Optimal Bit Complexity Randomised Distributed MIS and Maximal Matching Algorithms for Anonymous Rings

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Séminaire DALGO - LIF

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Content

Context

Model

Contribution

MIS

Maximal Matching

Conclusion

Bibliography
Case study for many problems [AW04]

“rings are a convenient structure for message-passing systems and correspond to physical communication systems, for example, token rings.”

Classical Problems

- vertex colouring
- maximal independent set
- vertex cover
- maximal matching
- graph decomposition
The Problems

Maximal Independent Set (MIS)

Maximal Matching
The Network

Simple Unoriented Ring Graph

Model [Tel00]
- Message passing communication
- FIFO channels
- Reliable system
- Synchronous system

Hypotheses
- Anonymous network
- No knowledge about the size of the ring, position or distance information
- Processors distinguish channels
Theorem [Pel00]

There is no deterministic distributed algorithm for anonymous ring graphs for solving the MIS problem or the maximal matching problem assuming all vertices wake up simultaneously.

Definitions

- Probabilistic algorithm: random choices.
- Distributed probabilistic algorithm: collection of local probabilistic algorithms.
- Las Vegas Algorithm: probabilistic algorithm which terminates with a positive probability and which produces a correct result.

Remark

Anonymous ring \(\Rightarrow\) Identical local probabilistic algorithms
Complexity

Time Complexity [Pel00]

- Round for each processor:
  1. Send messages to (some) the neighbours,
  2. Receive messages from (some) the neighbours,
  3. Perform some local computations.

- Time complexity: maximum possible number of rounds needed until every node has completed its computation.

Bit complexity [KOSS06]

- Bit round: round such that each process can send/receive at most 1 bit to/from each neighbour.
- Bit complexity for \( A \): number of bit rounds to complete \( A \).
Contribution

Las Vegas distributed algorithms which compute a MIS or a maximal matching for anonymous rings.

Bit complexity and time complexity in $O(\sqrt{\log n})$ with high probability $(1 - o(n^{-1}))$.

Optimal algorithms ([FMRZ]).

Related Works


Contribution

Las Vegas distributed algorithms which compute a MIS or a maximal matching for anonymous rings.

- Bit complexity and time complexity in $O(\sqrt{\log n})$ with high probability $(1 - o(n^{-1}))$.
- Optimal algorithms ([FMRZ]).

Related Works [FMRZ, MRSDZ11, II86, KOSS06]

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**Algorithm** \textit{RingMIS}

### Local State for a vertex $v: \eta(v)$

- $1$: in the MIS (final state)
- $0$: not in the MIS (final state)
- $X_0$, $X_1$: intermediate states
- $?$: undetermined state

### Algorithm

- Initial State: every vertex is in state $?$.
- Sequence of phases \textit{DEF} denoted $(\textit{DEF})^*$:
  - Drawing ($D$),
  - Expansion ($E$),
  - Filling ($F$).
Algorithm \textit{RingMIS}

\begin{itemize}
\item \(v_2, v_3\) and \(v_4\) are in a final state with probability at least \((1/3)^7\).
\item No possible ambiguity.
\end{itemize}
Algorithm *RingMIS*

Expansion

```
0 -- 1 -- 0 -- X1 -- X0 -- ? -- ? -- ? -- ?

0 -- 1 -- 0 -- 1 -- 0 -- X1 -- X0 -- - -- -
```
### Algorithm *RingMIS*

#### Filling

<table>
<thead>
<tr>
<th>$F_0$</th>
<th>$\cdots \cdot 0 - 1 - 0 \cdot X_1 - X_0 - X_0 - X_1 \cdot 0 - 1 - 0 \cdot \cdots$</th>
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Algorithm \textit{RingMIS}

**Remarks**

- **Invariant I:**
  At the start of each stage, the ring is divided as:

  \[ ? \quad ? \quad X_0 \quad X_1 \quad 0 \quad 1 \quad 0 \quad X_1 \quad X_0 \quad ? \quad ? \]

  or

  \[ ? \quad ? \quad X_0 \quad X_1 \quad 0 \quad 1 \quad \cdots \quad 1 \quad 0 \quad X_1 \quad X_0 \quad ? \quad ? \]

  until each vertex is in final state.

