_b Cov(\mathbb{Q})$$

It then follows from Fact 55 that  $Cov(\mathbb{Q})(A_{\mathbb{Q}})^* H_b Cov(\mathbb{Q}) \subseteq Cov(\mathbb{Q})$  and therefore that  $Cov(\mathbb{S}) \subseteq H_b Cov(\mathbb{Q})$ . Using Lemma 14, we conclude that  $\langle \mathbf{L} | Cov(\mathbb{S}) \rangle \subseteq \langle \mathbf{L} | H_b \rangle \odot \langle \mathbf{L} | Cov(\mathbb{Q}) \rangle$  which terminates the proof.