



## Uniform election in trees and polyominoids<sup>☆</sup>

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### ABSTRACT

Election is a classical paradigm in distributed algorithms. This paper aims to design and analyze a distributed algorithm choosing a node in a graph which models a network. In case the graph is a tree, a simple schema of algorithm acts as follows: it removes leaves until the graph is reduced to a single vertex; the elected one. In Métivier et al. (2003) [7], the authors studied a randomized variant of this schema which gives the same probability of being elected to each node of the tree. They conjectured that the expected election duration of this algorithm is  $O(\ln(n))$  where  $n$  denotes the size of the tree, and asked whether it is possible to use the same algorithm to obtain a fair election in other classes of graphs.

In this paper, we prove their conjecture. We then introduce a new structure called polyominoid graphs. We show how a spanning tree for these graphs can be computed locally so that our algorithm, applied to this spanning tree, gives a uniform election algorithm on polyominoids.

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### 1. Introduction

We consider distributed networks of processors [13]. They are presented as connected graphs where vertices represent processors, and two vertices are connected by an edge if the corresponding processors have a direct communication link. Labels are attached to vertices and sometimes to edges. The aim of an election algorithm is to choose exactly one element in the set of processors. Thus, starting from a configuration where all processors are in the same state, we must obtain a configuration where exactly one processor is in the state “leader” and all other processors are in the state “lost”. The leader can be used subsequently to make decisions or to centralise some information. The election problem is well known and many solutions are available [1,3–5,10,13]. It was first proposed by Le Lann [3].

The networks studied in this paper are anonymous and have a tree topology or what we call a polyominoid topology. Polyominoids are certain subgraphs of the grid and have the property that a spanning tree can be computed locally (see Fig. 1).

The main motivation behind this study is to introduce a uniform probabilistic distributed election algorithm over trees and polyominoids. This algorithm is *totally fair*; i.e. it gives the same chance of being elected to all vertices of the graph considered. The algorithm removes vertices of the graph once their random lifetime delay has expired. The analysis of the algorithm reveals the surprising fact that, wherever the vertex is placed in the graph, it has the same probability of surviving as the others. This is a first step in extending the investigation of [8] to graphs other than trees. The previous proof for the uniformity of the election has been simplified. As a new result, we find a *common* distribution for the election duration, whose existence was conjectured in [8].

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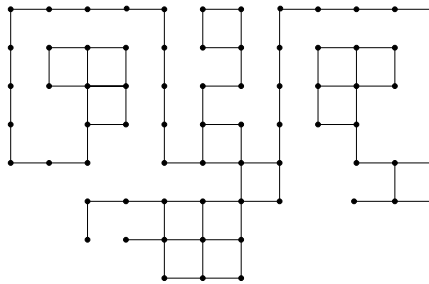


Fig. 1. An example of a polyominoid.

The modelling process is not a new one, but its application in the analysis of the election algorithm is quite new and presents an interest in further investigations on distributed algorithms, where random delays are involved. On the other hand, in an election algorithm the fairness assumption requires that no process should initially be deprived of being elected. The idealisation of this requirement leads to the search for algorithms giving the same chance to all processors.

Our distributed algorithm may be viewed as a randomised extension of a variant of [4], where random delays are introduced.

We consider cellular local computations which can modify the state (or label) of a vertex at each step. The new label of the vertex depends on the previous one and those of its neighbours. The novelty of our approach is the use of random delays for relabelling. These delays are exponential random variables defined independently for active vertices. The parameter of the random variable for a vertex is equal to the attributed value assigned to the vertex. The process of relabelling continues until no more transformations are possible; i.e. a final configuration is reached. In this configuration, there is only one E-labelled vertex, considered as elected.

## 2. Fair election in trees

In this section, we recall some results from [7]. We also give a simple proof of the fairness of their algorithm.

Let  $T = (V, E)$  be a tree in the graph theoretic sense:  $V$  is its set of vertices,  $E$  the set of edges. The size of  $T$ , denoted by  $n$ , is the number of vertices.

