

Topological Classification of Complex Polynomials: New Experimental Results

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Abstract

This is a preliminary and informal report on the current state of affairs in our experimental study of the topological classification of complex polynomials. In [2] a complete classification of polynomials up to degree 9 was given. Here we finish the degrees 10 and 11, and discuss some newly observed phenomena.

It may be difficult to read this text without having [2] at hand.

1 Introduction

In [7] it is shown that the problem of the topological classification of complex polynomials is equivalent to the computation of the orbits of an action of the braid group on the set of combinatorial structures called cacti. In [2], using the explicit computation of the above orbits and the enumerative results of [4] concerning the total number of cacti, we have achieved the complete classification of polynomials of degree $n \leq 9$.

Here we give a preliminary report of the continuation of this study for bigger orbits. For the general introduction to the subject the reader may see the paper [2].

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2 New algorithms: discussion

2.1 Method

Let us recall the general scheme of the algorithm of an orbit construction:

1. Put a cactus into the orbit.
2. For each cactus in the orbit:
 - (a) act on it by all the group generators;
 - (b) for each cactus thus obtained verify if it is already in the orbit;
 - (c) if not, add it to the orbit.
3. When for all the cacti in the orbit all the group generators produce nothing new (that is, they produce the elements already existing in the orbit), STOP: the orbit is complete.

2.2 Improvements

The above general scheme is preserved, but the following improvements are introduced into the programs:

(1) We introduced a “normal form” of a cactus: among the n isomorphic representations of a non-rooted cactus as a sequence of permutations (g_1, g_2, \dots, g_k) , $g_i \in S_n$, $g_1 g_2 \dots g_k = (1, 2, \dots, n)$, we choose the *smallest* one in the lexicographic order. This permits us, while producing a new cactus and verifying if it was already found before, to make a straightforward comparison with the previously found cacti instead of testing the isomorphism each time.

(2) The whole orbit is ordered according to the lexicographic order on the cacti. This permits us to make a binary search while looking for a cactus equal to a new one.

In fact, we use a more sophisticated (but also very classical) data structure which is called a “well balanced tree”, or an “AVL-tree”. This structure permits to make not only the search but also an addition of a newly found element in a time proportional to the logarithm of the orbit size.

(3) Previous programs (used in [2]) were implemented in Magma and GAP. The new ones are implemented in C++. The compiled languages usually give better results than the interpreted ones. They also assure better portability and permit us to use many minor computational shortcuts.

2.3 Comparison and current limits

A comparison of the new program with the old one gives, for example, the following: in order to compute an orbit with 12 600 elements, the old (Magma) program needed more than 183 hours, while the new (C++) one, being tested on the same machine, needs about 6 minutes.

In fact, our Magma system is installed on a specific machine, and a quicker machine was bought after buying the Magma licence. Nothing prevents us from running our C++ program on the quickest machine available, and then the same orbit is computed in less than 2.5 minutes.

The biggest orbit computed up to now is of size 1 639 792; its computation takes about 50 hours, but at least half of the time is taken by compression and uncompression of data (an operation that may be omitted for the orbits smaller than one million). Today the main obstacle to the further progress is the space available. Of course, it is possible to save parts of the orbits in files; but then we loose almost all the advantages of the AVL-trees.

For the degree $n = 12$ the biggest putative orbit that needs to be computed (its defect being > 13) corresponds to the passport

$$[4 \times 2^2 1^8, 3 \times 2^1 1^{10}]$$

and is of size $N = 12^5 \times (9/2)^4 = 102\,036\,672$. Even if we forget for the moment the space limitations and think only in terms of the computation time, then the old algorithm would have to work for about 4 million years, while the new one needs “only” about 2 years. Certainly it is a big progress, but from the practical point of view it does not change much, and the complete classification for the degree 12 remains unavailable.

It is, however, quite possible to continue our computations for the smaller orbits (let us say, not bigger than one million) and for bigger values of n . The examples collected up to now show very clearly that all the interesting phenomena take place for relatively small orbits.

2.4 Pure braid group

In most cases we construct an orbit under the action not of the full braid group B_k but of the pure braid group P_k . The reason is, the group P_k preserves the *ordered* passport, while the group B_k mixes up all the partitions in a passport and thus gives all the cacti corresponding to the unordered

passport. For example, to the following unordered passport ($n = 12, k = 7$)

$$[2 \times 3^1 1^9, 2 \times 2^2 1^8, 3 \times 2^1 1^{10}]$$

there correspond $\frac{7!}{2!2!3!} = 210$ ordered passports; therefore the B_k -orbit is at least 210 times bigger than the P_k -orbit.

But the pure braid approach has not only the advantages. There are three problems to struggle against.

(1) The group B_k has $k - 1$ generators, while the group P_k has $k(k - 1)/2$ generators. Thus the step 2.a of the algorithm of Section 2.1 needs $k/2$ times more applications of a generator to a cactus.

(2) The generators of P_k are more complicated than those of B_k . Namely, they are as follows:

$$a_{ij} = \sigma_i \sigma_{i+1} \dots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \dots \sigma_{i+1}^{-1} \sigma_i^{-1}, \quad 1 \leq i \leq j \leq k - 1.$$

We know how to apply *directly* only the generators σ_i ; therefore we need much more work in order to apply a single generator a_{ij} of the group P_k .

One may easily compute the total length of all the generators of the pure braid group: it is $(k^3 - k)/3$. Thus the construction of the pure braid orbit demands, for the treatment of a single cactus, $(k^2 + k)/3$ times more work than it would need for the B_k -orbit. In the above example, when $k = 7$, we need 6 applications of the σ_i if we deal with B_k , and 112 if we deal with P_k .

(3) When all the partitions in a passport are different, then the ordered passport is preserved *only* by the elements of the pure braid group. However this is not the case when there are some equal partitions in the passport: this time the subgroup of B_k that preserves the ordered passport is bigger. In such a case the pure braid group may artificially split a class of the topological equivalence of polynomials into several different orbits. An example of this kind is given in Section 3.4.

