

Hurwitz groups as monodromy groups of dessins: several examples

Gareth A. Jones and Alexander K. Zvonkin

December 31, 2016

Abstract

We present a number of examples to illustrate the use of small quotient dessins as substitutes for their often much larger and more complicated Galois (minimal regular) covers. In doing so we employ several useful group-theoretic techniques, such as the Frobenius character formula for counting triples in a finite group, pointing out some common traps and misconceptions associated with them. Although our examples are all chosen from Hurwitz curves and groups, they are relevant to dessins of any type.

MSC Classification: primary 14H57, secondary 20B25.

1 Introduction

When Grothendieck wrote his *Esquisse d'un Programme*, he famously expressed his delight that basic objects such as plane trees, and other simple drawings, could encode highly sophisticated mathematical structures, namely algebraic curves defined over number fields. Indeed, such was his pleasure in the elegance and power of this data compression that he used the childish-looking nature of these sketches in giving them the name of *dessins d'enfants*, which has now come to represent a vast mathematical theory (see [16, 22, 23] for instance).

The recent proof by González-Diez and Jaikin-Zapirain [17] that regular dessins provide a faithful representation of the absolute Galois group has focussed attention even more closely than before on these highly symmetric objects, those dessins for which the associated Belyĭ function is a regular covering. Every dessin \mathcal{D} is the quotient of some regular dessin $\mathcal{R} = \tilde{\mathcal{D}}/H$ where $\tilde{\mathcal{D}}$ is the Galois (minimal regular) cover of \mathcal{D} , by a subgroup H of the automorphism group G of \mathcal{R} ; this latter group is also realised as the

monodromy group of \mathcal{D} , or equivalently of its associated Belyĭ function, regarded as a branched covering of the sphere; this is a permutation group acting transitively on the fibre over a base-point, with H as a point-stabiliser.

A common situation is that in which a dessin \mathcal{D} , which is ‘small’ in some sense (having low genus, or few edges, for example) is used in this way as a substitute for a much larger regular dessin \mathcal{R} . The latter is uniquely determined as the Galois cover of \mathcal{D} provided G acts faithfully on the cosets of H , that is, the core of H in G is trivial, in which case we will call \mathcal{D} a *faithful* quotient of \mathcal{R} . For instance, one can see this idea in action, even before the era of dessins, in the use by Conder [5, 6] of Graham Higman’s technique of ‘sewing together coset diagrams’ to realise large alternating and symmetric groups as quotients of triangle groups. Similarly, Jendroĭ, Nedela and Škoviera [21] have used this idea to obtain new results and new proofs of old ones for graphs and for maps on surfaces.

Our aim in this paper is to explore this relationship between regular dessins \mathcal{R} and their faithful quotients \mathcal{D} , with its occasional unexpected subtleties. We do this through a series of examples, chosen with the secondary aim of illustrating some useful techniques from group theory (finite, discrete and computational) for enumerating, constructing and classifying dessins of various types associated with specific groups. Our examples are all Hurwitz groups and curves, those attaining Hurwitz’s upper bound [20] of $84(g - 1)$ for the number of automorphisms of a curve of genus $g \geq 2$. This choice is purely for personal and historical interest since the corresponding dessins are extremely rare. (Conder [10] listed all the regular dessins of genus 2 to 101; their total number is 19 029, and only seven of them attain the Hurwitz bound.) Nevertheless, the methods we describe can in fact be applied to dessins of any type.

There are no new theorems in this paper. Indeed, the only proof we offer is really a disproof, of the occasional and erroneous assertion that the Hurwitz group of genus 17 is isomorphic to the affine group $\text{AGL}_3(2)$. These two groups do indeed look very similar, both being extensions of an elementary abelian normal subgroup of order 8 by $\text{GL}_3(2)$; however, the affine group is a split extension, while the Hurwitz group is not. We also exhibit certain traps to avoid while using irreducible characters and the Frobenius formula to count the number of dessins with a given monodromy group.

We will use many well-known facts concerning the inner structure of various finite groups. Unfortunately (and obviously), we are unable to fill in all the necessary details, since otherwise the paper would be enlarged *ad infinitum*. In a few occasions we supply the reader with some details but in most cases he or she should consult other sources on group theory.

Acknowledgement The authors gratefully thank Jarke van Wijk for giving permission to use his beautiful pictures of maps (Figures 6 and 8). The second author is partially supported by the research grant GRAAL ANR-14-CE25-0014.

2 Background details

2.1 Dessins

By Belyĭ's Theorem, a compact Riemann surface \mathcal{S} , regarded as a complex projective algebraic curve, is defined over a number field if and only if it admits a Belyĭ function, a non-constant meromorphic function $f : \mathcal{S} \rightarrow \mathbb{P}^1(\mathbb{C})$ ramified over at most three points (which one can, without loss of generality, take to be $0, 1$ and ∞). The *monodromy group* G of the Belyĭ pair (\mathcal{S}, f) is the monodromy group of the covering of the thrice-punctured sphere $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ induced by f , that is, the permutation group on the sheets (more precisely on the fibre over a base-point) induced by unique lifting of closed paths. If f has degree n then G is a transitive subgroup of the symmetric group S_n , generated by the local monodromy permutations x, y and z around $0, 1$ and ∞ , satisfying $xyz = 1$. By the Riemann Existence Theorem, these three permutations define \mathcal{S} and f up to isomorphism.

Following Grothendieck, one can represent f by means of a bicoloured map on \mathcal{S} , called a dessin \mathcal{D} , with the fibres over 0 and 1 as the black and white vertices, and the unit interval lifting to the n edges, one on each sheet of the covering. Then the permutations x and y represent the rotations of the edges around their incident black and white vertices, while the permutation $z = (xy)^{-1}$ rotates edges, two steps at a time, around incident faces.

The *automorphism group* $\text{Aut}(\mathcal{D})$ of a dessin \mathcal{D} is the group of covering transformations of f , or equivalently the centraliser of G in S_n . A dessin \mathcal{D} is *regular* if the covering is regular, that is, if $\text{Aut}(\mathcal{D})$ acts transitively on the edges, in which case $\text{Aut}(\mathcal{D}) \cong G$.

The *type* of a dessin, or of a triple (x, y, z) , is the triple of orders of x, y and z . Rather more information is conveyed by the *passport*, by which we mean the ordered triple of partitions of n giving the cycle-structures of x, y and z . These correspond to the conjugacy classes of S_n containing these permutations, but in some cases we will give more precise information by referring to the passport as the triple of their conjugacy classes in the monodromy group G which they generate.

In cases where $y^2 = 1$ the white vertices of the dessin are redundant and may be omitted, so that fixed points of y are now represented by free edges

(one end of which is a vertex while the other remains free), and 2-cycles by traditional edges or loops. The edges of the original dessin now correspond to the half-edges of the resulting map, and the monodromy group of the dessin can be identified with that of the map, now permuting half-edges. In some of our examples, we will use this simplification without further comment.

For further background reading we suggest [16, 22] or [23].

2.2 The Frobenius formula

For a given finite group G , the regular dessins of type (p, q, r) with automorphism group G are in bijective correspondence with the torsion-free normal subgroups N of the triangle group

$$\Delta = \Delta(p, q, r) = \langle X, Y, Z \mid X^p = Y^q = Z^r = XYZ = 1 \rangle$$

with $\Delta/N \cong G$. These subgroups N correspond bijectively to the orbits of $\text{Aut}(G)$ on generating triples (x, y, z) of elements of orders p, q and r in G satisfying $xyz = 1$. Since the action of $\text{Aut}(G)$ on generating triples is semiregular, meaning that only the identity element has fixed points, the number of orbits, and hence of dessins, is equal to the number of such triples divided by $|\text{Aut}(G)|$. The first step in the calculation of this number is to use the following classical result [15] (for a modern treatment see [28, Ch. 7]):

Theorem 2.1 (Frobenius) *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be conjugacy classes in a finite group G . Then the number of solutions in G of the equation $xyz = 1$, where $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, is given by the formula*

$$\frac{|\mathcal{X}| \cdot |\mathcal{Y}| \cdot |\mathcal{Z}|}{|G|} \sum_{\chi} \frac{\chi(x)\chi(y)\chi(z)}{\chi(1)},$$

where the sum is over all irreducible complex characters χ of G .

Remark 2.2 The first part of this formula, omitting the character sum, can be regarded as a naive guess for the number of solutions, assuming that the values of xyz are evenly distributed over the elements of G . Of course, in general they are not, and the character sum can be regarded as a correction term, taking into account the particular structure of the chosen group G . In many cases the character sum is dominated by the contribution, equal to 1, from the principal character, in which case the naive guess is not far from the correct answer.

Remark 2.3 This is a particular case of a more general formula for the number of solutions of $x_1 \dots x_k = 1$ in G , where each x_i is chosen from a specific conjugacy class. The only part of the generalisation which is not obvious is the denominator in the character sum, which is $\chi(1)^{k-2}$. For this, and other similarly useful formulae, see [28, Ch. 7].