In the algorithm from [7], initially all vertices have the same weight 1. Each leaf has a lifetime which is an exponentially distributed random variable with a parameter equal to the weight of the leaf. Once the lifetime of the leaf has expired, it is removed with the incident edge and its weight is recovered by its father. The process continues until the tree is reduced to one vertex, which is considered as the elected vertex. The probability of being elected for a vertex  $v$  in  $T$  is shown to be  $\frac{1}{n}$ . We give here a new proof of this result, simpler than the one in [7].

We enumerate here intermediate results and give the outline of the proof.

We first introduce a slight modification of the leaf-removal model over  $T$ . We translate the model into a variant on directed trees. For a given vertex  $v$ , the unique tree rooted at  $v$  can be defined. These rooted trees can be used in a natural way to compute the absorption probabilities.

We consider forests of rooted trees. Let  $F$  be a forest of rooted trees. We introduce a death process on  $F$  as follows. Each leaf  $v$  has an exponentially distributed lifetime with a parameter equal to its weight; initially, all vertices of  $F$  are of weight 1. At any time interval  $[t, t + \Delta t]$ , if the lifetime of a leaf has expired, the leaf is removed with its unique incident edge. If the vanishing leaf has a father, then its father picks up its weight, adding it to its weight. The leaf-removal process goes on in the reduced forest until the forest totally disappears.

For a given forest  $F$ , let  $L(F)$  be the vanishing time; it is a positive-real-valued r.v. (random variable).

The following proposition is surprising. It asserts that  $L(F)$  depends only on the size of the forest and not on its structure. For the sake of simplicity, we first prove a lemma and a corollary from which we easily derive the proposition. Let  $\mathcal{E}(a)$  denote an exponential r.v. of parameter  $a$  ( $a > 0$ ) and let  $M_n = \max_{1 \leq i \leq n} \mathcal{E}_i(1)$  the maximum of  $n$  independent identical r.v. each one of the same distribution as  $\mathcal{E}(1)$ . For two r.v.  $X$  and  $Y$ , we write  $X \stackrel{d}{=} Y$  to assert that they are of the same distribution. We have:

**Lemma 1.** *The sum  $\mathcal{E}(1) + \dots + \mathcal{E}(n)$  of  $n$  independent r.v. has the same distribution as  $M_n$ .*

**Proof.** This is a classical result of probability theory. Consider  $(\hat{\mathcal{E}}_i, 1 \leq i \leq n)$ , the order statistics of  $n$  i.i.d.  $\mathcal{E}(1)$  random variables  $\mathcal{E}_1(1), \dots, \mathcal{E}_n(1)$ , that is the sequence  $(\mathcal{E}_i(1), 1 \leq i \leq n)$ , sorted in increasing order. The variable  $M_n = \max_{1 \leq i \leq n} \mathcal{E}_i(1)$  is also the sum of the random variables  $\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1}$ , for  $i = 1, \dots, n$  with the convention  $\hat{\mathcal{E}}_0 = 0$ . Using the memory-less properties of the exponential distribution, one has  $\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1} \stackrel{d}{=} \mathcal{E}(n + 1 - i)$  for all  $i \in \{1, \dots, n\}$ , and the variables  $(\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1})$  are independent, (see Proposition p. 19 in Feller [2]). This proves the lemma.  $\square$

From the lemma we easily derive:

**Corollary 2.** (i) Consider  $a_1, \dots, a_k$  ( $k \geq 1$ ) positive integers, summing to  $n$ . If the random variables  $M_{a_i}$  are independent, and independent of  $\mathcal{E}(n + 1)$  then:

$$M_{n+1} \stackrel{d}{=} \mathcal{E}(n + 1) + \max_{1 \leq i \leq k} M_{a_i}.$$

(Recall that the distribution of  $M_n$  is given by  $\Pr(M_n \leq x) = (1 - \exp(-x))^n$ .)

(ii) For any  $k \geq 1$  and  $n \geq 1$ , set:

$$Y_{n+1,k} \stackrel{d}{=} \mathcal{E}(n + 1) + \mathcal{E}(n + 2) + \dots + \mathcal{E}(n + k), \tag{1}$$

where the variables  $\mathcal{E}(n + i)$  are independent. We have:

$$M_{n+k} \stackrel{d}{=} M_n + Y_{n+1,k}.$$

**Proof.** (ii) is a direct consequence of the lemma. For (i), write  $\max_{1 \leq i \leq k} M_{a_i} \stackrel{d}{=} M_n$  and apply the lemma.  $\square$

We now state the main proposition on the lifetime of a forest.