If we “forget” the linking between the braid strings (or, equivalently, if we add the relations $\sigma_i^2 = 1$) and consider only the images of the string ends, then the group B_k projects onto the symmetric group S_k , while the group P_k projects onto the identity group $\{1\}$. But what we need in fact, if we consider once more the example

$$[2 \times 3^1 1^9, 2 \times 2^2 1^8, 3 \times 2^1 1^{10}],$$

is a subgroup of B_7 that projects onto the Young subgroup $S_2 \times S_2 \times S_3 < S_7$. Let us denote this group $B_{2,2,3} < B_7$. We don't know if such groups were

ever studied in detail. The group $B_{2,2,3}$ may be generated by the following elements:

$$\begin{aligned} \sigma_1; \quad \sigma_3; \quad \sigma_5, \sigma_6; \\ a_{22} = \sigma_2^2; \\ a_{44} = \sigma_4^2; \\ a_{24} = \sigma_2\sigma_3\sigma_4^2\sigma_3^{-1}\sigma_2^{-1}. \end{aligned}$$

We see that the total length of these generators is 14. It is worse than 6 (for B_7), but much better than 112 (for P_7). Also, we may now not worry about the problem invoked in (3). This approach is in the process of implementation.

2.5 Perspectives and dreams

We don't discuss here the topological classification of polynomials in general, but only its computational aspects.

Clearly we are approaching our limits. One may dream of more powerful computers, and probably of parallel computations, but both need some financial support without promising much of insight.

What would be a major advance both in computation and in understanding of the braid group action may be roughly described as follows. Let us take once more the biggest orbit of degree 12, with the passport

$$[4 \times 2^2 1^8, 3 \times 2^1 1^{10}]$$

and the Goulden-Jackson number $N = 12^5 \times (9/2)^4 = 102\,036\,672$. We would like to “get rid of” the factor 12^5 . That is, we would like to find a way to represent the whole orbit using only $(\frac{9}{2})^4 = 410\frac{1}{16}$ combinatorial “objects”. (These “objects” remain to be invented, and we cannot explain the meaning of the fractions.)

Alternatively, maybe it is possible to “get rid of” the factor $(9/2)^4$ and represent an orbit by means of $12^5 = 248\,832$ “objects”. The last number is much bigger than the previous one; but still it is within the realm of accessible orbit sizes.

Some other perspectives will be discussed in Section 5.5.

3 Degree 10

Below a complete list is given of all the polynomial passports of degree $n = 10$ for which there exists more than one orbit. Each time the reason of the splitting is indicated.

We remind to the reader that the symmetry is a particular case of a composition. But we use here the term “composition” only for the cases of a composition more complicated than a simple symmetry.

3.1 Symmetry

It is clear that for $n = 10$ the possible symmetry orders are only 2 and 5. But in fact a cactus having symmetry of order 5 corresponds to two critical values (a case which is “trivial” from the point of view of the topological classification). Hence in the list given below we observe only the symmetry of order 2.

Note that the center of such a symmetry does not have to have valency 2, but its valency must be divisible by 2. Below the centers of symmetry are underlined.

- (1) Passport: [4411, 22111111, 211111111].
Two orbits: 50 asymmetric cacti, and 5 symmetric ones.
- (2) Passport: [3322, 22111111, 211111111].
Two orbits: 50 asymmetric cacti, and 5 symmetric ones.
- (3) Passport: [2×331111 , 211111111].
Two orbits: 60 asymmetric cacti, and 5 symmetric ones.
- (4) Passport: [331111, 222211, 211111111].
Two orbits: 60 asymmetric cacti, and 5 symmetric ones.
- (5) Passport: [2×222211 , 211111111].
Two orbits: 60 asymmetric cacti, and 5 symmetric ones.
- (6) Passport: [4111111, 331111, 22111111].
Two orbits: 85 asymmetric cacti, and 5 symmetric ones.
- (7) Passport: [4111111, 222211, 22111111].
Two orbits: 85 asymmetric cacti, and 5 symmetric ones.

- (8) Passport: [61111, 2×22111111].
Two orbits: 120 asymmetric cacti, and 5 symmetric ones.
- (9) Passport: [331111, 2221111, 22111111].
Two orbits: 430 asymmetric cacti, and 15 symmetric ones.
- (10) Passport: [222211, 2221111, 22111111].
Two orbits: 430 asymmetric cacti, and 15 symmetric ones.
- (11) Passport: [42211, 2×22111111].
Two orbits: 730 asymmetric cacti, and 10 symmetric ones.
- (12) Passport: [33211, 2×22111111].
Two orbits: 730 asymmetric cacti, and 10 symmetric ones.
- (13) Passport: [331111, 2×22111111 , 211111111].
Two orbits: 3 050 asymmetric cacti, and 25 symmetric ones.
- (14) Passport: [222211, 2×22111111 , 211111111].
Two orbits: 3 050 asymmetric cacti, and 25 symmetric ones.
- (15) Passport: [4111111, 3×22111111].
Two orbits: 4 275 asymmetric cacti, and 25 symmetric ones.
- (16) Passport: [2221111, 3×22111111].
Two orbits: 21 400 asymmetric cacti, and 75 symmetric ones.
- (17) Passport: [4×22111111 , 211111111].
Two orbits: 150 000 asymmetric cacti, and 125 symmetric ones.

3.2 Composition

In [2] we explain why the composition may lead to multiple orbits: a polynomial is a composition of non-linear polynomials of smaller degrees if and only if its monodromy group is imprimitive (Ritt, 1922). Hence, if for the same passport there co-exist decomposable polynomials and indecomposable ones, then they have different monodromy groups and therefore belong to different classes of topological equivalence.

It turns out that in all the examples given below the pattern of splitting into multiple orbits is slightly different. For all the four passports of this

section, and also for the passport of Section 3.4, *all* the corresponding polynomials are decomposable; but they are decomposable “in different ways”, and this is why they belong to different orbits.