Remark 2.4 The ATLAS of Finite Groups [11] contains character tables of many finite simple groups, and of other associated groups. Maple can compute characters of symmetric groups, and GAP can compute character tables of arbitrary not-so-big groups.

A conjugacy class \mathcal{X} in G has order

$$|\mathcal{X}| = \frac{|G|}{|C_G(x)|}$$

where $C_G(x)$ is the centraliser in G of an element $x \in \mathcal{X}$, and similarly, for \mathcal{Y} and \mathcal{Z} . The ATLAS gives orders of centralisers, rather than those of conjugacy classes.

The Frobenius formula gives us the number of triples in G with passport $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$. To obtain the number of triples of type (p, q, r) , one simply takes the sum of these numbers over all triples of conjugacy classes \mathcal{X}, \mathcal{Y} and \mathcal{Z} consisting of elements of orders p, q and r .

Suppose we are given a triple $\pi = (\lambda, \mu, \nu)$ of partitions of number n , and we would like to know the number of dessins with the passport π . Enumerative combinatorics gives an explicit answer only in very specific cases, such as, for example, for plane trees. In more complicated cases, an invaluable, and in most cases the only source of information is the Frobenius formula. But it must be used with care.

Let us discuss first the case of the symmetric group $G = S_n$. Since in this case a cycle-structure uniquely determines the corresponding conjugacy class, it seems that we can then apply the Frobenius formula directly. There is, however, a trap to avoid: a triple of permutations (x, y, z) with cycle-structures (λ, μ, ν) does not necessarily generate a transitive subgroup of S_n . We must find a way to eliminate these non-transitive solutions.

The next difficulty to resolve is the fact that the edges of the same dessin \mathcal{D} may be labelled in many different ways. To be specific, the number of labellings is $n!/\text{Aut}(\mathcal{D})$. Therefore it is reasonable to divide the number of triples of permutations by $n!$: in this way we will get the “number” of non-isomorphic dessins, each one of them being counted with the weight $1/\text{Aut}(\mathcal{D})$.

Example 2.5 Let us take $n = 12$ and look for those maps with the passport $\pi = (6^1 3^1 2^1 1^1, 2^6, 6^1 3^1 2^1 1^1)$. Computing the corresponding characters using Maple, applying the Frobenius formula and dividing the result by $12!$ we get $19\frac{1}{2}$. In fact, the correct answer is 18. Two more “maps” are non-connected, and one of them has a non-trivial automorphism of order 2, so that its contribution to the sum is $1/2$.

Now suppose that we work inside a group G different from S_n . Then, the next difficulty arises: we can come across a cycle-structure corresponding to several different conjugacy classes. For example, in the group $\mathrm{PSL}_2(27)$ (in its natural representation of degree $n = 28$), which will be treated in Section 7, there are two conjugacy classes $\mathcal{X}_1, \mathcal{X}_2$ of elements with cycle-structure $3^9 1^1$, one conjugacy class \mathcal{Y} of elements with cycle-structure 2^{14} , and three conjugacy classes $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ with cycle-structure 7^4 . Thus, looking for dessins with passport $\pi = (3^9 1^1, 2^{14}, 7^4)$ we must take into account six possible combinations of conjugacy classes.

Return now to the problem with many possible labellings of edges of the same dessin. If we want to stay inside a given group G and not to be sent to one of its conjugate copies, division by $n!$ would be not a good idea. We are tempted to divide the number of triples (x, y, z) by $|G|$. But there is a trap which awaits us here, and it is much more subtle than the previous ones! A significant part of our paper is devoted to untangling the complications arising in this case. See in this respect the discussions in §3.4 and Sections 5 and 7.

Finally, a triple (x, y, z) of permutations belonging to a group G may generate not the entire group G but only a proper subgroup. If we are interested in triples generating G itself we need to do more work depending on the particular group and its structure. One possible technique for doing this is discussed in the next section.

2.3 Möbius inversion in groups

Instead of counting all triples of a given type (p, q, r) in G , we need to count *generating* triples of that type. Occasionally it is obvious which triples generate G (all of them in some cases), but in general a more systematic method, based on P. Hall’s theory of Möbius inversion in groups [18], is available.

Given a type (p, q, r) , let $\sigma(H)$ denote the number of triples of that type in each subgroup $H \leq G$, and let $\phi(H)$ denote the number of them which

generate H . Since each triple in G generates a unique subgroup, we have

$$\sigma(G) = \sum_{H \leq G} \phi(H). \quad (1)$$

As shown by Hall, equation (1) can be inverted, reversing the roles of the functions σ and ϕ , to give

$$\phi(G) = \sum_{H \leq G} \mu_G(H) \sigma(H), \quad (2)$$

where μ_G is the Möbius function on the lattice of subgroups of G , recursively defined by

$$\sum_{K \geq H} \mu_G(K) = \delta_{H,G}, \quad (3)$$

with δ denoting the Kronecker delta.

Although for many finite groups G , such as all but a few of the alternating and symmetric groups, the subgroup lattice is too complicated to allow μ_G to be calculated, this has been achieved for several classes of groups. For instance Hall dealt with nilpotent groups and the groups $\mathrm{PSL}_2(p)$ (p prime) in [18], and the latter calculation was extended by Downs [12] to $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ for all prime powers q . He and the first author have recently dealt with the Suzuki groups in [13], and Pierro with the ‘small’ Ree groups in [27]. In many cases (for instance, if H is not an intersection of maximal subgroups of G) we find that $\mu_G(H) = 0$, so such subgroups H can be omitted from the summation in equation (2). By using the Frobenius formula to evaluate $\sigma(H)$ for subgroups $H \leq G$ with $\mu_G(H) \neq 0$ one can determine $\phi(G)$, and hence obtain the number $\phi(G)/|\mathrm{Aut}(G)|$ of regular dessins of a given type with automorphism group G .

The examples we present later in this paper do not in fact require Möbius inversion, since in most cases it is easy to see that the relevant triples generate the whole group. However, there are more complicated cases, such as those considered in [13, 27], where it cannot be avoided.

Hall’s theory has much wider applications than that of counting triples described here. In its most general form, it can be used to count the normal subgroups of *any* finitely generated group with a given finite quotient group. For instance, Hall showed that the free group of rank 2 has 19 normal subgroups N with quotient isomorphic to the alternating group A_5 ; it follows that there are 19 regular dessins \mathcal{R} with $\mathrm{Aut}(\mathcal{R}) \cong A_5$ (described in [1]), and since this group has eight faithful transitive permutation representations, there are $19 \cdot 8 = 152$ dessins \mathcal{D} with monodromy group A_5 .

2.4 Hurwitz groups and surfaces

We have chosen all our examples from the historically important Hurwitz groups. A *Hurwitz group* G is a finite group which attains Hurwitz's upper bound [20] of $84(g-1)$ for the order of the automorphism group of a compact Riemann surface of genus $g \geq 2$. Equivalently, G is a non-trivial finite quotient Δ/N of the triangle group $\Delta = \Delta(2, 3, 7)$, acting as the automorphism group of the Riemann surface $\mathcal{S} = \mathbb{H}/N$ (called a *Hurwitz surface* or *Hurwitz curve*), where \mathbb{H} is the hyperbolic plane. As such, G is the automorphism group of a regular dessin \mathcal{R} on \mathcal{S} , called a *Hurwitz dessin*; regarded as a map, this is a trivalent tessellation by heptagons, or its dual, a 7-valent triangulation. The number of such surfaces and dessins associated with G , up to isomorphism, is equal to the number of normal subgroups N of Δ with $\Delta/N \cong G$. Note that since its three periods are mutually coprime, Δ is a perfect group (that is, it has no non-trivial abelian quotient groups), and hence so is every Hurwitz group G . In particular, this implies that a Hurwitz group cannot be solvable.

The choice of the parameters $(p, q, r) = (2, 3, 7)$ is explained as follows. A regular dessin of type (p, q, r) with an automorphism group G of order $|G| = n$ has n/p black vertices of valency p , n/q white vertices of valency q , n/r faces of valency r , and n edges. Its Euler characteristic is thus equal to

$$2 - 2g = \frac{n}{p} + \frac{n}{q} + \frac{n}{r} - n.$$

In order to get a negative Euler characteristic with the least possible genus (when n is given) we should have the sum $1/p + 1/q + 1/r$ less than 1 but as close to 1 as possible. The triple $(2, 3, 7)$ obviously provides the answer. Then

$$2 - 2g = n \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 \right) = -\frac{n}{42}$$

which gives $n = 84(g - 1)$.

Remark 2.6 For convenience of drawing, we will regard Δ as the triangle group $\Delta(3, 2, 7)$, so that \mathcal{R} and its quotient dessins \mathcal{D} have type $(3, 2, 7)$ rather than $(2, 3, 7)$. Writing the periods in that order means that the black and white vertices have valencies dividing 3 and 2 respectively; this allows us to represent dessins more simply as uncoloured maps, by deleting the now redundant white vertices (so that those of valency 1 give rise to free edges). However, from time to time we will show white vertices explicitly.