**Proposition 3.** Let  $F$  be a forest of size  $n$  then the distribution function  $G_F(t)$  of the r.v.  $L(F)$  is given by:

$$G_F(t) = \Pr(L(F) \leq t) = (1 - e^{-t})^n, \quad \forall t \geq 0.$$

**Proof.** By structural induction on  $F$ . The proposition obviously holds for a forest reduced to a vertex. If the proposition holds for  $F_1, \dots, F_k$ , then it clearly holds for their union. Finally, Corollary 2 allows us to assert that the addition of a new vertex  $v$  with an exponentially distributed lifetime with a parameter equal to the size of the obtained tree rooted at  $v$ , preserves the assertion of the proposition.  $\square$

Given two forests  $F_1$  and  $F_2$ , we say  $F_1$  beats  $F_2$ , if  $L(F_1) \geq L(F_2)$ . The next result easily follows from the above proposition.

**Corollary 4.** Let  $F_1$  and  $F_2$  be two forests of sizes  $n_1$  and  $n_2$  respectively. The probability that  $F_1$  beats  $F_2$  is  $\frac{n_1}{n_1+n_2}$ .

**Lemma 5.** Consider a vertex  $v$  in  $T$  with the adjacent vertices  $v_1, \dots, v_k$ . Let  $F$  consist of two trees  $A$  and  $B$  obtained by the suppression of edge  $\{v, v_1\}$  rooted at  $v$  and  $v_1$  respectively. Then, the probability that  $v$  is removed before the whole tree factor on the side of  $v_1$  (i.e. undirected  $B$ ) in the election process over  $T$  is the same as the probability of  $v_1$  beating  $v$  in  $F$ .

**Proof.** The events whose probabilities are to be calculated can be represented as sequences of leaves being removed  $\sigma = \langle l_1, \dots, l_k \rangle$ , where  $l_i, 1 \leq i \leq k$ , are leaves or vertices which become leaves in  $T$  (or in  $F$  respectively) after the removal of some previous vertices in the sequence, such that  $l_k = v$  and  $v_1$  does not figure in  $\sigma$ . On the one hand, it is easy to see that any sequence satisfying the above conditions in  $T$  does it in  $F$  and vice-versa. On the other hand, the probability of such  $\sigma$  according to (1) is:

$$P(\sigma) = \prod_{1 \leq i \leq k} q_i,$$

with  $q_i = \frac{\lambda(l_i)}{\lambda(T_i)}$ , where  $T_i$  is the residual tree (forest resp.) before the  $l_i$  removal. In each step of the leaf removal along  $\sigma$ ,  $T$  and  $F$  have the same set of leaves and, hence, the quantities involved on  $T$  are the same as the corresponding ones on  $F$ . The lemma follows.  $\square$

**Proposition 6.** Let  $q(v)$  denote the probability of being elected in  $T$  for a vertex  $v$ . We have  $q(v) = \frac{1}{n}$ .

**Proof.** For  $n = 1$  or  $n = 2$  the proposition is obvious. Otherwise, let  $v_1, \dots, v_k$  be the vertices adjacent to  $v$ . Let, on the other hand,  $A_1, \dots, A_k$  be disjoint trees rooted at  $v_1, \dots, v_k$  of sizes  $n_1, \dots, n_k$  respectively. Clearly,  $v$  fails iff it vanishes before one of the factors situated on the  $v_i$  side for  $1 \leq i \leq k$ . These last events are pairwise disjoint and therefore, according to the previous lemma, the failure probability of  $v$  is the sum of the probabilities of  $v$  being beaten by one of its neighbours  $v_i$  in the forest consisting of the tree rooted at  $v$  and  $v_i$  respectively. Hence, according to Corollary 4, we have:

$$1 - q(v) = \sum_{i=1}^k \frac{n_i}{n}.$$

Since  $\sum_{i=1}^k n_i = n - 1$ , the proposition follows.  $\square$

### 3. Election duration through a tree

In [8], the authors study the uniform election duration for particular classes of tree graphs. The study seems interesting since it leads to a conjecture suggesting that it is logarithmic on average. Also, some results hint that there is a deeper common feature of this r.v. for different trees of the same size. We prove here that the uniform election duration over all trees of the same size has the same distribution.