- Once a final area is formed, the number of vertices in final state is increased, at each phase, by at least 4.

- The algorithm is available only for graphs with at least 7 vertices.
  For fewer than 7 vertices one can apply an alternative \(O(1)\) algorithm.
**Analysis**

**Lemma**

Let \( v \) be a vertex that enters the MIS or its complement at time \( t_0 > 0 \).
Then, for any \( t \geq 0 \), after \( t \) phases, all the vertices at distance at most \( 2t \geq 0 \) from \( v \) are in the MIS or in its complement.

**Theorem**

Algorithm *RingMIS* computes a MIS in a ring of size \( n \) w.h.p. in \( O\left(\sqrt{\log n}\right) \) rounds.

**Corollary**

Algorithm *RingMIS* computes a MIS in a ring of size \( n \) on average in \( O\left(\sqrt{\log n}\right) \) rounds.
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Algorithm *RingMM*

**Local State for a vertex *v*: **$\eta(v)$

- 1: in the Maximal Matching (final state)
- 0: not in the Maximal Matching (final state)
- $X$: intermediate states
- ?: undetermined state

Edge entering in the maximal matching: $1 \ast 1$.

**Algorithm**

As for MIS,

- Initial State: every vertex is in state ?.

- Sequence of phases $DEF$ denoted $(DEF)^*$:
  - Drawing ($D$),
  - Expansion ($E$),
  - Filling ($F$).
The pair \((Y \cdots Y)\) can be one of these patterns: \((-? \rightarrow? \rightarrow), \text{ or } (\leftarrow? \leftarrow? \leftarrow), \text{ or } (\leftarrow?-\rightarrow), (X \ast X), \text{ or } (? \rightarrow X), \text{ or } (\leftarrow?- X)\) etc.

**Remarks**

- An edge \(1 \ast 1\) is in the maximal matching and remains in it.
Algorithm \textit{RingMM}

\begin{center}
\begin{tikzpicture}

\node[draw, rounded corners] (X) at (0,0) {
\begin{tabular}{cccccccc}
1 & * & 1 & --- & X & * & X & --- \\
\end{tabular}
};
\node[draw, rounded corners] (Y) at (0,-1.5) {
\begin{tabular}{cccccccc}
1 & * & 1 & --- & 1 & * & 1 & --- \ X & * & X & --- & --- & --- & --- & --- \\
\end{tabular}
};
\draw[->] (X) -- (Y);
\end{tikzpicture}
\end{center}
Algorithm *RingMM*

| F<sub>0</sub> | \[\cdots 1 \ast 1 - X \ast X \ast X \ast X - 1 - 1 \cdots\]  
| F<sub>1</sub> | \[\cdots 1 - 1 - X \ast X - \ast X - X \ast X - 1 - 1 \cdots\]  
| F<sub>2</sub> | \[\cdots 1 \ast 1 - X \ast X - \ast X - X \ast X - 1 - 1 \cdots\]  
| F<sub>3</sub> | \[\cdots 1 - 1 - X \ast X - \ast X - X \ast X - 1 - 1 \cdots\]  
| F<sub>4</sub> | \[\cdots 1 \ast 1 - X \ast X - \ast X - X \ast X - 1 - 1 \cdots\]  
| F<sub>5</sub> | \[\cdots 1 - 1 - X \ast X - \ast X - X \ast X - 1 - 1 \cdots\]
Algorithm *RingMM*

**Remarks**

- **Invariant** $I$:

  $X \xrightarrow{*} X \xrightarrow{*} 1 \xrightarrow{*} 1 \xrightarrow{*} X \xrightarrow{*} X$

  $X \xrightarrow{*} X \xrightarrow{*} 1 \xrightarrow{*} 1 \xrightarrow{*} [((1 \times 1) + ((1 \times 1) + (0 - 1 \times 1)))] - 1 \xrightarrow{*} 1 \xrightarrow{*} X \xrightarrow{*} X$

- The number of vertices in final state is increased, at each phase, by at least 4.
- The algorithm is available only for graphs with at least 6 vertices.
Analysis

Theorem

Algorithm \textit{RingMM} computes a maximal matching in a ring of size \( n \) in \( O(\sqrt{\log n}) \) rounds w.h.p. and on average.
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Akka Zemmari
Optimal Bit Complexity, MIS, MM, Anonymous Rings
Generalisation

2-MIS

A 2-independent set is a subset $K$ of $V$ such that the distance between any two vertices of $K$ is at least 3. A 2-MIS is s.t. any vertex of $G$ is in $K$ or at distance 1 or 2 of an element of $K$.

