

On the join of two pseudovarieties

Marc Zeitoun

LITP, IBP, Université Paris 7

e-mail: mz@litp.ibp.fr

The aim of this lecture is to survey some recent developments in the theory of finite semigroups. More precisely, we shall consider the following problem about pseudovarieties of semigroups: given two pseudovarieties \mathbf{V} and \mathbf{W} , find a description of their join $\mathbf{V} \vee \mathbf{W}$ (that is, of the pseudovariety they generate).

This question is motivated by the theory of rational languages: it appears in a natural way when considering parallel operation of automata.

The lattice of semigroup varieties (in Birkhoff's sense) has been studied for a long time. In particular, it was proved that the join of two finitely based varieties might not be finitely based (see for instance Taylor [21]). Other important contributions in this area were given by Biryukov [11], Fennemore [12, 13] and Gerhard [14] who described the lattice of idempotent semigroup varieties, and by Polák [18] who described the lattice of varieties of completely regular semigroups.

The problems appearing in the study of the lattice of pseudovarieties are analogous. Reiterman's theorem [19] is the starting point of an equational theory for pseudovarieties: just as varieties are defined by identities, pseudovarieties are defined by pseudoidentities.

Numerous algorithmic problems on pseudovarieties were proposed, for instance by Rhodes [20], Almeida [5] or Kharlampovich and Sapir [15]. Most of these problems are still open. Given a pseudovariety \mathbf{V} , two important problems appear:

Problem 1. *Find a finite pseudoidentity basis for \mathbf{V} , or prove that it is not finitely based.*

Problem 2. *Solve the membership problem for \mathbf{V} : given a finite semigroup S , does it exist an algorithm testing the membership of S in \mathbf{V} ?*

A positive answer to problem 1 frequently gives the solution for problem 2. We first survey some of the solutions given to these problems when \mathbf{V} is a join. In the sequel, we will denote by $[\Sigma]$ the pseudovariety defined by a set Σ of pseudoidentities. We will mention the following pseudovarieties:

- \mathbf{A} , the pseudovariety of finite aperiodic semigroups.
- \mathbf{J} , the pseudovariety of finite \mathcal{J} -trivial semigroups.
- \mathbf{B} , the pseudovariety of finite bands.
- \mathbf{G} , the pseudovariety of finite groups.

• In the general case, nothing is known about the pseudoidentity basis describing a join, and the problem seems to be very difficult. A recent result of Albert, Baldinger and Rhodes [1] states that even the join of two simple decidable pseudovarieties might not be decidable:

Theorem 3. *One can find a finite set Σ of identities such that the join $[\Sigma] \vee [xy = yx]$ is not decidable.*

• Sometimes, one can only determine whether the join has a finite basis. On the positive side, one of the few existing results is due to Almeida [4]:

Theorem 4. *Every commutative pseudovariety is finitely based.*

On the negative side, Volkov [23] and Trotter and Volkov [22] gave examples of non finitely based joins:

Theorem 5. *Let A_2 be the syntactic semigroup of $(a+b)^*a^2(a+b)^*$. Let \mathbf{V} be a semigroup pseudovariety containing A_2 , and let \mathbf{H} be a non trivial group pseudovariety. Then $\mathbf{V} \vee \mathbf{H}$ is not finitely based. In particular, $\mathbf{A} \vee \mathbf{G}$ is not finitely based.*

Theorem 6. *The pseudovariety $\mathbf{J} \vee \mathbf{G}$ is not finitely based.*

We shall give another example of a non finitely based join in this lecture.

• Finally, in some cases, computations can be done explicitly. Few methods are known at present, but Almeida [5] used the theory of implicit operations to find a description of some joins. His method is based on Reiterman's theorem and on topological arguments. Thus, Almeida, Azevedo and Weil computed some difficult joins, using at some point some *ad hoc* facts. Notice that algebraic methods fail so far for most of these computations.

A famous example is the description of $\mathbf{R} \vee \mathbf{L}$ given by Almeida and Azevedo in [6], that was conjectured by König [16]. Recall that $\mathbf{R} = \llbracket (xy)^\omega x = (xy)^\omega \rrbracket$ is the pseudovariety of \mathcal{R} -trivial semigroups while $\mathbf{L} = \llbracket y(xy)^\omega = (xy)^\omega \rrbracket$ is the one of \mathcal{L} -trivial semigroups.