In the sequel,  $T = (V, E)$  is a tree of size  $n$ . Let  $D_n$  be the uniform election duration on  $T$ ; i.e. the instant at which  $T$  is reduced to a unique vertex.

We first prove the following:

**Lemma 7.** For a tree  $T$  of size  $n \geq 2$ , all  $n - 1$  edges have the same probability of surviving until the last step; or equivalently all pairs  $u$  and  $v$  of neighbour vertices have the same chance of being the finalists.

**Proof.** It is similar to the proof of Proposition 3. Let  $u_i, 1 \leq i \leq k$ , denote neighbour vertices of  $u$  other than  $v$  and  $v_j, 1 \leq j \leq l$ , those of  $v$  other than  $u$ . Denote by  $F$  the set of trees rooted at the vertices  $u_i$  and  $G$  the set of trees rooted at the vertices  $v_j$ ; each of these forests can be empty. Suppose that  $F$  and  $G$  are of the sizes  $n_1$  and  $n_2$  respectively with  $n_1 + n_2 = n - 2$ . One of the vertices  $\{u, v\}$  is removed before the final step in the election iff:

- either  $u$ , as the root of the arborescence consisting of  $u$  and  $F$ , is beaten by one of the vertices  $v_j$  in  $G$ ,
- or  $v$ , as the root of the arborescence consisting of  $v$  and  $G$ , is beaten by one of the vertices  $u_i$  in  $F$ .

These events are disjoint; the first is of probability  $\frac{n_2}{n_1+n_2+1}$  and the second  $\frac{n_1}{n_1+n_2+1}$ . Hence the probability of having  $\{u, v\}$  in the last step is  $1 - \frac{n_1+n_2}{n_1+n_2+1} = \frac{1}{n-1}$ .  $\square$

**Theorem 8.** The density function of the r.v.  $D_n$  is:

$$f(t) = n(n - 1)e^{-2t}(1 - e^{-t})^{n-2}.$$

**Proof.** Consider an arbitrary edge  $a = \{v_1, v_2\}$  in  $T$ . Let the sub-tree  $T_i$  of  $T$  rooted at  $v_i$  be of size  $n_i$  ( $i = 1, 2$ ). The probability that at time  $t$ , only  $a$  survives is, according to Proposition 3, is equal to  $\prod_{i=1}^2 [(1 - e^{-t})^{n_i-1} - (1 - e^{-t})^{n_i}] = e^{-2t}(1 - e^{-t})^{n-2}$ . This probability does not depend on the edge  $a = \{u, v\}$ . Hence, according to the previous lemma, the probability that the election process is in the state of one edge is  $(n - 1)e^{-2t}(1 - e^{-t})^{n-2}$ . Therefore, if  $F(t)$  denotes the probability of being in an absorbing state at time  $t$  (i.e.  $F(t)$  is the distribution function of  $D_n$ ), we have:

$$\frac{dF(t)}{dt} = n(n - 1)e^{-2t}(1 - e^{-t})^{n-2}.$$

The theorem follows.  $\square$

**Corollary 9.** The expected value of  $D_n$  is  $\sum_{i=2}^n \frac{1}{i} = H_n - 1$ , where  $H_n$  is the  $n$ th harmonic number, and its variance is equal to  $\sum_{i=2}^n \frac{1}{i^2}$  which tends to  $\frac{\pi^2}{6} - 1$  as  $n$  tends to  $\infty$ .