Many non-abelian finite simple groups are now known to be (or known not to be) Hurwitz groups: see the surveys by Conder [8, 9]. For example, we have the following theorem [26]:

Theorem 2.7 (Macbeath) *The group $G = \mathrm{PSL}_2(q)$ is a Hurwitz group if and only if one of the following holds:*

1. $q = 7$, or
2. q is a prime $p \equiv \pm 1 \pmod{7}$, or
3. $q = p^3$ for some prime $p \equiv \pm 2$ or $\pm 3 \pmod{7}$.

In cases (1) and (3) the Hurwitz surface and dessin associated with G are unique, but in case (2) there are three of each, corresponding to three normal subgroups N of Δ with quotient G .

Of course, by Dirichlet's theorem on primes in arithmetic progressions there are infinitely many examples satisfying each of the congruences in cases (2) and (3).

3 Klein's curve

In this section we take \mathcal{R} to be the regular dessin of type $(3, 2, 7)$ on Klein's quartic curve

$$u^3v + v^3w + w^3u = 0,$$

the Hurwitz surface of least genus, namely $g = 3$. The automorphism group G of both \mathcal{R} and the curve is the smallest Hurwitz group, namely $\mathrm{GL}_3(2) = \mathrm{SL}_3(2) = \mathrm{PSL}_3(2)$, of order 168, also isomorphic to $\mathrm{PSL}_2(7)$. We will try to represent \mathcal{R} as the minimal regular cover $\tilde{\mathcal{D}}$ of a smaller dessin $\mathcal{D} = \mathcal{R}/H$ where $H \leq G$.

3.1 $\mathrm{PSL}_3(2)$

Since G is simple, it acts faithfully on the cosets of any proper subgroup. The subgroups of G are well-known, and the smallest index of any proper subgroup H is 7, with H , the stabiliser of a point in the Fano plane $\Pi := \mathbb{P}^2(2)$, isomorphic to the symmetric group S_4 . We therefore first look for quotient dessins $\mathcal{D} = \mathcal{R}/H$ of degree 7, arising from the action of G as the automorphism group of Π .

Let us construct Π by using the *difference set* $\{0, 1, 3\}$ (equivalently, of the set $\{1, 2, 4\}$ of quadratic residues) in the additive group of the field \mathbb{F}_7 :

every non-zero residue modulo 7 appears exactly once as the difference of two elements of this set. Then the lines are the translates of this set, see Figure 1.

0	1	3
1	2	4
2	3	5
3	4	6
4	5	0
5	6	1
6	0	2

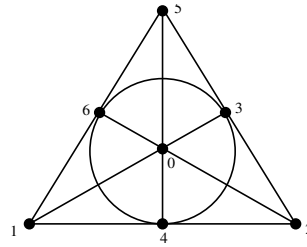


Figure 1: The Fano plane.

By construction, this geometry has the following automorphism:

$$\gamma = (0, 1, 2, 3, 4, 5, 6) : a \mapsto a + 1 \pmod{7}.$$

Consider a clockwise rotation of our figure through 120° . The point 0 is fixed, so we get the permutation x which we will use as a black vertex permutation of a dessin we are looking for:

$$x = (1, 5, 2)(3, 4, 6).$$

This permutation gives us the left tree in Figure 2. The permutation z defining the (only) face in this case is equal to

$$z = \gamma^3 = (0, 3, 6, 2, 5, 1, 4) : a \mapsto a + 3 \pmod{7}.$$

The tree on the right in the same figure is given by permutations x^{-1} and $z^{-1} = \gamma^4$.

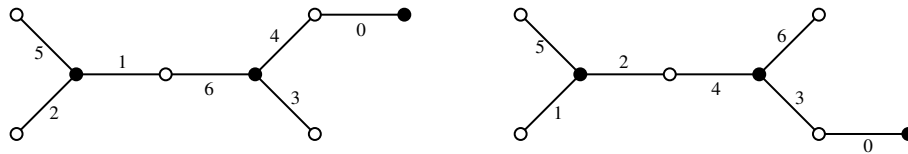


Figure 2: Trees providing a $(3, 2, 7)$ -generation of the group $\mathrm{PSL}_3(2)$.

The involutions which correspond to the white vertices may be interpreted as mirror symmetries of our geometry with respect to a line. Indeed,

the involution $y = (0, 4)(1, 6)$, corresponding to the left tree, preserves point-wise the line $\{2, 3, 5\}$ and preserves set-wise the lines $\{5, 0, 4\}$ and $\{5, 1, 6\}$. Hence, it is the “mirror symmetry” with respect to the line $\{2, 3, 5\}$. Analogously, the involution of the right tree is the symmetry with respect to the line $\{5, 6, 1\}$.

Remark 3.1 We did not prove that the monodromy group G of the trees is indeed the whole group $\mathrm{PSL}_3(2)$ and not a proper subgroup. However, G clearly has order divisible by $3 \cdot 2 \cdot 7 = 42$, so it has index $m \leq 4$ in $\mathrm{PSL}_3(2)$. Hence its core (intersection of all conjugates) is a normal subgroup of index at most $4!$, since it is the kernel of the homomorphism $\mathrm{PSL}_3(2) \rightarrow S_m$ induced by the action of $\mathrm{PSL}_3(2)$ on the cosets of G . Since $\mathrm{PSL}_3(2)$ is simple and of order greater than $4!$, the core, and hence also G , must be the whole group.

Remark 3.2 It is interesting to note that Klein’s curve is a remarkable (and complicated) object. A book [24] of 340 pages has been devoted to the study of various properties of this curve. However, our simple picture *proves the existence* of such a curve.

3.2 Character table and Frobenius’s formula for $\mathrm{PSL}_3(2)$

The above construction is consistent with the information provided to us by the character table of the group $\mathrm{PSL}_3(2)$. The table, computed by GAP, is shown in Table 1. (It can also be found in the ATLAS [11], where this group is denoted by $L_2(7)$.)

The six conjugacy classes of elements of the group are denoted in the table by $1a$, $2a$, etc. The notation used in the ATLAS, which we adopt below, is $1A, 2A, \dots$. The first digit gives the order of the elements of the class. We see, for example, that there are two conjugacy classes $7A$ and $7B$ of elements of order 7. The notation $X.1, X.2$, etc. is used for the irreducible characters; we will use more standard ATLAS notation χ_1, χ_2, \dots for characters. A dot in the table means zero; $E(7)$ is the primitive 7th root of unity; finally, $/A$ means the complex conjugate of A .

The character of the permutation representation of degree 7 is $\chi_1 + \chi_4$; its value on a given class is the number of fixed points. We see that for the class $2A$ it is equal to 3, thus giving us the cycle-length partition $2^2 1^3$; for $3A$ this value is 1, which gives us the partition $3^2 1^1$; and for the classes $7A$ and $7B$ the number of fixed points is, naturally, zero.

The sizes of the classes $3A$, $2A$ and $7A$, also computed by GAP, are, respectively, 56, 21 and 24. Therefore, the Frobenius formula for the passport

	1a	2a	4a	3a	7a	7b
X.1	1	1	1	1	1	1
X.2	3	-1	1	.	A	/A
X.3	3	-1	1	.	/A	A
X.4	6	2	.	.	-1	-1
X.5	7	-1	-1	1	.	.
X.6	8	.	.	-1	1	1

$$\begin{aligned}
A &= E(7)^{\wedge 3} + E(7)^{\wedge 5} + E(7)^{\wedge 6} \\
&= (-1 - \text{Sqrt}(-7))/2 = -1 - b7
\end{aligned}$$

Table 1: Character table of the group $\text{PSL}_3(2) \cong \text{PSL}_2(7)$ computed by GAP.

$(3A, 2A, 7A)$ gives us 168 triples (x, y, z) with entries in these classes and $xyz = 1$. We obtain the same number for the passport $(3A, 2A, 7B)$, so the total number of triples of type $(3, 2, 7)$ is $336 = 2|G|$. As shown in Remark 3.1 they all generate G . Now $\text{Aut}(G)$ is an extension of G by a cyclic group C_2 , induced by duality of the plane Π . (Alternatively, if we identify G with $\text{PSL}_2(7)$ as in Section 3.3, then $\text{Aut}(G)$ can be identified with $\text{PGL}_2(7)$; see Section 3.4 for more details.) Dividing by $|\text{Aut}(G)| = 2|G|$ we see that there is one regular dessin \mathcal{R} of type $(3, 2, 7)$ with automorphism group G .