Theorem 7. *The pseudovariety $\mathbf{R} \vee \mathbf{L}$ is defined by $\llbracket (xy)^\omega x(zx)^\omega = (xy)^\omega (zx)^\omega \rrbracket$.*

Let $\mathcal{P}_S(\mathbf{V})$ denote the lattice of subpseudovarieties of \mathbf{V} . The lattice $\mathcal{P}_S(\mathbf{V} \vee \mathbf{W})$ can sometimes be decomposed in the direct product $\mathcal{P}_S(\mathbf{V}) \times \mathcal{P}_S(\mathbf{W})$. Azevedo [9] gave an example of such a decomposition:

Theorem 8. *Let $\mathbf{U} = \llbracket (xy)^\omega = x^\omega y^\omega, x^{\omega+1} = x \rrbracket$ and let Φ and Ψ defined by*

$$\begin{aligned} \Phi : \mathcal{P}_S(\mathbf{U}) &\longrightarrow \mathcal{P}_S(\mathbf{G}) \times \mathcal{P}_S(\mathbf{B}) \\ \mathbf{V} &\longmapsto (\mathbf{V} \cap \mathbf{G}, \mathbf{V} \cap \mathbf{B}) \\ \Psi : \mathcal{P}_S(\mathbf{G}) \times \mathcal{P}_S(\mathbf{B}) &\longrightarrow \mathcal{P}_S(\mathbf{U}) \\ (\mathbf{H}, \mathbf{P}) &\longmapsto \mathbf{H} \vee \mathbf{P} \end{aligned}$$

Then, Φ and Ψ are mutually inverse isomorphisms. In particular, $\mathbf{G} \vee \mathbf{B} = \mathbf{U}$. Furthermore, if $\mathbf{V} \subseteq \mathbf{G} \vee \mathbf{B}$, then \mathbf{V} is decidable if and only if $\mathbf{V} \cap \mathbf{G}$ is decidable.

Recent results involve the pseudovariety \mathbf{G} , and use more difficult geometrical arguments. For example, Almeida and Weil [7, 8] proved the following. Recall that if \mathbf{H} is a pseudovariety of groups, $\bar{\mathbf{H}}$ denotes the pseudovariety of all monoids whose subgroups lie in \mathbf{H} .

Theorem 9. *Let \mathbf{H} be a group pseudovariety satisfying $(\mathbf{H} \cap \llbracket xy = yx \rrbracket) * \mathbf{H} = \mathbf{H}$. Then*

$$\begin{aligned} (\mathbf{A} \cap \llbracket xy = yx \rrbracket) \vee \mathbf{H} &= \llbracket x^\omega y = yx^\omega \rrbracket \cap \bar{\mathbf{H}} \\ (\mathbf{J} \cap \llbracket x^\omega y^\omega = y^\omega x^\omega \rrbracket) \vee \mathbf{H} &= \mathbf{DH} \cap \llbracket x^\omega y^\omega = y^\omega x^\omega \rrbracket \end{aligned}$$

where \mathbf{DH} denotes the pseudovariety of all semigroups whose regular \mathcal{D} -classes lie in \mathbf{H} . This holds in particular for $\mathbf{H} = \mathbf{G}$.

Other results giving an idea of the state of the art for such computations can be found in [3, 2, 6, 7, 8, 10, 9].

We finally present the solutions given in [24, 25] of two problems proposed by Almeida [5]. The first question is the computation of the join $\mathbf{LI} \vee \mathbf{B}$ of the pseudovariety $\mathbf{LI} = \llbracket x^\omega y x^\omega = x^\omega \rrbracket$ of locally trivial semigroups and of the pseudovariety \mathbf{B} . We find a finite basis and a natural algebraic interpretation for this join:

Theorem 10. *The pseudovariety $\mathbf{LI} \vee \mathbf{B}$ is the pseudovariety of all finite semigroups whose idempotents form an ideal. In other terms, $\mathbf{LI} \vee \mathbf{B}$ is defined by*

$$\mathbf{LI} \vee \mathbf{B} = \llbracket (x^\omega y)^\omega = x^\omega y, \quad (yx^\omega)^\omega = yx^\omega \rrbracket$$

The second question is the computation of the join $\mathbf{J} \vee \mathbf{B}$. It is more complex.