**Proof.** Since the r.v.  $D_n$  is the same for all trees of size  $n$ , it suffices to calculate its expectation for a chain of size  $n$ . In this case, the expected duration for the  $i$ th removal is  $\frac{1}{i+1}$ . This establishes the first claim of the corollary. To prove the second claim, we use the same method since the transition times, from a chain of size  $n - i$  to a that of size  $n - i - 1$  are independent and each one is exponentially distributed with parameter  $i + 1$ .  $\square$

**Corollary 10.** Let  $X_n$  denote the deviation between the election duration and  $\log n$ :  $X_n = D_n - \log n$ . As  $n$  tends to  $\infty$ , the density of  $X_n$  tends to  $e^{-2x}e^{-e^{-x}}$ , for  $x \in \mathbb{R}$ , and a simple computation yields its expectation:  $\mathbb{E}(X_n) = \gamma - 1 + o(1)$ .

We have also:

**Corollary 11.** Let  $D_n^u$  be the election time conditioned on the vertex  $u$  being elected. Then the density function of the r.v.  $D_n^u$  is also:

$$f(t) = n(n - 1)e^{-2t}(1 - e^{-t})^{n-2}.$$

**Proof.** Let the neighbours of  $u$  be  $v_1, v_2, \dots, v_d$  and let  $n_i$  be the size of the tree rooted at  $v_i$  not including  $u$ . We have that the probability of  $\{u, v_i\}$  being the sole remaining edge at time  $t$  is  $e^{-2t}(1 - e^{-t})^{n-2}$ . In this case  $u$  will be elected at time  $t$  with probability density  $n_i e^{-2t}(1 - e^{-t})^{n-2}$ . Summing over  $i$  gives that the probability density of  $u$  being elected at time  $t$  is  $\sum_{i=1}^d n_i e^{-2t}(1 - e^{-t})^{n-2}$  or  $(n - 1)e^{-2t}(1 - e^{-t})^{n-2}$ . Since the probability of  $u$  being elected is  $1/n$ , the result follows.  $\square$

## 4. Polyominoids

### 4.1. Preliminaries and notation

There are many definitions for polyominoes and grid-like graphs in the literature, see [11,6]. Traditionally, a polyomino is the set of cells situated in the interior of an orthogonal polygon drawn on a grid. We define polyominoids as finite graphs whose nodes are points from  $\mathbb{Z} \times \mathbb{Z}$ , where  $\mathbb{Z}$  denotes the set of integers, possibly linked by the neighbourhood relationship, defined in the sequel.

Throughout this paper, the vertices are points from  $\mathcal{Z} = \mathbb{Z} \times \mathbb{Z}$ . The edges are links between pairs of points, i.e. sets of pairs of points of one of the forms  $\{(x, y), (x + 1, y)\}$  or  $\{(x, y), (x, y + 1)\}$ , for some  $x \in \mathbb{Z}, y \in \mathbb{Z}$ . Two vertices  $v = (x, y)$  and  $v' = (x', y')$  of  $\mathcal{Z}$  are *neighbours* if either  $x = x'$  and  $|y - y'| = 1$  (*vertical*) or else  $y = y'$  and  $|x - x'| = 1$  (*horizontal*). We refer to each element of an edge  $e$  as its *end*. Let  $\mathcal{T}$  be the set of all these edges and set  $\mathbf{U} = (\mathcal{Z}, \mathcal{T})$ . A *cell* is a sub-graph of  $\mathbf{U}$ , induced by a set  $\{(x, y), (x + 1, y), (x + 1, y + 1), (x, y + 1)\}$  of four pairwise neighbour vertices. Two horizontal (vertical resp.) edges  $e = \{(x, y), (x + 1, y)\}$  and  $e' = \{(x', y'), (x' + 1, y')\}$  ( $e = \{(x, y), (x, y + 1)\}$  and  $e' = \{(x', y'), (x', y' + 1)\}$  resp.) are said to be *transposable*, if  $x = x'$  ( $y = y'$  resp.). A *path* is a finite alternated sequence  $\sigma = v_0, e_1, \dots, e_k, v_k$  of  $k + 1$  vertices and  $k$  *different* edges ( $k \geq 0$ ), such that each edge  $e_i$  has one end in  $v_{i-1}$  and the other one in  $v_i$ . We should note that a path may pass up to three times through a vertex but cannot take an edge more than once. The *length* of a path is its number of edges,  $k$ . For the sake of brevity in a path, we may drop edges, identifying  $\sigma$  by the sequence of vertices  $v_0, v_1, \dots, v_k$ . If so, any pair of two successive terms  $v_i$  and  $v_{i+1}$  should constitute a unique set. A *cycle* is a path of length  $k \geq 4$  for which the first vertex  $v_0$  and the last one  $v_k$  coincide.  $\mathbf{U}$  is bipartite i.e. all its cycles are of even length.