We badly need a convenient notation for a composition. Here we propose one, but we are not sure that it is the best possible.

Notation 3.1 Let $n = pq$, and consider a polynomial $R(z) = P \circ Q(z) = P(Q(z))$, where $\deg R = n$, $\deg P = p$ and $\deg Q = q$. We will explain our notation by an example. Let us take $p = 7$, and let the polynomial P have 3 critical values. Thus, its passport consists of 3 partitions; in front of each partition we put its “color” (1, 2 and 3):

$$\pi = [1 : 3211; \quad 2 : 22111; \quad 3 : 211111].$$

First of all, we permit ourselves to add, at the end of a passport, one or several partitions of the type 1^p . These valencies correspond to non critical points; we may need them because some of them may turn out to be critical values of the polynomial Q . (Note that adding such a partition to a passport does not change its Goulden-Jackson number and defect, does not interfere with the planarity condition, etc.) For our example, let us take the new version of π as follows:

$$\pi = [1 : 3211; \quad 2 : 22111; \quad 3 : 211111; \quad 4 : 1111111].$$

The parts of the partitions belonging to π are the valencies of the vertices of a cactus. We call it a P -cactus. Now suppose that some of the vertices of the P -cactus are critical values of the polynomial Q . Let us denote these values a, b, c, \dots , and let us mark the corresponding parts of the passport:

$$\pi = [1 : 3_a 211; \quad 2 : 221_b 1_c 1; \quad 3 : 2_d 111111; \quad 4 : 1_e 1111111].$$

In the above example there are 5 critical values of Q , namely, a, b, c, d, e , and, for example, the value c is a vertex of the P -cactus, of color 2 and of degree 1.

Suppose now that the polynomial Q is of degree $q = 10$; denote its passport by τ . There are 5 partitions in τ , their “colors” being labelled by a, b, c, d, e . Let us take as an example the following passport:

$$\tau = [a : 4^1 1^6; \quad b : 3^1 1^7; \quad c : 2^2 1^6; \quad d : 2^1 1^8; \quad e : 2^1 1^8].$$

Our goal is to reconstruct the passport ρ of the polynomial $R = P \circ Q$ from the above data. In fact it is very easy:

- All the vertices of the P -cactus that are not labelled by a, b, c, \dots , are just repeated 10 times (with the same color and degree). We may also say that they are “multiplied by the partition” 1^{10} corresponding to a non critical value.
- The vertex 3_a (of color 1) is “multiplied” by the partition $4^1 1^6$ which corresponds to a ; this gives $12^1 3^6$ (all these vertices are of color 1). The vertex 1_b is “multiplied by $3^1 1^7$, and so on.

The reader will easily verify the result:

$$[1 : (12^1 3^6)_a 2^{10} 1^{20}; \quad 2 : 2^{20} (3^1 1^7)_b (2^2 1^6)_c 1^{10}; \quad 3 : (4^1 2^8)_d 1^{50}; \quad 4 : (2^1 1^8)_e 1^{60}].$$

Here we have given not only the colors and degrees of the vertices of the R -cactus, but also indicated which of them are preimages of a , of b , etc. If we forget this additional information, the same passport will be rewritten as

$$\rho = [1 : 12^1 3^6 2^{10} 1^{20}; \quad 2 : 2^{22} 3^1 1^{23}; \quad 3 : 4^1 2^8 1^{50}; \quad 4 : 2^1 1^{68}].$$

The passport ρ being presented in this form, it is not easy at all to see that it admits a composition.

The above introduced notation is used below in order to describe how the passports are decomposed. The compositions for the degree 10 are enumerated in the next two sections.

3.3 List of compositions for degree 10

There are five passports for which the reason for the existence of multiple orbits is a composition. Four of them are presented in this section, and the remaining case is explained in detail in the next section.

(1) Passport: $[22222, 31111111, 22111111]$.

Two orbits, of size 5 and 2 (both decomposable).

The decomposition for the orbit of size 5 looks as follows: the P -cactus labelled as

$$[1 : 2; \quad 2 : 1_a 1; \quad 3 : 1_b 1_c]$$

is composed with the Q -cactus

$$[a : 311; \quad b : 2111; \quad c : 2111],$$

which gives as a result

$$[1 : 22222; 2 : (311)_a 11111; 3 : (2111)_b (2111)_c].$$

As for the orbit of size 2, here the P -cactus

$$[1 : 2; 2 : 1_a 1; 3 : 1_b 1]$$

is composed with the Q -cactus

$$[a : 311; b : 221],$$

which gives for the resulting passport

$$[1 : 22222; 2 : (311)_a 11111; 3 : (221)_b 11111].$$

A representative of the orbit of size 5 is

$$\begin{aligned} g_1 &= (1,2)(3,4)(5,6)(7,10)(8,9) \\ g_2 &= (3,5,7) \\ g_3 &= (1,3)(8,10) \end{aligned}$$

and a representative of the orbit of size 2 is

$$\begin{aligned} g_1 &= (1,2)(3,4)(5,6)(7,8)(9,10) \\ g_2 &= (3,5,9) \\ g_3 &= (1,3)(7,9) \end{aligned}$$

The order of the monodromy group is 14 400 for the first orbit, and 7 200 for the second one. Cf. Section 3.5.

(2) Passport: [22222, 3211111, 21111111].

Two orbits, of size 10 and 2 (both decomposable).