Finally, the presence of $\sqrt{-7}$ in the character table suggests that the trees, being considered as *dessins d'enfants*, should be defined over the field $\mathbb{Q}(\sqrt{-7})$. And, indeed, the computation of Belyı̄ functions confirms this hypothesis:

$$f(t) = K \cdot (t^2 + 7a)^3 (t - 7),$$

$$f(t) - 1 = K \cdot (t^2 - 6t + a)^2 (t^3 + 5t^2 + (19a + 24)t + (83a + 108)),$$

where

$$K = -\frac{1}{2^6 3^3 (7a + 17)},$$

and a is a root of

$$a^2 + 3a + 4,$$

that is,

$$a = -\frac{3}{2} \pm \frac{1}{2} \sqrt{-7}.$$

Choosing one of the two values of a we get one of the two trees. Namely, its two black vertices of valency 3 are roots of $t^2 + 7a$; its only black vertex of

valency 1 lies at the point $t = 7$; its two white vertices of valency 2 are roots of $t^2 - 6t + a$; and, finally, its three white vertices of valency 1 are roots of $t^3 + 5t^2 + (19a + 24)t + (83a + 108)$. Being a polynomial, f has a pole of multiplicity 7 at infinity: it corresponds to the (only) face of valency 7. For any $s \notin \{0, 1, \infty\}$ the equation $f(t) = s$ does not have multiple roots.

An interesting observation is that the cubic factor

$$P(t) = t^3 + 5t^2 + (19a + 24)t + (83a + 108)$$

in the function $f - 1$ factorizes over the field $\mathbb{Q}(\sqrt{-7})$. Namely, $P = Q \cdot R$ where

$$Q(t) = t^2 + (1 + \sqrt{-7})t + \frac{-31 + 13\sqrt{-7}}{2}, \quad R(t) = t + (4 - \sqrt{-7}).$$

Thus, one of the three white vertices of valency 1 is separated from the other two: it does not belong to the same Galois orbit. This vertex is shown in Figure 3.

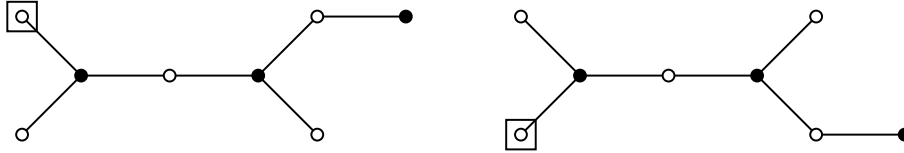


Figure 3: White vertex of valency 1 which is not Galois-conjugate to the other two.

3.3 $\mathrm{PSL}_2(7)$

Ever since the time of Galois it has been known that $\mathrm{PSL}_3(2)$ is isomorphic (as an abstract group) to $\mathrm{PSL}_2(7)$: the former group acts on the seven points of the Fano plane Π , whereas the latter acts on the eight points of the projective line $\mathbb{P}^1(7)$ over the field \mathbb{F}_7 , that is, on the set

$$\{0, 1, 2, 3, 4, 5, 6, \infty\}.$$

The character of this permutation representation is $\chi_1 + \chi_5$, see Table 1. Therefore, the permutations in the class $3A$ have two fixed points, those in the class $2A$ have none, and for the classes $7A$ and $7B$ the number of fixed points is one. We thus obtain the passport $(3^2 1^2, 2^4, 7^1 1^1)$. It is easy to see that there exists only one map with this passport: it is shown in Figure 4.

Once again, we can only marvel at the fact that this simple dessin provides us a $(3, 2, 7)$ -presentation of the group $\mathrm{PSL}_2(7)$ and thus, in the bud, with all the information we need in order to construct the Klein curve.

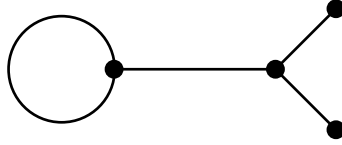


Figure 4: A map with the monodromy group $\mathrm{PSL}_2(7)$. White vertices in the middle of the edges are implicit.

Labels shown in Figure 5 (left) give the following permutations obviously belonging to $\mathrm{PSL}_2(7)$:

$$y : a \mapsto -\frac{1}{a}, \quad z : a \mapsto a + 1,$$

while x can be easily computed using the relation $xyz = 1$:

$$x = (yz)^{-1} : a \mapsto -\frac{1}{a-1}.$$

Remark 3.3 We multiply permutations from left to right, as is usual in symbolic calculation systems. But if we want, in our case, to represent the above fractional linear functions x, y, z as 2×2 -matrices X, Y, Z , then these matrices should be multiplied in the inverse order: $xyz = 1$ but $ZYX = I$.

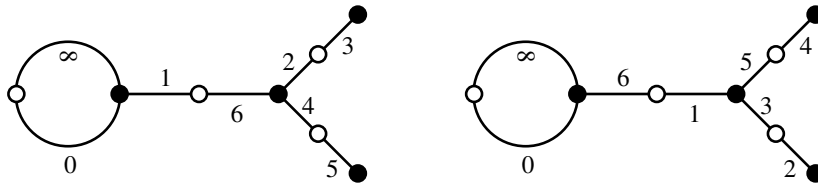


Figure 5: $(3, 2, 7)$ -generators of the group $\mathrm{PSL}_2(7)$. In the right map, as compared with the left one, the permutation $z \in 7A$ is replaced with $z^{-1} \in 7B$. Notice that the involution y remains the same.

3.4 Uniqueness of quotients

Why, in the case of $\mathrm{PSL}_2(7)$, is the quotient dessin unique (and defined over \mathbb{Q}), whereas in the case of $\mathrm{PSL}_3(2)$ we obtain a chiral pair defined over $\mathbb{Q}(\sqrt{-7})$? The answer is that G has only one conjugacy class of subgroups H of index 8, but two of index 7, and non-conjugate subgroups give non-isomorphic quotients. The subgroups of index 7 are the stabilisers of points or of lines in the Fano plane Π , forming two conjugacy classes of size 7, all isomorphic to S_4 . These two classes are transposed by the outer automorphism group $\mathrm{Out}(G)$ (corresponding to the duality of Π), just as this group transposes the conjugacy classes $7A$ and $7B$ of elements of order 7 in G . We therefore have four possible choices when forming quotient dessins of degree 7: there are two possibilities (up to conjugacy) for the subgroup H , and two (again up to conjugacy) for the element z . Since $\mathrm{Out}(G)$ transposes the two choices in each case, it has two orbits on these four possibilities, corresponding to two non-isomorphic dessins, transposed by changing our choice for H or for z (but not both!). In the case of quotients of degree 8, however, there is a single conjugacy class of subgroups H of this index, namely the stabilisers of points in $\mathbb{P}^1(7)$; our two possible choices for the class of z are transposed by $\mathrm{Out}(G)$, so they lead to isomorphic dessins.

In general, the number of faithful quotients of degree n of a regular map \mathcal{R} is the number of conjugacy classes of subgroups of index n in $G = \mathrm{Aut}(\mathcal{R})$ with trivial core. Of course, when $n = |G|$ there is a unique quotient, namely \mathcal{R} itself, corresponding to the identity subgroup.

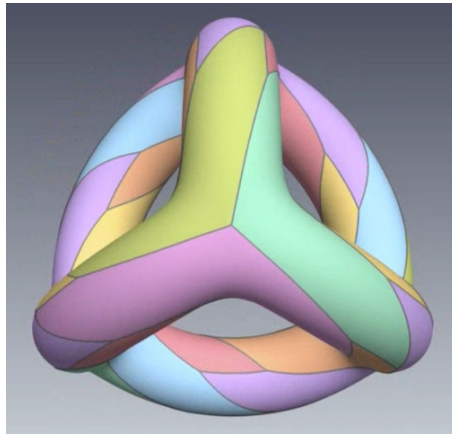


Figure 6: The Klein map of genus 3. The author of the picture is Jarke van Wijk.

We finish this section with a picture of the Klein map on the surface of genus 3. The map contains 24 heptagonal faces and 56 vertices of degree 3. The picture, see Figure 6, was created by Jarke van Wijk from the Eindhoven University of Technology.

4 The Fricke-Macbeath curve

In 1899, in the same issue of the journal “Mathematische Annalen”, two papers were published. In the first one [2], by Burnside, it was shown that the simple group $\mathrm{PSL}_2(8)$ has a $(3, 2, 7)$ -presentation; according to a tradition of that era, no motivation for this result was given. In the second paper [14], Fricke constructed a Riemann surface of genus 7 with the automorphism group $\mathrm{PSL}_2(8)$ of order $504 = 84 \cdot 6$, that is, of maximal size for that genus. Later on this surface was rediscovered by Macbeath [25].

Thus, for us, in order to prove the result of Burnside and Fricke, it suffices to produce a $(3, 2, 7)$ -presentation of the group $\mathrm{PSL}_2(8)$. Such a presentation is shown in Figure 7, by which we mean that this map of type $(3, 2, 7)$ has monodromy group $\mathrm{PSL}_2(8)$.

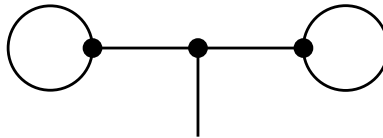


Figure 7: A map with monodromy group $\mathrm{PSL}_2(8)$.

It would not be an easy task to give a “purely mathematical” proof of the above statement, that is, one which is not computer-assisted. Indeed, since this dessin of degree 9 corresponds to the natural representation of G , the edge labels should be the elements of the projective line $\mathbb{P}^1(8)$ over \mathbb{F}_8 , and the elements of this field should, in turn, be represented as polynomials of degree 2 with coefficients in \mathbb{F}_2 . But today, using Maple or GAP, we can in a fraction of a second compute the order of the group, which is 504, and then look at the catalogue [4] and see that there is only one permutation group of degree 9 and of order 504.