Given a cycle  $\gamma$ , one can easily define the vertices or edges inside it, see [12]. In the sequel we need the definition for edges.

It is clear that the number of edges in  $\gamma$  which are transposable to a given one is even. Let now  $e = \{(x, y), (x + 1, y)\}$  be a horizontal edge not included in  $\gamma$ ; it is said to be *inside*  $\gamma$ , if the number of edges of  $\gamma$  transposable to  $e$  with ordinate less than  $y$  is odd. The same definition can be made for a vertical edge in a similar way.

If no edge of the cycle  $\gamma$  is inside another cycle then  $\gamma$  is called *external*.

A *polyominoid* is a partial sub-graph  $\mathbf{P} = (V, E)$  of  $\mathbf{U}$  subject to the following conditions:

- (i)  $V$  is finite,
- (ii)  $\mathbf{P}$  is connected and
- (iii)  $\mathbf{P}$  does not contain any hole, i.e. for any cycle  $\gamma$  in  $\mathbf{P}$ , the edges inside  $\gamma$  are contained in  $E$  (Fig. 3).

The size of  $\mathbf{P} = (V, E)$  is the cardinal of  $V$ .

A polyominoid  $\mathbf{Q} = (V', E')$  is called a *sub-polyominoid* of a polyominoid  $\mathbf{P} = (V, E)$  if  $V' \subseteq V, E' \subseteq E$  and  $E' = E \cap \{(u, v) \mid u \in V', v \in V'\}$ .

The class of polyominoids can be defined on  $\mathbf{U}$  by induction in a distributive fashion as follows. The construction is totally distributive in that the application of rewriting rules requires only the knowledge of the neighbouring areas in a ball of radius 2. We define the set  $\mathcal{P}$  of partial sub-graphs of  $\mathbf{U}$  by the following inductive rules:

- (a) For any  $(x, y) \in \mathcal{Z}, \mathbf{P} = (\{(x, y)\}, \emptyset)$  is in  $\mathcal{P}$ .
- (b) Let  $\mathbf{P} = (V, E) \in \mathcal{P}$ . Consider two neighbouring vertices  $v$  and  $v'$  such that  $v \in V$  and  $v' \notin V$ . Then,  $\mathbf{Q} = (V \cup \{v'\}, E \cup \{(v, v')\})$  is in  $\mathcal{P}$ .
- (c) Let  $\mathbf{P} = (V, E) \in \mathcal{P}$ . Suppose  $V$  contains 4 neighbouring vertices  $v_1 = (x, y), v_2 = (x + 1, y), v_3 = (x + 1, y + 1), v_4 = (x, y + 1)$ , situated on a cell in  $\mathbf{U}$ , such that three edges of the cell on them are in  $E$  and the fourth one, say  $e$ , is not. Then,  $\mathbf{Q} = (V, E \cup \{e\})$  is in  $\mathcal{P}$ .

Thus, the set of polyominoids can be generated by a context-free-like grammar. The following proposition shows the equivalence of the two definitions.

**Proposition 12.** *A partial sub-graph  $\mathbf{P} = (V, G)$  of  $\mathbf{U}$  is a polyominoid iff it belongs to  $\mathcal{P}$ .*