For the orbit of size 10 we have: the P -passport is

$$[1 : 2; 2 : 1_a 1_b; 3 : 1_c 1],$$

and the Q -passport is

$$[a : 311; b : 2111; c : 2111].$$

The result of the composition is

$$[1 : 22222; 2 : (311)_a(2111)_b; 3 : (2111)_c11111].$$

For the other orbit, of size 2, we have the P -passport equal to

$$[1 : 2; 2 : 1_a1; 3 : 1_b1]$$

and the Q -passport equal to

$$[a : 32; b : 2111],$$

which gives the composition

$$[1 : 22222; 2 : (32)_a11111; 3 : (2111)_b11111].$$

A representative of the orbit of size 10 is

$$\begin{aligned} g_1 &= (1,2)(3,4)(5,6)(7,10)(8,9) \\ g_2 &= (3,5,7)(8,10) \\ g_3 &= (1,3) \end{aligned}$$

and a representative of the orbit of size 2 is

$$\begin{aligned} g_1 &= (1,2)(3,4)(5,6)(7,8)(9,10) \\ g_2 &= (1,3)(5,7,9) \\ g_3 &= (1,5) \end{aligned}$$

The monodromy group is the same for both orbits; its order is 28 800.
Cf. Section 3.5.

(3) Passport: [4222, 22111111, 21111111].

Two orbits, of size 25 and 10 (both decomposable).

A representative of the orbit of size 25 is

$$\begin{aligned} g_1 &= (1,2)(3,4)(5,6,7,10)(8,9) \\ g_2 &= (3,5)(8,10) \\ g_3 &= (1,3) \end{aligned}$$

and a representative of the orbit of size 10 is

$$\begin{aligned}
g_1 &= (1,2)(3,4)(5,6)(7,8,9,10) \\
g_2 &= (1,3)(5,7) \\
g_3 &= (1,5)
\end{aligned}$$

The monodromy group is the same for both orbits; its order is 28 800.

(4) Passport: $[22222, 221111111, 2 \times 211111111]$.

Two orbits, of size 50 and 20 (both decomposable).

A representative of the orbit of size 50 is

$$\begin{aligned}
g_1 &= (1,2)(3,4)(5,6)(7,10)(8,9) \\
g_2 &= (5,7)(8,10) \\
g_3 &= (3,5) \\
g_4 &= (1,3)
\end{aligned}$$

and representative of the orbit of size 20 is

$$\begin{aligned}
g_1 &= (1,2)(3,4)(5,6)(7,8)(9,10) \\
g_2 &= (3,5)(7,9) \\
g_3 &= (3,7) \\
g_4 &= (1,3)
\end{aligned}$$

The monodromy group is the same for both orbits; its order is 28 800.

Remark 3.2 There are several other passports giving rise to a composition of the corresponding cacti. But in these cases *all* the cacti having such a passport are decomposable, and, moreover, they are decomposable “in the same way”. Therefore the composition in question does not create a separate orbit. Below we give the list of these passports together with the orbit sizes:

$[2^5, 4^1 1^6, 2^1 1^8]$	2
$[2^5, 2 \times 3^1 1^7]$	2
$[4^1 2^3, 3^1 1^7, 2^1 1^8]$	10
$[2^5, 2^3 1^4, 2^1 1^8]$	10
$[2^5, 3^1 1^7, 2 \times 2^1 1^8]$	20
$[4^1 2^3, 3 \times 2^1 1^8]$	100
$[2^5, 4 \times 2^1 1^8]$	200

3.4 A more complicated case

We consider the remaining example separately, because it is very interesting. It corresponds to the passport

$$[22222, 2 \times 22111111].$$

There are 27 cacti with this passport, while the Goulden-Jackson number is equal to $N = 10 \times \frac{4!}{5!} \times \left(\frac{7!}{2!6!}\right)^2 = 24\frac{1}{2} = 22 + \frac{1}{2} \times 5$.

There are *four* classes of the topological equivalence of polynomials having this passport, and two of them illustrate some new phenomena that we did not observe before.

All the four orbits correspond to a composition $P(Q(z))$ where the polynomial P is always the same; $\deg P = 2$, and $\deg Q = 5$. The passport of P is

$$[2, 2 \times 11].$$

Of course, it is a “fake” passport: the two partitions 11 are redundant. But we need them in order to be able to label the critical values of Q .

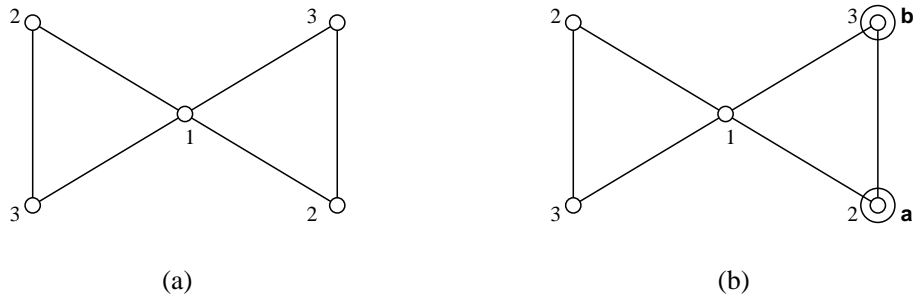


Figure 1: Labelling of the vertices of the P -cactus

First orbit The vertices of the P -cactus are labelled as follows (see also Figure 1):

$$[1 : 2; \quad 2 : 1_a 1; \quad 3 : 1_b 1].$$

The Q -cactus has only two critical values, a and b , and its passport is

$$[a : 221; \quad b : 221].$$

Then the passport of the composition is

$$[1 : 22222; \quad 2 : (221)_a 11111; \quad 3 : (221)_b 11111].$$

The orbit of this composition consists of 2 cacti. The monodromy group is of order 200 (cf. Section 3.5). A representative of this orbit is

$$\begin{aligned} g_1 &= (1, 2)(3, 4)(5, 6)(7, 8)(9, 10) \\ g_2 &= (3, 9)(5, 7) \\ g_3 &= (1, 3)(5, 9) \end{aligned}$$

Second “orbit” The vertices of the P -cactus are labelled as follows:

$$[1 : 2; \quad 2 : 1_a 1_b; \quad 3 : 1_c 1].$$

The Q -cactus has three critical values a , b and c ; the corresponding passport is

$$[a : 2111; \quad b : 2111; \quad c : 221].$$

Then the passport of the composition is

$$[1 : 22222; \quad 2 : (2111)_a(2111)_b; \quad 3 : (221)_c 11111].$$

We see that the internal structure of the colors 2 and 3 is different. This is why the pure braid group gives in this case two orbits (of size 5), while the full braid group, instead of also giving two orbits (of size 15), gives a single orbit (of size 30).