Filling

$F_0 : 100X_1X_0X_0X_0X_1001 \rightarrow 100100010001$
$F_1 : 100X_1X_0X_0?X_0X_0X_1001 \rightarrow 1001001001001$
$F_2 : 100X_1X_0X_0??X_0X_0X_1001 \rightarrow 0100100010010$
$F_3 : 100X_1X_0X_0???X_0X_0X_1001 \rightarrow 10010010010001001$
$F_4 : 100X_1X_0X_0?????X_0X_0X_1001 \rightarrow 1001001001001001$
$F_5 : 100X_1X_0X_0?????X_0X_0X_1001 \rightarrow 1001000100010001$
$F_6 : 100X_1X_0X_0??????X_0X_0X_1001 \rightarrow 10010010001001$
$F_7 : 100X_1X_0X_0??????X_0X_0X_1001 \rightarrow 1001001001001001$
$F_8 : 100X_1X_0X_0????????X_0X_0X_1001 \rightarrow 1001000010010001$
$F_9 : 100X_1X_0X_0????????X_0X_0X_1001 \rightarrow 1001000100010001$
$F_{10} : 100X_1X_0X_0????????X_0X_0X_1001 \rightarrow 100100001000010001$

Drawing

Expansion

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
Open Questions

1. An oriented ring seems to be more constrained than a non-oriented ring but our results suggest that the orientation of the ring helps the design of distributed algorithms such as MIS or maximal matching. Is this due to the fact that the orientation already breaks the symmetry of the ring?

2. If IDs are allowed to the nodes (or knowledge of an upper-bound of the size of the ring is given) how significantly will such hypotheses help the design of MIS/maximal matching algorithms?

3. Can we apply the same schema to design algorithms:
   - to solve the same problems in other classes of graphs?
   - to solve other problems?

4. Can we do the same in other models, e.g., the beeping model?
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Algorithm \textit{RingMIS} computes a MIS in a ring of size $n$ w.h.p. in $O\left(\sqrt{\log n}\right)$ rounds.

For any $u \in V$, and any $t > 1$, let $P(u, t)$ denote the path of length $2t + 1$ centred on $u$ and for any $i \in \{1, \cdots, t\}$, let $E(u, i)$ (resp. $\overline{E}(u, i)$) denote the event “a vertex in the path $P(u, 2(t - i + 1))$ enters the set in phase $i$” (resp. “no vertex in the path $P(u, 2(t - i + 1))$ enters the set in phase $i$”).

Let $v$ be a vertex which is not yet in the MIS nor in its complement. By Lemma 22, $v$ will enter the set or its complement within $t$ phases unless none of the events $E(v, i)$, for any $i \in \{1, \cdots, t\}$ happen.

On the other hand, any vertex $u$ in $P(v, 2t)$ distant 6 or more from all determined areas will enter the set in a phase if it is at the centre of the pattern in Figure 22, and so with probability at least $\rho = \left(\frac{1}{3}\right)^7$ independently of what happens outside these seven vertices.

Thus : $P\left(E(v, i)\right) \leq (1 - \rho) \frac{4(t-i+1)}{7}$.

Note that this is an upper-bound, and any more accurate (and tedious) computation will just affect the constants in the following formulas.

Now, if we denote $p > t(v)$ the probability that $v$ does not enter the set, or its complement, within $t$ phases, then we have :

\begin{align*}
  p > t(v) & \leq \prod_{i=1}^{t} P\left(E(v, i)\right) \\
           & \leq \prod_{i=1}^{t} (1 - \rho) \frac{4(t-i+1)}{7} \\
           & = (1 - \rho) \frac{2}{7} (t^2 + t) < (1 - \rho) \frac{2}{7} t^2.
\end{align*}

Adding over all vertices we see that the probability that some vertex remains undetermined after $124\sqrt{\log n}$ phases is $o\left(\frac{1}{n^{2}}\right)$ as $n \to \infty$.