**Proof.** 1. Let  $\mathbf{P}$  be in  $\mathcal{P}$ . We prove that it is a polyominoid. Clearly, the graph defined by (a) above is a polyominoid. Suppose that  $\mathbf{P}$  is a polyominoid and prove that  $\mathbf{Q}$  obtained by (b) or (c) is a polyominoid too. The claim is obvious for (b). To prove that (c) preserves polyominoidness, let  $\mathbf{P} = (V, E)$  and  $\mathbf{Q} = (V, E')$ , where  $E'$  is obtained from  $E$  by a (c) application. Let  $e$  be an edge inside a cycle  $\gamma$  in  $\mathbf{Q}$ . If all edges of  $\gamma$  are in  $E$ , then  $e$  should be in  $E$ . Otherwise,  $\gamma$  uses one of the edges of a cell over the set of vertices  $S = \{v_1, v_2, v_3, v_4\}$ , say  $\{v_1, v_2\}$ , which does not belong to  $E$  (we have  $E' = E \cup \{v_1, v_2\}$ ). In this case, it is possible to transform  $\gamma$  to another cycle  $\gamma'$  included in  $E$ , avoiding  $v_1$  by using other vertices from  $S$ , and the edges inside  $\gamma'$  are exactly the same as those inside  $\gamma$  apart from some edges of the cell.

2. We now prove, by induction on the number of edges that if  $\mathbf{Q} = (V, E)$  is a polyominoid then it belongs to  $\mathcal{P}$ . For  $|E| = 0$ ,  $\mathbf{Q}$  is obviously in  $\mathcal{P}$ . Let it be true for polyominoids with number of edges less than or equal to  $m$  and prove it for  $\mathbf{Q} = (V, E)$  with  $|E| = m + 1$ . If  $\mathbf{Q}$  has one vertex of degree 1, then the property clearly holds. Otherwise, it contains at least a cycle. Let  $\gamma$  be an external cycle of  $\mathbf{Q}$ . Clearly, if we drop an edge of  $\gamma$  from  $\mathbf{Q}$ , the resulting graph  $\mathbf{R}$  preserves properties (i)–(iii) and therefore, by the induction hypothesis on  $|E|$ , is in  $\mathcal{P}$ . An application of rule (c) to  $\mathbf{R}$  constructs  $\mathbf{Q}$  as a member of  $\mathcal{P}$ .  $\square$

#### 4.1.1. Right-most spanning tree

Let  $\mathbf{P} = (V, E)$  be a polyominoid. We define a graph  $T = (V, F)$  as follows:

- if  $e = \{(x, y), (x + 1, y)\}$  is an edge in  $E$  then  $e$  belongs to  $F$ , i.e. each horizontal edge in  $E$  belongs to  $F$  :  $e = \{(x, y), (x + 1, y)\} \in E \implies e \in F$ ,
- if  $e = \{(x, y), (x, y + 1)\}$  belongs to  $E$  and  $e$  is not a left side edge of a cell in  $\mathbf{P}$  then  $e$  belongs to  $F$ , i.e. each vertical edge, which is not a left edge of a cell, belongs to  $F$ :  $e = \{(x, y), (x, y + 1)\} \in E \implies e \in F$  iff  $\{(x, y), (x + 1, y), (x + 1, y + 1), (x, y + 1)\}$  is not a cell in  $\mathbf{P}$ .

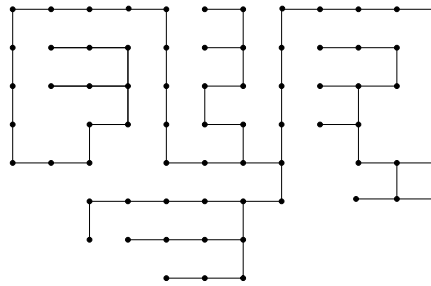


Fig. 2. Right-most spanning tree of the polyomino given in Fig. 1.

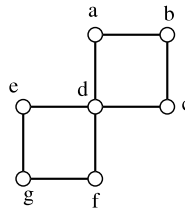


Fig. 3. A polyomino.