Theorem 11. *The pseudovariety $\mathbf{J} \vee \mathbf{B}$ is not finitely based.*

Theorem 12. *The pseudovariety $\mathbf{J} \vee \mathbf{B}$ is decidable.*

We shall sketch the techniques used for the proofs of these theorems. They illustrate the fact that the knowledge of implicit operations on $\mathbf{V} \vee \mathbf{W}$ may be difficult even when implicit operations on \mathbf{V} and on \mathbf{W} are known. We conclude by recalling two important problems that are still open:

Problem 13. *Is the pseudovariety $\mathbf{A} \vee \mathbf{G}$ decidable?*

Problem 14. *Is the pseudovariety $\mathbf{J} \vee \mathbf{G}$ decidable?*

We conjecture that the answer to problem 14 is positive. Azevedo [9], followed by several authors proposed a basis for this pseudovariety:

$$x^{\omega+1} y_1^\omega \dots y_k^\omega x^\omega = x^\omega y_1^\omega \dots y_k^\omega x^{\omega+1} \quad (k > 0), \quad (xy)^\omega = (yx)^\omega$$

We now present an array to summarize some of the known results and some of the open problems. In this array, $[k]$ stands for a complete answer, (k) stands for a partial answer, $\langle k \rangle$ stands for a guess.

We recall the following standard notations of pseudovarieties.

Nil	: nilpotent semigroups.
K (resp D)	: semigroups in which every idempotent is a left (resp. a right) zero
K₁ (resp. D₁)	: left zero (resp right zero) semigroups.
LI₁	: rectangular bands.
G	: groups.
G_{com}	: commutative groups.
Sl	: commutative and idempotent semigroups.
Com	: commutative semigroups.
ZE	: semigroups in which idempotents are central.
Inv	: semigroups in which idempotents commute.
Lcom	= $\llbracket x^\omega yz = x^\omega zy \rrbracket$
Rcom	= $\llbracket yzx^\omega = zy x^\omega \rrbracket$
Perm	: semigroups satisfying a non trivial permutation identity.
H	denotes a pseudovariety of groups

When possible, we give general references containing many examples, such as [5], [9], [17] or [24] rather than original articles to keep the bibliography in a reasonable size.