Remark 4.1 An ambitious question we would like to ask is as follows: what kind of information about a “big” regular map with an automorphism group G can we extract from a “small” quotient map with a monodromy

group G ? In the above example, it is easy to see that Figure 7 is the only possible $(3, 2, 7)$ -map of degree 9. Therefore, it is defined over \mathbb{Q} . We may (or may we, indeed?) infer from this that a $(3, 2, 7)$ -presentation of the group $\mathrm{PSL}_2(8)$ is unique up to an automorphism, and therefore the Hurwitz map of genus 7 is also unique and defined over \mathbb{Q} . However, Manfred Streit [30] showed that the latter map cannot be *realized* over \mathbb{Q} , and Rubén Hidalgo [19] managed to realize it over $\mathbb{Q}(\sqrt{-7})$. Can this be seen from Figure 7?

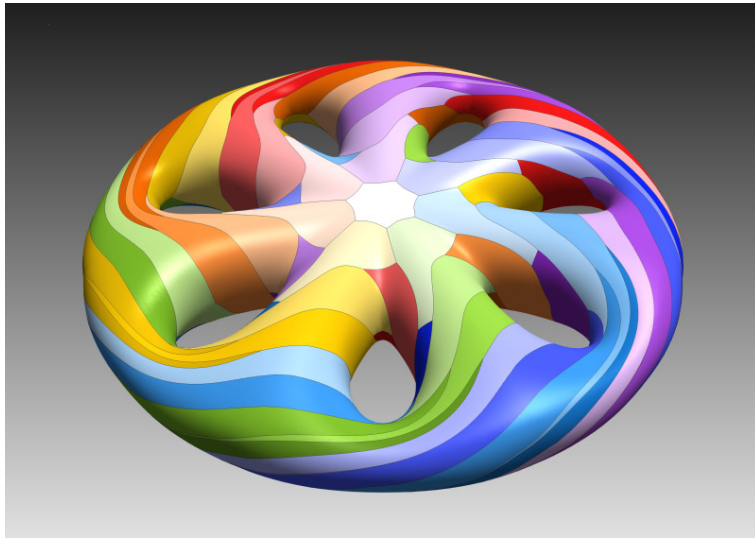


Figure 8: The Fricke-Macbeath map of genus 7 with the automorphism group $\mathrm{PSL}_2(8)$ of order $504 = 84 \cdot 6$. The author of the picture is Jarke van Wijk.

Figure 8 shows the Fricke-Macbeath map. This chef-d’oeuvre of computer graphics was made by Jarke van Wijk [31]. According to van Wijk, the pictures of this kind are produced by *tubification* of graphs [32]. We take a graph, chosen to have Betti number $E - V + 1$ equal to the genus of the map, where V and E are the numbers of vertices and edges of the graph. This graph is then embedded in three-dimensional space, if possible exhibiting some group of symmetries which it shares with the map. We then replace its vertices and edges with spheres and tubes, thus creating a surface on which the desired map may be drawn. The main difficulty of this approach is that there are infinitely many non-isomorphic graphs with a given Betti number (and not all of them planar, by the way). Permutations representing the map give us complete information about the map itself, but tell us ab-

solutely nothing about a convenient structure of a graph to be tubified and about its possible embedding into \mathbb{R}^3 .

Figures 2, 4 and 7 constitute a complete list of $(3, 2, 7)$ -maps with a single face of degree 7. In subsequent sections we will examine all the $(3, 2, 7)$ -maps with two faces of degree 7.

5 Three Hurwitz maps of genus 14

For our next example we take G to be the Hurwitz group $\mathrm{PSL}_2(13)$ of genus 14 and order $84 \cdot 13 = 1092$. As for all groups $\mathrm{PSL}_2(q)$ with $q > 11$, the transitive permutation representation of least degree is the natural representation, of degree $q + 1 = 14$, so we look for possible quotient dessins \mathcal{D} of this degree. The point stabilisers are isomorphic to $C_{13} \rtimes C_6$. The group G has unique conjugacy classes $3A$ and $2A$ of elements of order 3 and 2, but it has three self-inverse conjugacy classes, $7A$, $7B$ and $7C$, of elements of order 7; those in $7B$ and $7C$ are the squares and fourth powers of those in $7A$. (This applies to all the Hurwitz groups $\mathrm{PSL}_2(q)$ with $q \neq 7$.) For each of these classes, if we combine it with the classes $3A$ and $2A$, the Frobenius formula gives $2|G|$ triples, all generating G since no proper subgroup of G is a Hurwitz group. (There are four conjugacy classes of maximal subgroups of G , isomorphic to $C_{13} \rtimes C_6$, D_7 , D_6 and A_4 ; none of them has order divisible by 2, 3 and 7.) Since $\mathrm{Aut}(G) = \mathrm{PGL}_2(13)$ has order $2|G|$, leaving each of the three classes of elements of order 7 invariant, we obtain three regular dessins \mathcal{R} , one for each class. As Streit [30] has shown, they form a Galois orbit, defined over the field $\mathbb{Q}(\cos(2\pi/7))$. (The entries in the character table of G also belong to this field.)

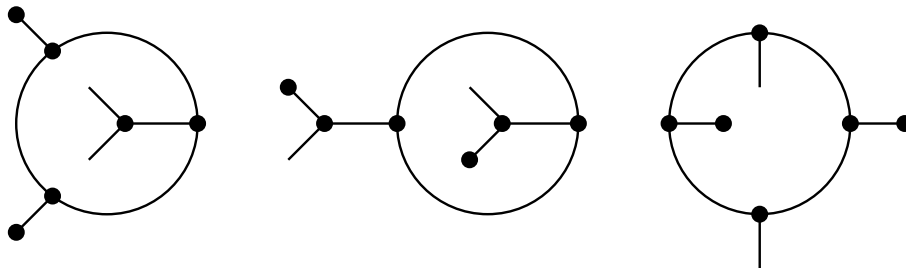


Figure 9: Hurwitz generation of $\mathrm{PSL}_2(13)$.

The corresponding quotient dessins \mathcal{D} of degree 14 are shown in Figure 9. They also form a single Galois orbit, so at first it is a little disturbing to see

that the map on the left is invariant under a reflection, whereas the other two are not, and nor are they mirror images of each other. However, if we draw these two as maps on the sphere, then in each case the antipodal involution provides the expected orientation-reversing isomorphism. These two maps also illustrate the Couveignes-Filimonenkov phenomenon: they are defined, but cannot be realized over a real field. Does their antipodal symmetry mean something interesting for the corresponding regular maps \mathcal{R} of genus 14?

This example illustrates a general phenomenon, in which all the Macbeath-Hurwitz groups $\mathrm{PSL}_2(q)$ for $q \neq 7$ or 27 have unique classes of elements of orders 3 and 2, but three of order 7; one can show by direct calculation with Möbius transformations or by the Frobenius formula that each choice of classes of elements of these orders gives $2|G|$ generating triples of type $(3, 2, 7)$, resulting in a total of $6|G|$ triples.

Here we obtained three Hurwitz dessins \mathcal{R} , one for each class of elements of order 7, so why did this not happen in our earlier example, where $G = \mathrm{PSL}_2(8)$? The answer is that in that case, and indeed in all examples of case (3) of Theorem 2.7, where $q = p^3$ for some prime p , $\mathrm{Aut}(G)$ is not $\mathrm{PGL}_2(q)$ but the larger group $\mathrm{P}\Gamma\mathrm{L}_2(q)$. This is an extension of $\mathrm{PGL}_2(q)$ by a cyclic group of order 3 induced by the Galois group $\mathrm{Gal}(\mathbb{F}_q) \cong \mathbf{C}_3$ of the field \mathbb{F}_q . It acts by permuting the three classes of elements of order 7 transitively, so that the $6|G|$ triples form a single orbit under $\mathrm{Aut}(G)$ and hence correspond to a single Hurwitz dessin. In case (2) of Theorem 2.7, with $q = p$ prime, the Galois group of the field is trivial, so the triples form three orbits, corresponding to three Hurwitz dessins.

Pictures of the corresponding regular maps of genus 14 are not yet available, certainly because of the problem with their spatial arrangement.

6 Genus 17

There remain nine $(3, 2, 7)$ -maps with two faces of degree 7. Six of them are shown in Figure 10, three more are given in Figure 16.

All six maps in Figure 10 are imprimitive: they cover the $\mathrm{PSL}_3(2)$ -trees we have seen before. Figure 11 proposes a labelling of the maps in the first row of Figure 10 for which the blocks are the same. These blocks are shown in Table 2, while the action of the permutations on blocks and the ramification points may be seen in Figure 12.