**Proposition 13.** *The graph  $T = (V, F)$  described above is a spanning tree of the polyomino  $P$ .*

**Proof.** We show that  $T$  spans  $P$  and that  $T$  contains no cycles, both by contradiction.

1.  $T$  spans  $P$ . Suppose the contrary. There must be at least 1 pair of vertices adjacent in  $P$  but not connected in  $T$ . Let  $\{u, v\}$  be a rightmost such pair. Since  $\{u, v\}$  is not in  $T$ ,  $\{u, v\}$  must be a vertical edge which is the leftmost edge of a cell in  $P$ . The two horizontal edges of this cell must be in  $T$ , so, since  $u$  and  $v$  are not connected in  $T$ , the two ends of the other vertical edge of this cell are not connected in  $T$ , contradicting the supposition that  $\{u, v\}$  is a rightmost pair adjacent in  $P$  but not connected in  $T$ .
2.  $T$  contains no cycles. Suppose to the contrary that  $T$  contains a cycle  $\gamma$  and let  $e = \{(x, y), (x, y + 1)\}$  be a leftmost vertical edge in  $\gamma$ . Then each of the edges  $\{(x, y), (x + 1, y)\}$ ,  $\{(x + 1, y), (x + 1, y + 1)\}$  and  $\{(x, y + 1), (x + 1, y + 1)\}$  must be either in  $\gamma$  or inside  $\gamma$  and so in  $P$ , so that  $e$  is not in  $T$ .  $\square$

**Remark 1.** The above proposition does not hold in general for finite connected sub-graphs of  $U$ : see Fig. 4.

**Definition.** The spanning tree  $T = (V, F)$  is called the *right-most spanning tree* of the polyomino  $P$ .

**Example 1.** Fig. 2 gives the right-most spanning tree of the polyomino given in Fig. 1.

We now apply our election algorithm to polyominoes with a processor at each vertex. As in the case of trees, we suppose that all processors start execution at the same time, that execution is synchronous and that communication delay can be ignored. We require that a processor knows, for each of its ports, its direction in the grid (North, South, East or West). A vertex can decide which of its incident edges are in the right-most spanning tree if it knows to which cells it belongs. Thus one exchange of messages, in which each vertex tells all its neighbours which other neighbours it has, suffices to compute this tree.

Our uniform election algorithm on polyominoes can now be simply stated as:

```

Every vertex decides which are its neighbours in the right-most spanning tree
while  $P$  is not reduced to a unique vertex
do
    • any vertex which is or becomes a leaf
      generates its lifetime according to its weight,
    • once the lifetime of a leaf has expired, it is removed
      with incident edges and its remaining neighbour in the right-most spanning
      tree collects its weight.
od
    
```

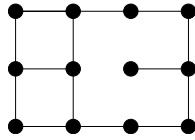


Fig. 4. A finite connected graph with holes.

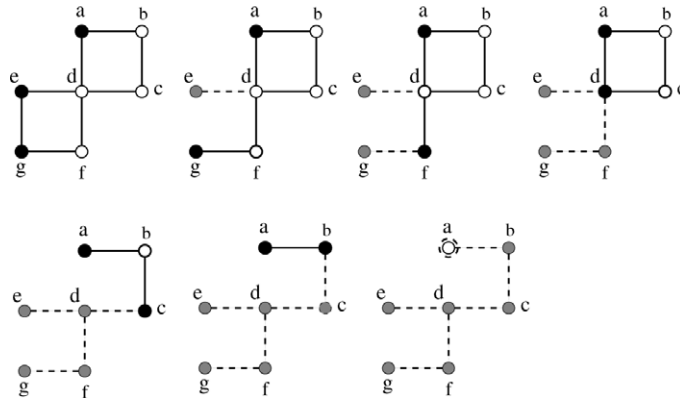


Fig. 5. Election in a polyomino graph: the black vertices are the active ones, the grey-coloured vertices and the dashed edges are those of the right-most spanning tree and the doubly circled vertex is the elected one.

Since this is an application of the tree algorithm to the right-most spanning tree, we draw two conclusions: firstly that the election is, as claimed, uniform, and secondly that the distribution of the duration is exactly the same as for trees of the same number of vertices (Fig. 5).

## 5. Conclusion and further investigations

In a recent study, Marckert, Saheb-Djahromi and Zemmari extend the algorithm to elect vertices with probabilities proportional to weights initially assigned; see [9]. The authors introduce Markovian and non-Markovian models and derive some identities.

The case of probability distributions assigned on the set of vertices in terms of their positions in graphs remains to be studied. This involves the problem of feasibility: some distributions are realisable by variants of this algorithm, while others do not seem to be. A thorough investigation in this direction remains to be carried out.

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