- (1) $\mathbf{G} \vee \mathbf{A}$ is not finitely based (Volkov [23])
(2), (3), (4), (5) $\mathbf{V} \vee \mathbf{H}$ is not finitely based if $\mathbf{J} \subseteq \mathbf{V} \subseteq \mathbf{A}$ and if \mathbf{H} is not commutative (Trotter and Volkov, [22])
[6] $\mathbf{G} \vee (\mathbf{J} \cap \mathbf{Inv}) = \mathbf{DG} \cap \mathbf{Inv}$ (Almeida, Weil [7, 8])
[7] $\mathbf{G} \vee \mathbf{Nil} = \llbracket x^\omega = y^\omega \rrbracket$, (8) $\mathbf{H} \vee \mathbf{Nil} = (\mathbf{G} \vee \mathbf{Nil}) \cap \bar{\mathbf{H}}$ decidable, (Pin [17])
[9] $\mathbf{G} \vee \mathbf{K} = \llbracket x^\omega y^\omega = x^\omega \rrbracket$, (10) $\mathbf{H} \vee \mathbf{K} = (\mathbf{G} \vee \mathbf{K}) \cap \bar{\mathbf{H}}$ *idem*
[11] $\mathbf{G} \vee \mathbf{D} = \llbracket y^\omega x^\omega = x^\omega \rrbracket$, (12) $\mathbf{H} \vee \mathbf{D} = (\mathbf{G} \vee \mathbf{D}) \cap \bar{\mathbf{H}}$ *idem*
[13] $\mathbf{G} \vee \mathbf{LI} = \llbracket x^\omega y^\omega x^\omega = x^\omega \rrbracket$, (14) $\mathbf{H} \vee \mathbf{LI} = (\mathbf{G} \vee \mathbf{LI}) \cap \bar{\mathbf{H}}$ *idem*
[15] $\mathbf{G} \vee \mathbf{K}_1 = \llbracket xy^\omega = x \rrbracket$, (16) $\mathbf{H} \vee \mathbf{K}_1 = (\mathbf{G} \vee \mathbf{K}_1) \cap \bar{\mathbf{H}}$ *idem*
[17] $\mathbf{G} \vee \mathbf{D}_1 = \llbracket y^\omega x = x \rrbracket$, (18) $\mathbf{H} \vee \mathbf{D}_1 = (\mathbf{G} \vee \mathbf{D}_1) \cap \bar{\mathbf{H}}$ *idem*
[19] $\mathbf{G} \vee \mathbf{LI}_1 = \llbracket xy^\omega x^\omega = x = x^\omega y^\omega x \rrbracket$, (20) $\mathbf{H} \vee \mathbf{LI}_1 = (\mathbf{G} \vee \mathbf{LI}_1) \cap \bar{\mathbf{H}}$ *idem*
[21] $\mathbf{G}_{\text{com}} \vee \mathbf{Nil} = \llbracket x^\omega = y^\omega, x^\omega y = yx^\omega, x^\omega yz = x^\omega zy \rrbracket$, (Almeida [5])
[22] $\mathbf{G} \vee \mathbf{Sl} = \llbracket (xy)^\omega = x^\omega y^\omega = (yx)^\omega, x^{\omega+1} = x^\omega \rrbracket$, (Almeida [5], Azevedo [9])
[23] $\mathbf{G} \vee \mathbf{Com} = \llbracket x^\omega y = yx^\omega \rrbracket$ (Almeida [5], Azevedo [9])
[24] $\mathbf{G} \vee \mathbf{B} = \llbracket (xy)^\omega = x^\omega y^\omega, x^{\omega+1} = x^\omega \rrbracket$ (Almeida [5], Azevedo [9])
[25] $\mathbf{R} \vee \mathbf{L} = \llbracket (xy)^\omega x(zx)^\omega = (xy)^\omega (zx)^\omega \rrbracket$ (Almeida [5], Azevedo [9])
⟨26⟩ $\mathbf{R} \vee \mathbf{D}_1 = \mathbf{R} \vee \mathbf{LI}_1 \stackrel{?}{=} \llbracket (xy)^\omega xz = (xy)^\omega z \rrbracket$ ⟨27⟩ $\mathbf{L} \vee \mathbf{K}_1 = \mathbf{L} \vee \mathbf{LI}_1 \stackrel{?}{=} \llbracket zx(yx)^\omega = z(yx)^\omega \rrbracket$

\vee	G	A	R	L	J	J∩Inv	Nil	K	D	LI
G	G	(1,2)	(3)	(4)	(5)	[6]	[7]	[9]	[11]	[13]
A		A	A	A	A	A	A	A	A	A
R			R	[25]	R	R	R	R		
L				L	L	L	L		L	
J					J	J	J	[32]	[33]	[34]
J∩Inv						J∩Inv	J∩Inv			
Nil							Nil	K	D	LI
K								K	LI	LI
D									D	LI
LI										LI
Sl										
B										
K₁										
D₁										
LI₁										
G_{com}										
Com										
Rcom										
Lcom										
Perm										