Indeed, let us write the permutations describing the maps of Figure 11 and their action on the blocks. Here are permutations which correspond to the left map, and also to the left tree of Figure 12. We see that the

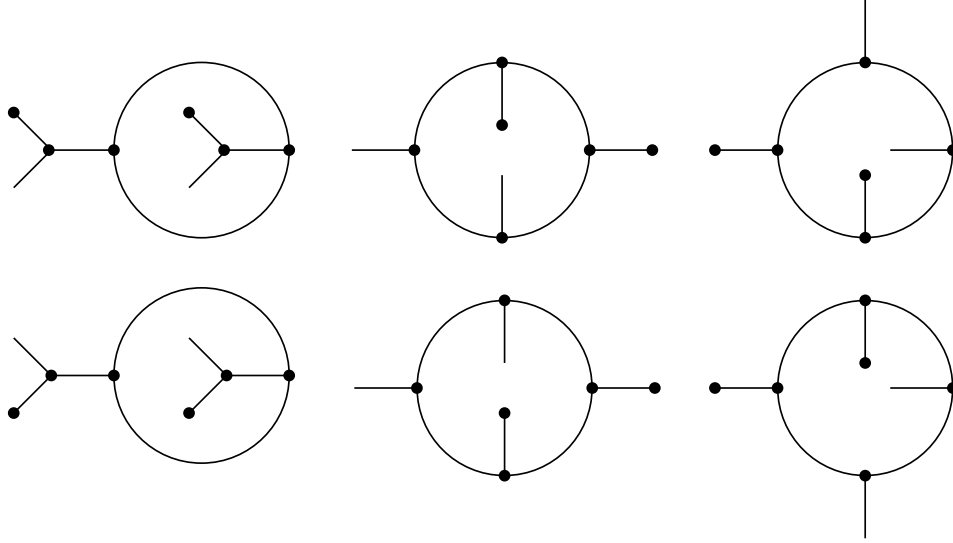


Figure 10: Six imprimitive maps. There are seven blocks of size 2.

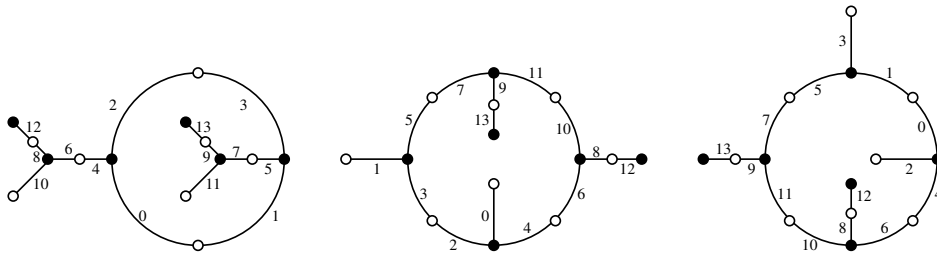


Figure 11: Labelling of the three upper maps in Figure 10.

a	b	c	d	e	f	g
0	2	4	6	8	10	12
1	3	5	7	9	11	13

Table 2: Blocks of the permutations describing maps in Figure 11.

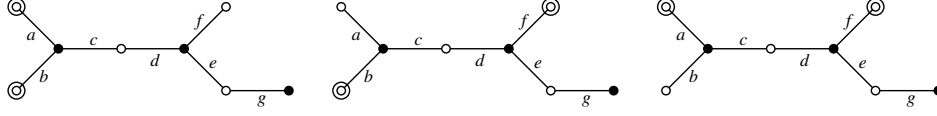


Figure 12: $\mathrm{PSL}_3(2)$ -trees, labelled with blocks for the maps in Figure 11.

permutations x and z are unramified while y is ramified over the vertices a and b of the tree. The corresponding cycles of y are underlined and the vertices of the tree are singled out. It is clear that the map which is mirror symmetric to this one will be ramified over the mirror symmetric tree.

$$\begin{aligned}
x &= (0, 2, 4)(1, 3, 5)(6, 8, 10)(7, 9, 11)(12)(13) \\
&= (a, b, c)(a, b, c)(d, e, f)(d, e, f)(g)(g), \\
y &= (0, 1)(2, 3)(4, 6)(5, 7)(8, 12)(9, 13)(10)(11) \\
&= (\underline{a, a})(\underline{b, b})(c, d)(c, d)(e, g)(e, g)(f)(f), \\
z &= (0, 5, 11, 9, 13, 7, 3)(1, 4, 10, 8, 12, 6, 2) \\
&= (a, c, f, e, g, d, b)(a, c, f, e, g, d, b).
\end{aligned}$$

Permutations for the map in the middle: they are ramified over b and f and thus correspond to the tree in the middle of Figure 12.

$$\begin{aligned}
x &= (0, 2, 4)(1, 3, 5)(6, 8, 10)(7, 9, 11)(12)(13) \\
&= (a, b, c)(a, b, c)(d, e, f)(d, e, f), (g)(g) \\
y &= (0)(1)(2, 3)(4, 6)(5, 7)(8, 12)(9, 13)(10, 11) \\
&= (a)(a)(\underline{b, b})(c, d)(c, d)(e, g)(e, g)(\underline{f, f}), \\
z &= (0, 4, 10, 9, 13, 7, 3)(1, 5, 11, 8, 12, 6, 2) \\
&= (a, c, f, e, g, d, b)(a, c, f, e, g, d, b).
\end{aligned}$$

Finally, the permutations for the map on the right: they are ramified over a and f .

$$\begin{aligned}
x &= (0, 2, 4)(1, 3, 5)(6, 8, 10)(7, 9, 11)(12)(13) \\
&= (a, b, c)(a, b, c)(d, e, f)(d, e, f), (g)(g) \\
y &= (0, 1)(2)(3)(4, 6)(5, 7)(8, 12)(9, 13)(10, 11) \\
&= (\underline{a, a})(b)(b)(c, d)(c, d)(e, g)(e, g)(\underline{f, f}), \\
z &= (0, 4, 10, 9, 13, 7, 3)(1, 5, 11, 8, 12, 6, 2) \\
&= (a, c, f, e, g, d, b)(a, c, f, e, g, d, b).
\end{aligned}$$

Recall that the three white vertices of Figure 12 split into two Galois orbits: the vertex b is separated from the other two (see Figure 3). Therefore we may suppose that the third map, for which the ramification points avoid the vertex b , might not behave in the same way as the other two maps. And, indeed, computing the order of the monodromy group for all three maps we find out that it is equal to 1344 for the first and second map while it is 168 for the third. Of course, the same is true for their mirror images.

The group of size 168 can only be $\mathrm{PSL}_3(2)$. The identification of the Hurwitz group G of size 1344 is a more subtle matter. The catalogue [3] shows that there are three non-isomorphic permutation groups of degree 14 and of order 1344.

One of these three groups cannot be projected onto $\mathrm{PSL}_3(2)$, so it cannot arise here. This group is a semidirect product $A \rtimes B$ of an elementary abelian normal subgroup $A \cong (\mathbb{C}_2)^6$ by a complement $B \cong \mathrm{AHL}_1(7) \cong \mathbb{C}_7 \rtimes \mathbb{C}_3$, the subgroup of index 2 in $\mathrm{AGL}_1(7)$ consisting of the affine transformations $t \mapsto at + b$ of \mathbb{F}_7 for which a is a non-zero square. (Here ‘H’ stands for ‘half’.) The group acts on the 14 points of the cartesian product $\mathbb{F}_2 \times \mathbb{F}_7$, with elements of A acting on pairs (s, t) by preserving t and changing an even number of coordinates s , while elements of B preserve s while acting naturally on t . This action is imprimitive, since the group permutes the seven pairs $\{(0, t), (1, t)\}$. Since the group is solvable, it cannot be a Hurwitz group.

The other two groups are also imprimitive: each is an extension of an elementary abelian normal subgroup $T \cong (\mathbb{C}_2)^3$ by $\mathrm{GL}_3(2) = \mathrm{PSL}_3(2)$, where T is the kernel of the action on seven blocks of size 2. The obvious example of such a group is the affine group $\mathrm{AGL}_3(2)$, the group of all affine transformations of a 3-dimensional vector space V over \mathbb{F}_2 : this acts on the 14 affine planes in V , with T as the group of translations, complemented by the subgroup $\mathrm{GL}_3(2)$ fixing the vector 0. However, the following argument shows that our Hurwitz group G must be isomorphic to the third group, which is a *non-split* extension of T by $\mathrm{GL}_3(2)$ while $\mathrm{AGL}_3(2)$ is a split extension.

If T has a complement C in G , then C lifts to a subgroup M of index 8 in $\Delta = \Delta(3, 2, 7)$. Since C being isomorphic to $\mathrm{GL}_3(2)$, is simple, its core in G is trivial, so the core of M in Δ is a normal subgroup N of index 1344 in Δ . This subgroup M corresponds to a dessin of degree 8 and type $(3, 2, 7)$. It is easy to see that any dessin of this degree and type must have passport $(3^2 1^2, 2^4, 7^1 1^1)$, and we saw in §3.3 that the only possibility is the dessin in Figure 4. However, the monodromy group of this dessin is isomorphic to $\mathrm{GL}_3(2)$, so N has index 168 in Δ , a contradiction. Thus G is a non-split extension of T by $\mathrm{GL}_3(2)$, and in particular it cannot be isomorphic to $\mathrm{AGL}_3(2)$, as is sometimes assumed. (As confirmation of this, one can use

GAP to check that G has elements of order 8, whereas $\text{AGL}_3(2)$ does not.)