Note that it is the full braid group action that actually corresponds to the topological classification of polynomials. Therefore the 10 cacti of 30, that correspond to the *ordered* passport above, must be put together into a common “orbit”. We would say that its size is not 10 but 5+5. The monodromy group for this orbit is of order 14 400 (cf. Section 3.5), and its representative is

$$\begin{aligned} g_1 &= (1, 2)(3, 4)(5, 6)(7, 10)(8, 9) \\ g_2 &= (1, 3)(5, 7) \\ g_3 &= (1, 5)(8, 10) \end{aligned}$$

Third and fourth orbits The vertices of the P -cactus are labelled as follows (see also Figure 2):

$$[1 : 2; \quad 2 : 1_a 1_c; \quad 3 : 1_b 1_d].$$

The Q -cactus has four critical values a , b , c and d ; the corresponding passport is

$$[a : 2111; \quad b : 2111; \quad c : 2111; \quad d : 2111].$$

Then the passport of the composition is

$$[1 : 22222; \quad 2 : (2111)_a(2111)_c; \quad 3 : (2111)_b(2111)_d].$$

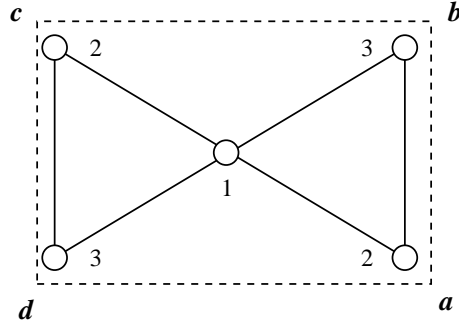


Figure 2: Labelling of the vertices of the P -cactus

Here we observe once more a new phenomenon: while the polynomial P remains the same, and polynomials Q_1 and Q_2 are topologically equivalent, their compositions $P \circ Q_1$ and $P \circ Q_2$ may be topologically non equivalent. Indeed, the Q -cacti obviously make a single orbit: all theorems say so, even the most classical one (due to Luroth and Clebsch). But geometrically some of them look somewhat different: they are sort of “symmetric”, with a “fake” symmetry around a center of one of the squares: see Figure 3.

This fake symmetry, however, becomes a real one (that is, a symmetry around a vertex) after the composition: see Figure 4.

The two corresponding orbits are of size 10 (for asymmetric cacti) and 5 (for the symmetric ones). The monodromy groups are of order 14 400 and 240 respectively (cf. Section 3.5). The representatives of the two orbits are given below:

$$\begin{array}{l}
 g_1 = (1, 2)(3, 4)(5, 10)(6, 7)(8, 9) \\
 g_2 = (3, 5)(6, 8) \\
 g_3 = (1, 3)(6, 10) \\
 \hline
 g_1 = (1, 2)(3, 4)(5, 10)(6, 7)(8, 9) \\
 g_2 = (3, 5)(8, 10) \\
 g_3 = (1, 3)(6, 8)
 \end{array}$$