- [28] $\mathbf{J} \vee \mathbf{B}$ is non finitely based but decidable. A basis is easy to get [24, 25].
- [29] $\mathbf{J} \vee \mathbf{K}_1 = \llbracket x(yz)^\omega = x(zy)^\omega, x^{\omega+1} = x^\omega \rrbracket$
- [30] $\mathbf{J} \vee \mathbf{D}_1 = \llbracket (yz)^\omega x = (zy)^\omega x, x^{\omega+1} = x^\omega \rrbracket$
- [31] $\mathbf{J} \vee \mathbf{LI}_1 = \llbracket x(yz)^\omega t = x(zy)^\omega t, x^{\omega+1} = x^\omega \rrbracket$
- [32] $\mathbf{J} \vee \mathbf{K} = \llbracket a^\omega x(yz)^\omega = a^\omega x(zy)^\omega, x^\omega = x^{\omega+1} \rrbracket$ Almeida [5]
- [33] $\mathbf{J} \vee \mathbf{D} = \llbracket (yz)^\omega x a^\omega = (zy)^\omega x a^\omega, x^\omega = x^{\omega+1} \rrbracket$ Almeida [5]
- [34], [35], [36], [37], [38]: see Azevedo [10].
- [39] $\mathbf{Nil} \vee \mathbf{B} = \llbracket x^\omega y = (xy)^\omega = xy^\omega \rrbracket$ [40] $\mathbf{SI} \vee \mathbf{Nil} = (\mathbf{Nil} \vee \mathbf{B}) \cap \mathbf{ZE}$ [5, 24]
- [41] $\mathbf{K} \vee \mathbf{B} = \llbracket x^\omega y = x^\omega y^2, xy^\omega z = (xy^\omega z)^2 \rrbracket$ [42] $\mathbf{SI} \vee \mathbf{K} = (\mathbf{K} \vee \mathbf{B}) \cap \mathbf{Lcom}$ [24]
- [43] $\mathbf{D} \vee \mathbf{B} = \llbracket yx^\omega y = y^2 x^\omega, xy^\omega z = (xy^\omega z)^2 \rrbracket$ [44] $\mathbf{SI} \vee \mathbf{D} = (\mathbf{D} \vee \mathbf{B}) \cap \mathbf{Rcom}$ *idem*
- [45] $\mathbf{LI} \vee \mathbf{B} = \llbracket xy^\omega z = (xy^\omega z)^2 \rrbracket$ [46] $\mathbf{SI} \vee \mathbf{LI} = (\mathbf{LI} \vee \mathbf{B}) \cap \mathbf{Perm}$ *idem*
- [47] $\mathbf{Nil} \vee \mathbf{Com} = \llbracket x^\omega y = yx^\omega, x^{\omega+1} = x^\omega \rrbracket$
- [48] $\mathbf{SI} \vee \mathbf{K}_1 = \llbracket xyz = xzy, x^2 = x \rrbracket$
- [49] $\mathbf{SI} \vee \mathbf{D}_1 = \llbracket yzx = zyx, x^2 = x \rrbracket$
- [50] $\mathbf{SI} \vee \mathbf{LI}_1 = \llbracket xyzx = xzyx, x^2 = x \rrbracket$
- [51] $\mathbf{SI} \vee \mathbf{G}_{com} = \llbracket xy = yx, x^\omega = x \rrbracket$
- $\langle 52 \rangle \mathbf{K} \vee \mathbf{D}_1 = \mathbf{K} \vee \mathbf{LI}_1 \stackrel{?}{=} \llbracket x^\omega yz = x^\omega z \rrbracket$, $\langle 53 \rangle \mathbf{D} \vee \mathbf{K}_1 = \mathbf{D} \vee \mathbf{LI}_1 \stackrel{?}{=} \llbracket zyx^\omega = zx^\omega \rrbracket$

\vee	SI	B	K₁	D₁	LI₁	G_{com}	Com	Rcom	Lcom	Perm
G	[22]	[24]	[15]	[17]	[19]	G	[23]			
A	A	A	A	A	A					
R	R		R	$\langle 26 \rangle$	$\langle 26 \rangle$					
L	L		$\langle 27 \rangle$	L	$\langle 27 \rangle$					
J	J	[28]	[29]	[30]	[31]		[35]	[36]	[37]	[38]
J \cap Inv	J \cap Inv									
Nil	[40]	[39]				[8,21]	[47]	Rcom	Lcom	Perm
K	[42]	[41]	K	$\langle 52 \rangle$	$\langle 52 \rangle$	[10]	Lcom	Perm	Lcom	Perm
D	[44]	[43]	$\langle 53 \rangle$	D	$\langle 53 \rangle$	[12]	Rcom	Rcom	Perm	Perm
LI	[46]	[45]	LI	LI	LI	[14]	Perm	Perm	Perm	Perm
SI	SI	B	[48]	[49]	[49]	[51]	Com	Rcom	Lcom	Perm
B		B	B	B	B					
K₁			K₁	LI₁	LI₁	[16]				Perm
D₁				D₁	LI₁	[18]				Perm
LI₁					LI₁	[20]				Perm
G_{com}						G_{com}	Com	Rcom	Lcom	Perm
Com							Com	Rcom	Lcom	Perm
Rcom								Rcom	Perm	Perm
Lcom									Lcom	Perm
Perm										Perm

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