One can construct this Hurwitz group G homologically as follows. The normal subgroup K of index 168 in Δ corresponding to Klein's quartic curve is the fundamental group of this surface S of genus 3, so its commutator quotient group K/K' is the abelianised fundamental group, that is, the first integer homology group $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^6$. Similarly, if K^2 denotes the group generated by the squares in K then $K/K'K^2$ is the reduction of $H_1(S, \mathbb{Z})$ mod 2, namely $H_1(S, \mathbb{F}_2) \cong (\mathbb{F}_2)^6$. This is a 6-dimensional module for the automorphism group $A \cong \text{GL}_3(2)$ of this dessin, and it decomposes as a direct sum of two irreducible 3-dimensional A -submodules, corresponding to two normal subgroups N_1 and N_2 of Δ lying between K and $K'K^2$ (see Figure 13). These subgroups N_i are conjugate in the extended triangle group $\Delta[3, 2, 7]$, so they correspond to a chiral pair of regular dessins \mathcal{R}_i of type $(3, 2, 7)$ and genus 17 (the duals of the chiral pair of maps C17.1 in Conder's catalogue [10]). The quotient groups Δ/N_i give two realisations of the Hurwitz group G as $\text{Aut}(\mathcal{R}_i)$, each having a normal subgroup $T = K/N_i \cong \text{C}_2^3$ with quotient $\Delta/K \cong \text{GL}_3(2)$. The minimal common cover of \mathcal{R}_1 and \mathcal{R}_2 is a Hurwitz dessin of genus 129 with automorphism group $\Delta/K'K^2$, an extension of C_2^6 by $\text{GL}_3(2)$.

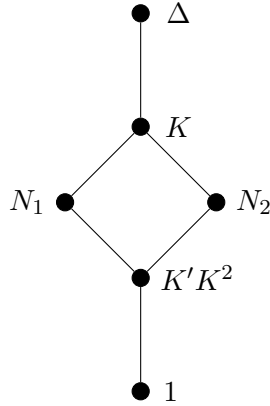


Figure 13: The subgroups N_i of Δ

The six dessins in Figure 10 correspond to six conjugacy classes of subgroups of index 14 in Δ , as follows. The two trees in Figure 2 correspond to two conjugacy classes of subgroups $H_i \cong \text{S}_4$ in $\text{GL}_3(2)$ for $i = 1, 2$, the stabilisers of points and lines in the Fano plane. These lift to two conjugacy classes of subgroups M_i of index 8 in Δ . These are Fuchsian groups whose

signatures $(0; 2, 2, 2, 3)$ can be deduced from Singerman's Theorem [29]: they have genus 0, since this is the genus of the trees, and their elliptic periods correspond to the three fixed points of the generator y of order 2 (the three white vertices of valency 1), and the unique fixed point of the generator x of order 3 (the black vertex of valency 1). These groups M_i therefore have presentations

$$\langle X_1, X_2, X_3, X_4 \mid X_1^2 = X_2^2 = X_3^2 = X_4^3 = X_1 X_2 X_3 X_4 = 1 \rangle,$$

from which it is clear that they each have three subgroups M_{ij} of index 2, the normal closures in M_i of $\{X_j, X_4\}$ for $j = 1, 2, 3$ (see Figure 14).

For each $i = 1, 2$, one of the three subgroups M_{ij} contains K , which is therefore its core: this is the lift to Δ of the commutator subgroup $H'_i \cong A_4$ of H_i , giving rise to a dessin (the third in each row) with monodromy group $\mathrm{GL}_3(2) \cong \Delta/K$. The other two subgroups M_{ij} give the first and second dessins in each row, with monodromy groups $G \cong \Delta/N_i$ of order 1344.

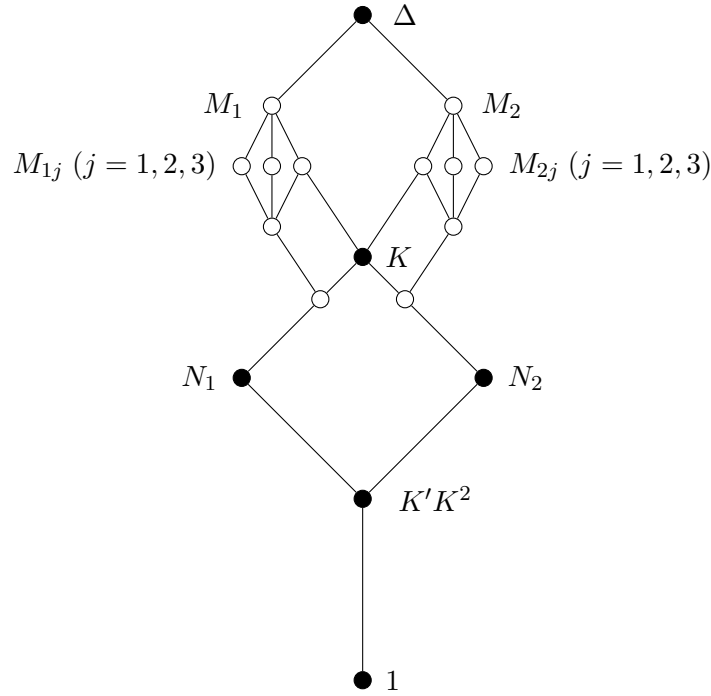


Figure 14: The subgroups M_{ij} of Δ (normal and non-normal subgroups are indicated in black and white)

7 Genus 118

Conder [7] has listed all the Hurwitz groups of order up to one million. The next genus after 17 for which there exist Hurwitz maps is $g = 118$, with the automorphism group $G = \mathrm{PSL}_2(27)$. As shown by Macbeath (see Theorem 2.7), there is just one Hurwitz curve associated with this group.

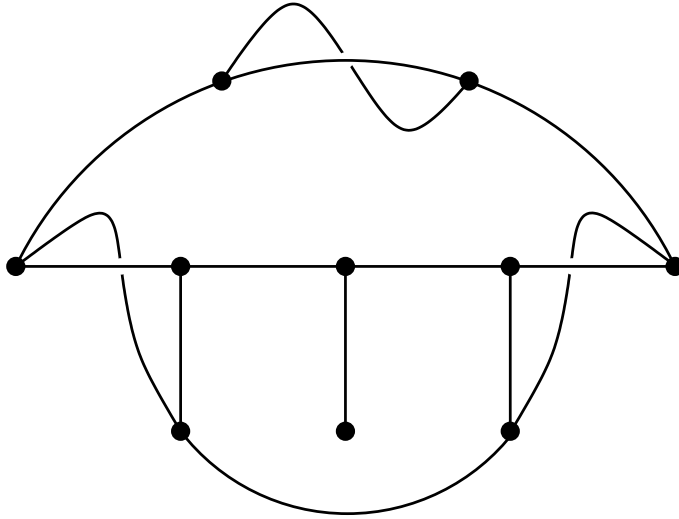


Figure 15: A map of type $(3, 2, 7)$ with the monodromy group $\mathrm{PSL}_2(27)$. This map is of genus $g = 1$.

The natural representation of G has degree 28. The corresponding quotient map is shown in Figure 15. It is drawn in the plane with three crossings, but it is easy to see that these can be removed by adding a single handle, so its genus is 1. Alternatively, one can check that there are four faces, all of degree 7, so the Euler characterisitic is 0.

The following permutations describe the map in Figure 15 and generate the group $\mathrm{PSL}_2(27)$:

$$\begin{aligned}
 x &= (1, 2, 4)(5, 8, 24)(6, 21, 10)(7, 16, 15)(9, 25, 28)(11, 13, 14)(12, 27, 23) \\
 &\quad (17, 26, 18)(19, 20, 22), \\
 y &= (1, 13)(2, 25)(3, 27)(4, 23)(5, 16)(6, 12)(7, 26)(8, 22)(9, 11)(10, 17) \\
 &\quad (14, 18)(15, 21)(19, 24)(20, 28), \\
 z &= (1, 11, 28, 19, 8, 20, 25)(2, 9, 14, 26, 15, 6, 23)(3, 12, 10, 18, 13, 4, 27) \\
 &\quad (5, 7, 17, 21, 16, 24, 22).
 \end{aligned}$$

The label of the half-edge attached to the vertex of valency 1 is 3; the positions of the other labels can easily be derived. Recall that the labels rotate around vertices in the counterclockwise direction.

8 Higher genus examples

8.1 Calculating the genus

More generally, when the group $G = \mathrm{PSL}_2(q)$ is a Hurwitz group (as in Theorem 2.7), the genus of the corresponding Hurwitz dessin (or dessins) \mathcal{R} is

$$g = \frac{|G|}{84} + 1 = \frac{q(q^2 - 1)}{168} + 1$$

for $q \neq 8$, and $g = 7$ for $q = 8$. Asymptotically, we thus have

$$g \sim \frac{q^3}{168} \quad \text{as } q \rightarrow \infty.$$

However, the natural representation of G provides a quotient dessin \mathcal{D} of much lower genus, which we will now calculate.