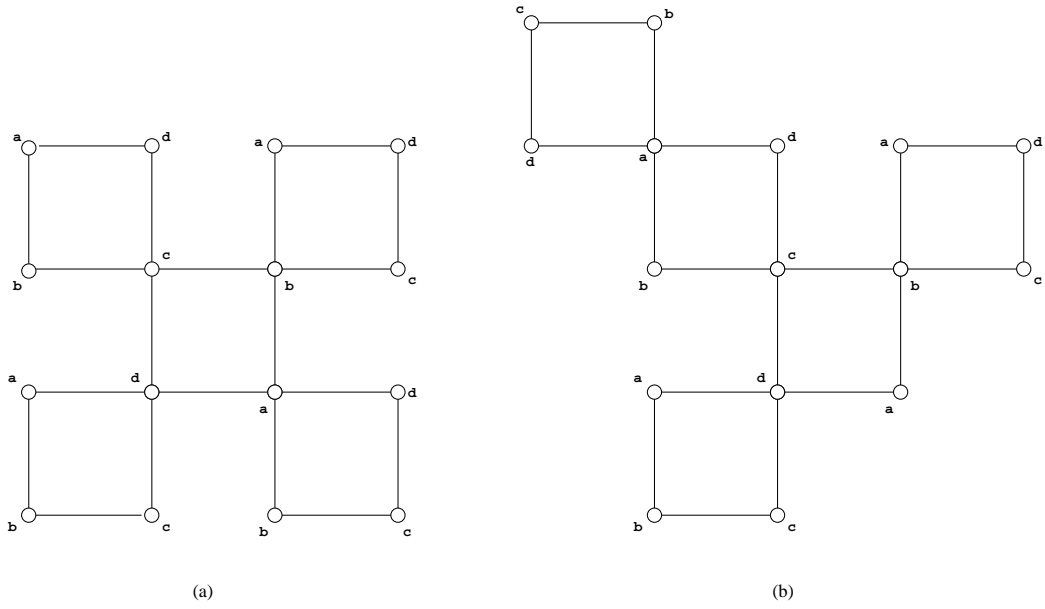


Figure 3: These cacti ($n = 5, k = 4$) belong to the same orbit

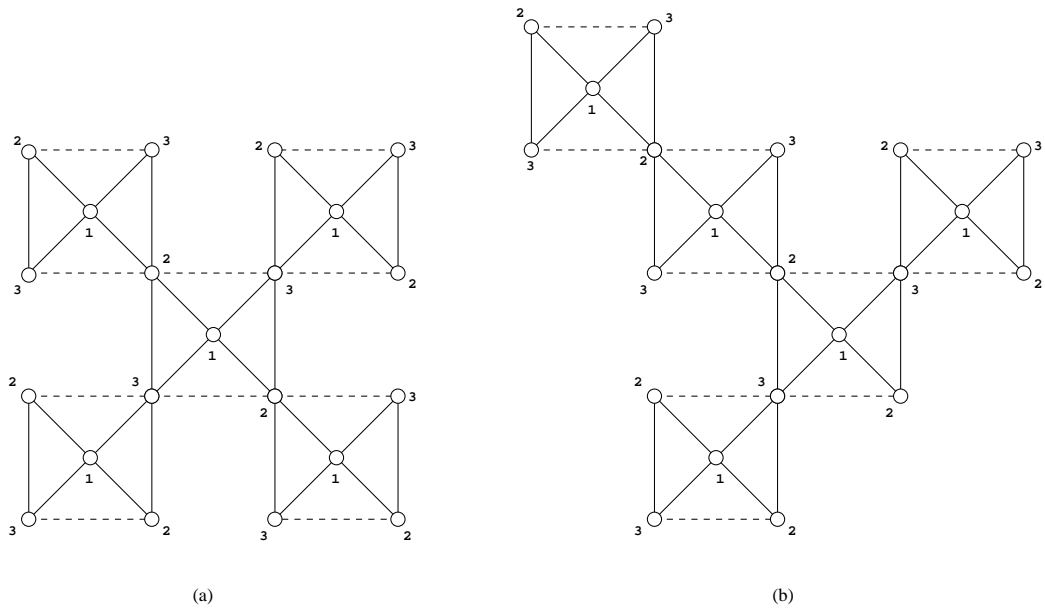


Figure 4: These cacti ($n = 10, k = 3$) belong to different orbits

3.5 Remarks on monodromy groups

Recall that the monodromy groups for compositions are imprimitive, i.e., there exists a system of blocks. For $n = 10$ the only possible block systems are of type 5^2 and 2^5 . Now, the presence of the permutation

$$\gamma = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$$

fixes the block system of type 5^2 to

$$\{1, 3, 5, 7, 9\} \quad \text{and} \quad \{2, 4, 6, 8, 10\},$$

and the block system of type 2^5 to

$$\{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}, \{5, 10\}.$$

We have obtained as monodromy groups the groups of order 28 800, 14 400, 7 200, 240 and 200. To characterize the groups themselves, the catalog [1] may be of great help.

1. There exists a unique (transitive) group of degree 10 and of order 28 800: it is denoted $T43$ in [1]. We have $28800 = 120^2 \times 2$, and therefore this group could be nothing else but the wreath product $S_5 \wr S_2$. It is unique as a subgroup of S_{10} (and not only up to a conjugation) because of the above remark concerning the unicity of the blocks.

2. There are two groups of order 14 400 (both are subgroups of index 2 in the previous group). One of them is an even group $T42 = (S_5 \wr S_2) \cap A_{10}$; it does not fit, because in all the cases we have had a monodromy group of order 14 400, at least one of the permutations in the cactus was odd. Therefore the monodromy group that fits is $T41$, which is an extension by S_2 of the ground group S_5^2 . Note that this group does not have permutations of the cyclic structures $2^1 1^8$, $4^1 2^3$ and $3^1 2^1 1^5$ (present in the examples 2, 3 and 4 of Section 3.3).

3. There exists a unique group of order 7 200; it is denoted $T40$ in [1]. It is a subgroup of index 2 in the previous group. Having in mind that $7200 = 60^2 \times 2$, this group could be nothing else but $A_5 \wr S_2$.

4. There exists a unique group of order 240; it is denoted $T22$ in [1]. This is the biggest group having both block systems of type 5^2 and 2^5 . Therefore it is $(S_5 \wr S_2) \cap (S_2 \wr S_5)$. We don't know any better characterization of this group.

5. There are five different groups of order 200 in [1]; but only one of them contains permutations of type $2^2 1^6$, namely, $T21$ (see the passport above). This is the group $D_5 \wr S_2$, where D_5 is the dihedral group of order 10 (thus the order of the wreath product is $10^2 \times 2 = 200$).

4 Degree 11

4.1 A single orbit for any passport

In [2] we have discovered three mechanisms that give rise to a splitting of a family of cacti into more than one orbit: (1) composition, and symmetry as its particular case; (2) exceptional monodromy groups (see Theorem 5.13 of [2]); and (3) cyclic structures that split into different conjugacy classes in the alternating group A_n (see Section 5.4 of [2]).

We don't dare to conjecture that these are the *only* possible reasons to have multiple orbits (see also Remark 5.3), even though the case $n = 10$ gave us nothing new in this respect.

If, however, this pattern corresponds to reality (at least for the smaller degrees), we may note that among the above mentioned mechanisms no one works for $n = 11$. Indeed, (1) to have a composition we need at least a composite number n ; (2) there are only three examples of exceptional monodromy groups, one for $n = 7$, one for $n = 13$ and one for $n = 15$ (see once more Theorem 5.13 of [2]); (3) finally, the last mentioned mechanism can only work for $n \geq 25$.

Our computations for $n = 11$ confirm this observation: for the degree 11 a passport always completely characterizes the topological equivalence class of polynomials. The following conjecture is a “one-dollar” one; that is, it is very tentative: the data available today make it plausible, but they are not at all sufficient.

Conjecture 4.1 For the degree 13 the only passport for which there is more than one orbit is that of Section 5.2. For the degrees 17, 19 and 23 a passport completely characterizes the topological equivalence class of polynomials.

4.2 The biggest orbit of degree 11

We have computed all the orbits of degree 11 except the biggest one:

$$[4 \times 2^2 1^7, 2 \times 2^1 1^9]: \quad N = 11^4 \times 4^4 = 3\,748\,096$$

For this remaining case (which is too big for the memory available) we have *proved* that there is a single orbit. Here we explain how the proof works.

Step 1 Let us multiply the last two permutations: $h = g_5 g_6$. The cyclic structure of h may be $3^1 1^8$ or $2^2 1^7$. Note that the sequence of permutations $[g_1, g_2, g_3, g_4, h]$ is also a cactus (with $k = 5$), as it verifies the planarity condition, and the product

$$g_1 g_2 g_3 g_4 h = g_1 g_2 g_3 g_4 g_5 g_6 = \gamma = (1, 2, \dots, 11).$$

The cactus $[g_1, g_2, g_3, g_4, h]$ may have two possible passports: $[4 \times 2^2 1^7, 3^1 1^8]$ and $[4 \times 2^2 1^7, 2^2 1^7] = [5 \times 2^2 1^7]$.

Step 2 The orbits with the passports $[4 \times 2^2 1^7, 3^1 1^8]$ and $[5 \times 2^2 1^7]$ are *already computed*. Each time there is a single orbit, of size $11^3 \times 4^4$ for the first passport, and $11^3 \times 4^5$ for the second one. (The total does not give the desirable number $11^4 \times 4^4$, but it must not: see below.)

Step 3 A permutation h with the cyclic structure $3^1 1^8$ can be factored into two permutations g_5, g_6 with cyclic structure $2^1 1^9$ in 3 different ways. For example,

$$\begin{aligned} (1, 2, 3) &= (1, 2) \times (1, 3) \\ &= (1, 3) \times (2, 3) \\ &= (2, 3) \times (1, 2), \end{aligned}$$

and these factorizations are obtained from one another under the action of the braid group generator σ_5 .

For an h with the cyclic structure $2^2 1^7$ there are two factorizations possible, as in the example below,

$$\begin{aligned} (1, 2)(3, 4) &= (1, 2) \times (3, 4) \\ &= (3, 4) \times (1, 2), \end{aligned}$$

and they are also “connected” by σ_5 .

Step 4 We have found two connected blocks, of size $3 \times 11^3 \times 4^4$ and $2 \times 11^3 \times 4^5$ respectively. Note that this time the total gives the desirable number $11^4 \times 4^4$. The only thing that remains to be proved is that the cacti $[g_1, \dots, g_6]$ having g_5g_6 of the type 3^11^8 , and those having g_5g_6 of the type 2^21^7 , coexist in the same orbit. The following example shows that this is true: take the cactus

$$[(8, 11)(9, 10); (2, 3)(4, 5); (2, 4)(9, 11); (1, 2)(6, 8); (1, 6); (7, 8)],$$

for which $g_5g_6 = (1, 6)(7, 8)$, and apply σ_4^2 : you will get

$$[(8, 11)(9, 10); (2, 3)(4, 5); (2, 4)(9, 11); (1, 8)(2, 6); (2, 8); (7, 8)],$$

for which $g_5g_6 = (2, 7, 8)$. The statement is proved.

Some aspects of the above method may be generalized; but the generalization is not direct, and there remain yet many obstacles to overcome in order to get a more or less “algorithmic” procedure to follow. It would be premature to discuss this subject here.

4.3 What about degree 12?

Our colleague physicist gives his own “physical definition” of a computationally hard problem. A problem is hard if it is more difficult to compute the next term than all the previous terms together. This is exactly what happens in our problem.

(1) In order to complete the classification of polynomials of all the degrees up to 11, we have computed the orbits for 644 passports. For the degree 12 there are 833 passports to process.

(2) The cases of symmetry and composition for the degree 12 must be abundant.

(3) As we have already mentioned before, the biggest putative orbit of degree 12 is of the size slightly bigger than 100 million. Our current program should take about two years to compute such an orbit, and also the memory limitations prevent us from dealing with the orbits bigger than 2 million.

It is obvious, therefore, that we need radically new ideas and approaches in order to treat the case $n = 12$.

5 Exceptional orbits

We remind to the reader that there exist exactly three polynomial passports (with more than two critical values), one for each of the degrees 7, 13 and 15 respectively, which give rise to a primitive monodromy group different from S_n and A_n . In [2] (see Theorem 5.13) we gave an example of a cactus generating each group. Below we complete this information by giving each time the number of orbits and their structure.

5.1 Degree 7

Passport: $[3 \times 2^2 1^3]$. The total number of the cacti is $N = 7 \times \left(\frac{4!}{2!3!}\right)^3 = 56$. There are four orbits:

- two A_7 -orbits of size 21;
- and two $PSL_3(2)$ -orbits of size 7.

The following two cacti are representatives of the A_7 -orbits:

$$\begin{array}{ll} g_1 = (4, 5)(6, 7) & g_1 = (3, 4)(6, 7) \\ g_2 = (2, 6)(3, 4) & \text{and } g_2 = (2, 6)(3, 5) \\ g_3 = (1, 2)(3, 6) & g_3 = (1, 2)(3, 6) \end{array}$$

and the following two are representatives of the $PSL_3(2)$ -orbits:

$$\begin{array}{ll} g_1 = (3, 4)(5, 7) & g_1 = (3, 5)(6, 7) \\ g_2 = (2, 5)(6, 7) & \text{and } g_2 = (2, 6)(4, 5) \\ g_3 = (1, 2)(3, 5) & g_3 = (1, 2)(3, 6) \end{array}$$

5.2 Degree 13

Passport: $[3 \times 2^4 1^5]$. The total number of cacti is $N = 13 \times \left(\frac{8!}{4!5!}\right)^3 = 35\,672$. There are five orbits:

- one A_{13} -orbit of size 35 620;
- four $PSL_3(3)$ -orbits of size 13.