For any prime power $q > 11$ the least index of any proper subgroup of $\mathrm{PSL}_2(q)$ is $q+1$, attained only by the conjugacy class of point-stabilisers H in the natural representation on $\mathbb{P}^1(q)$. A non-identity Möbius transformation (over any field) has at most two fixed points, so it follows that in this representation of G , any elements of orders 3 and 7 must have cycle-structures $3^a 1^{q+1-3a}$ and $7^c 1^{q+1-7c}$ where $a = \lfloor (q+1)/3 \rfloor$ and $c = \lfloor (q+1)/7 \rfloor$. In addition, the simplicity of G implies that any element of order 2 must induce an even permutation, so it has cycle-structure $2^b 1^{q+1-2b}$ where $b = 2\lfloor (q+1)/4 \rfloor$.

If G is a Hurwitz group, and we set aside the cases $q = 7, 8$ and 27 when q is a power of 7, 2 or 3, we find that the numbers of black vertices, non-free edges and faces of the associated quotient map $\mathcal{D} = \mathcal{R}/H$ are therefore

$$V = \frac{q+3+2\alpha}{3}, \quad E = \frac{q-\beta}{2} \quad \text{and} \quad F = \frac{q+7+6\gamma}{7},$$

where

$$q \equiv \alpha \pmod{3}, \quad q \equiv \beta \pmod{4} \quad \text{and} \quad q \equiv \gamma \pmod{7}$$

with $\alpha, \beta, \gamma = \pm 1$. Thus the Euler characteristic of \mathcal{D} is

$$V - E + F = 2 - \frac{1}{42}(q - 28\alpha - 21\beta - 36\gamma)$$

and its genus is

$$\bar{g} = \frac{1}{84}(q - 28\alpha - 21\beta - 36\gamma) \sim \frac{q}{84}.$$

(We have already seen that $\bar{g} = 0$ when $q = 7$ or 8 , and that $\bar{g} = 1$ when $q = 27$.)

In naive terms, whereas the information carried by the Hurwitz dessin \mathcal{R} increases cubically with q , that in \mathcal{D} increases only linearly, even though it is sufficient to determine \mathcal{R} uniquely.

It also follows from Theorem 2.7 and the above calculation that the only cases giving planar maps \mathcal{D} are $q = 7, 8, 13, 29$ and 43 . Of the remaining values $q \leq 100$, we obtain torus maps for $q = 27, 41, 71$ and 97 , while for $q = 83$ the genus is 2.

(A similar calculation with the Riemann-Hurwitz formula, now not restricted to the groups $\mathrm{PSL}_2(q)$, shows that *any* dessin of type $(3, 2, 7)$ and degree n has genus

$$\frac{n - 28u - 21v - 36w}{84} + 1,$$

where u, v and w are the numbers of fixed points of x, y and z , that is, the numbers of black vertices, white vertices and faces which have degree 1).

8.2 Failure of monotonicity

Intuition might lead one to suppose that, among all faithful quotients of a given regular dessin, the genus should be a non-decreasing function of the degree. Indeed this follows from the Riemann-Hurwitz formula if one compares two quotients, one of which covers the other. However, there are counterexamples in which the two quotients are not comparable in this way.

For instance, let \mathcal{R} be a Hurwitz dessin with automorphism group $G = \mathrm{PSL}_2(q)$ for some prime power $q = p^e$ coprime to 2, 3 and 7 (equivalently $q \neq 7, 8$ or 27 , see Theorem 2.7). If we factor out a Sylow p -subgroup H of G , then since $|H| = q$ the resulting faithful quotient $\mathcal{D} = \mathcal{R}/H$ has degree $n = |G : H| = (q^2 - 1)/2$. By the choice of q , none of the canonical generators x, y or z of G has fixed points in the action on the cosets of H , so \mathcal{D} has genus

$$g = \frac{n}{84} + 1 = \frac{q^2 + 167}{168}.$$

However, if instead we factor out a dihedral subgroup $H' \leq G$ of order $q - 1$ (there is a single conjugacy class of these maximal subgroups in G), then the quotient dessin \mathcal{D}' has degree $n' = q(q + 1)/2 > n$. The generators

x and z of orders 3 and 7 each lie in at most one conjugate of H' (namely the normaliser of their centraliser in G if $q \equiv 1 \pmod{3}$ or $\pmod{7}$), so they each have at most one fixed point in the action of G on the cosets of H' . A simple double counting argument (also applicable to x and z) deals with y : since $H' = N_G(H')$ there are $|G : H'| = q(q+1)/2$ conjugates of H' in G , each containing $(q \pm 1)/2$ involutions as $q \equiv \pm 1 \pmod{4}$; there is a single conjugacy class of $|G|/(q \mp 1) = q(q \pm 1)/2$ involutions in G , so the number of conjugates of H' containing any one of them is

$$\frac{q(q+1)/2 \cdot (q \pm 1)/2}{q(q \pm 1)/2} = \frac{q+1}{2},$$

and hence this is the number of fixed points of y . It then follows that \mathcal{D}' has genus

$$g' \leq \frac{q(q+1)/2 - 21(q+1)/2}{84} + 1 = \frac{q^2 - 20q + 147}{168} < g.$$

For example, if $q = 13$, so that $n = 84$ and $n' = 91$, then $g = 2$ whereas $g' = 0$.

(The inequality $g' < g$ fails in the three excluded cases: if $q = 7$ then $g = g' = 0$; if $q = 8$, taking $H' \cong C_7$, we have $g = 0$ and $g' = 1$; if $q = 27$ then $g = 1$ and $g' = 2$.)

9 A_{15} – the first alternating group to arise as a Hurwitz group

Although we have concentrated here on Hurwitz groups of the form $\mathrm{PSL}_2(q)$, there are many other examples of Hurwitz groups (see [8, 9] for detailed surveys). The smallest alternating group which is a Hurwitz group is A_{15} . Figure 16 gives its $(3, 2, 7)$ -presentations, using the natural representation: as for all alternating groups A_n with $n > 6$, this is the unique non-trivial representation of least degree, and by the simplicity of A_n it is faithful. There are three maps representing A_{15} . The genus of the corresponding regular Hurwitz map is

$$g = \frac{|A_{15}|}{84} + 1 = \frac{15!}{2 \cdot 84} + 1 = 7\,783\,776\,001.$$

More generally, Conder [5] has shown that all the alternating groups A_n for $n \geq 168$ are Hurwitz groups, while for $n \leq 167$ there are exactly 64 exceptions.

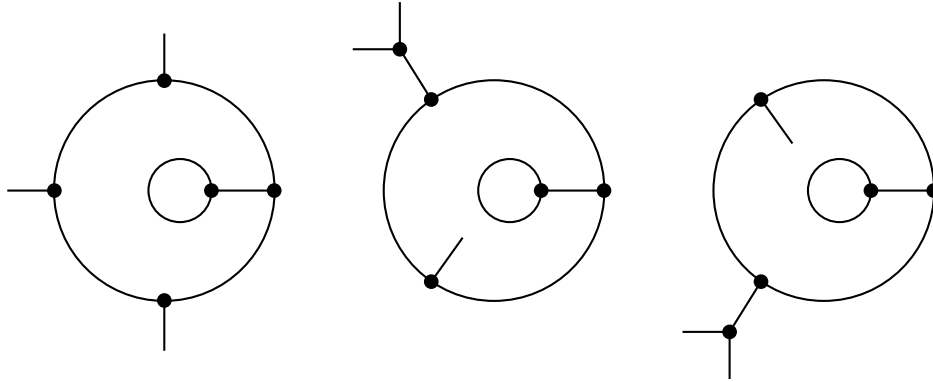


Figure 16: Three maps with the passport $(3^5, 2^6 1^3, 7^2 1^1)$ and with the monodromy group A_{15} .

An elementary case-by-case analysis, which is tedious both to perform and to describe, shows that there are just twelve $(3, 2, 7)$ -maps with two faces of degree 7. These are the three with monodromy group $G = \text{PSL}_2(13)$ in Figure 9, the six in Figure 10 (the two on the right with $G = \text{PSL}_3(2)$, the other four with $|G| = 1344$), and the three in Figure 16 with $G = A_{15}$. Similarly (and this is much easier to see) the only such maps with just one face of degree 7 are the two trees in Figure 2 with $G = \text{PSL}_3(2)$, and one map each in Figures 4 and 7 with $G = \text{PSL}_2(7)$ and $\text{PSL}_2(8)$. As a by-product of this analysis we have also proved that any group which has a faithful quotient of degree $n = 16, 17, \dots, 20$, including A_n , is not a Hurwitz group: one needs $n \geq 21$ in order to have three faces of degree 7.

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School of Mathematics
University of Southampton
Southampton SO17 1PF
UK
`G.A.Jones@maths.soton.ac.uk`

LaBRI
Université de Bordeaux
351 Cours de la Libération
F-33405 Talence Cedex
France
`zvonkin@labri.fr`