The following four cacti are representatives of the four $PSL_3(3)$ -orbits:

$$\begin{array}{l}
g_1 = (3, 4)(5, 7)(9, 13)(10, 11) \\
g_2 = (2, 9)(3, 8)(6, 7)(10, 12) \\
g_3 = (1, 2)(3, 9)(5, 8)(10, 13) \\
\hline
g_1 = (3, 5)(6, 13)(7, 12)(8, 9) \\
g_2 = (2, 6)(4, 5)(8, 10)(11, 12) \\
g_3 = (1, 2)(3, 6)(7, 13)(8, 11) \\
\hline
g_1 = (4, 5)(6, 7)(8, 13)(9, 11) \\
g_2 = (2, 8)(3, 6)(9, 13)(10, 11) \\
g_3 = (1, 2)(3, 8)(4, 6)(12, 13) \\
\hline
g_1 = (4, 9)(6, 8)(10, 11)(12, 13) \\
g_2 = (2, 12)(3, 10)(5, 9)(7, 8) \\
g_3 = (1, 2)(3, 12)(4, 10)(6, 9)
\end{array}$$

and the following cactus is a representative of the A_{13} -orbit:

$$\begin{array}{l}
g_1 = (6, 7)(8, 9)(10, 11)(12, 13) \\
g_2 = (2, 12)(3, 10)(4, 8)(5, 6) \\
g_3 = (1, 2)(3, 12)(4, 10)(5, 8)
\end{array}$$

Remark 5.1 It is very easy to find a representative of the A_{13} -orbit: practically any randomly chosen cactus will do. It is more difficult to compute the whole orbit of size 35 620: our previous program was unable to do that. The most difficult part is to find the representatives of the remaining four small orbits.

5.3 Degree 15

Passport: $[2^6 1^3, 2 \times 2^4 1^7]$. The total number of cacti is $N = 15 \times \frac{8!}{6!3!} \times \left(\frac{10!}{4!7!}\right)^2 = 126\,000$. There are four orbits:

- one A_{15} -orbit of size 125 945;
- two $PSL_4(2)$ -orbits of size 5;
- one “imprimitive” (or composition) orbit of size 45, with the monodromy group $(S_3 \wr S_5) \cap A_{15}$.

The following two cacti are representatives of the two $PSL_4(2)$ -orbits:

$$\begin{aligned}
g_1 &= (2, 3)(4, 5)(6, 15)(7, 9)(10, 13)(11, 12) \\
g_2 &= (2, 6)(7, 14)(8, 9)(11, 13) \\
g_3 &= (1, 2)(4, 6)(7, 15)(10, 14) \\
\hline
g_1 &= (2, 3)(4, 15)(5, 9)(7, 8)(10, 14)(12, 13) \\
g_2 &= (2, 4)(5, 15)(6, 7)(11, 14) \\
g_3 &= (1, 2)(6, 9)(10, 15)(12, 14)
\end{aligned}$$

A representative of the orbit of decomposable cacti is

$$\begin{aligned}
g_1 &= (2, 3)(4, 5)(7, 8)(9, 10)(12, 13)(14, 15) \\
g_2 &= (2, 14)(4, 12)(6, 11)(7, 9) \\
g_3 &= (1, 2)(4, 14)(6, 12)(7, 11)
\end{aligned}$$

The “passport form” of the decomposition is as follows. The P -cactus is of degree 5, and its passport is

$$[1 : 221; \quad 2 : 21_a 11; \quad 3 : 21_b 11].$$

The Q -cactus is of degree 3, with two critical values a and b , and with the passport

$$[a : 21; \quad b : 21].$$

The passport of the composition is therefore

$$[1 : 2222221111; \quad 2 : 222(21)_a 1111111; \quad 3 : 222(21)_b 1111111].$$

Finally, a representative of the “big” A_{15} -orbit is

$$\begin{aligned}
g_1 &= (4, 5)(6, 7)(8, 9)(10, 11)(12, 13)(14, 15) \\
g_2 &= (2, 14)(3, 12)(4, 10)(6, 8) \\
g_3 &= (1, 2)(3, 14)(4, 12)(6, 10)
\end{aligned}$$

5.4 A new invariant

One may ask why there exist several orbits having the same monodromy group? The answer was found by Gareth Jones (Southampton). In fact, the monodromy group is an invariant not only as an abstract group, but also as a particular permutation group, that is, as a particular subgroup of S_n . The same group G may have several conjugate copies inside S_n . But we must not forget as well that we had fixed once and for all the product $g_1 \cdots g_k = \gamma_n = (1, \dots, n) \in S_n$.

Proposition 5.2 *There are two conjugate copies of the group $PSL_3(2)$ inside S_7 that contain the permutation γ_7 in their intersection; four copies of $PSL_3(3)$ inside S_{13} that contain γ_{13} ; and two copies of $PSL_4(2)$ inside S_{15} that contain γ_{15} .*

The details of this construction will be explained elsewhere.

Remark 5.3 The reasons why for the passport $[3 \times 2^2 1^3]$ there are two A_7 -orbits remain mysterious. This is, up to now, the only case when we cannot propose a clear combinatorial invariant that would explain such a splitting. Just mention that the orbits themselves are not isomorphic: there are no bijection between them that would commute with the braid group action.

5.5 Once more about the perspectives

After the polynomial case, what should be the next case to study? Previously our project (outlined in [2]) was to attack the topological classification of the rational functions. Today we think that a “more convenient” class of functions (at least for our approach) is the class of meromorphic functions of genus $g > 0$ having a single pole. The reasons for this change of the point of view are the following:

(1) Many particular features of our programs are based on the fact that the product $g_1 \dots g_k = \gamma = (1, \dots, n) \in S_n$ is a cyclic permutation. The case of an arbitrary permutation γ would necessitate a re-programming of almost everything, while the case of a positive genus needs only a very little adjustment.

(2) There are some enumerative results for the factorizations of a “big cycle” γ in a product of permutations of given cyclic structures [5], while the similar results for an arbitrary γ are very scarce.

(3) An approach based on the formula of Frobenius, that uses the characters of the symmetric group, permits in principle to compute the number of above factorizations for an arbitrary γ . But this method has another flaw: it counts also “non-transitive cacti”, and therefore necessitates a cumbersome inclusion-exclusion procedure, while the presence of a big cycle automatically guarantees the transitivity.

(4) Last but not least, the permutation groups containing a big cycle are classified [3] while the “planar monodromy groups” are not [6].

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