

Plane Trees and Algebraic Numbers

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ABSTRACT. Every combinatorial plane tree can be obtained as the inverse image of a segment via a complex polynomial, the coefficients of the polynomial being algebraic numbers. This polynomial, which we call a *generalized Chebyshev polynomial*, is uniquely determined up to linear transformations of the complex plane. Thus, every plane tree has a canonical geometric form.

To find the above polynomial one must solve a system of algebraic equations. Algebraic number fields thus arising have profound relations with the combinatorics of the corresponding trees. These relations, in a more general setting, were first indicated in the famous, though unpublished manuscript by Alexandre Grothendieck [23].

Generalized Chebyshev polynomials form a very interesting class of polynomials. With minor additional conditions, they are closed with respect to composition, it is possible to study their iterations. They produce an infinite number of solutions of the polynomial Pell equation first considered by Abel in 1826 [1]. These properties, as well as many others, are discussed in the present paper.

The authors have tried to make their exposition as elementary as possible.

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1. Introduction. Generalized Chebyshev polynomials

1.1. Critical points and critical values of complex polynomials. We begin with something that everybody remembers from his or her high school years. Consider a polynomial $P(z)$ of degree n with complex coefficients. It maps complex plane of the variable z onto another one, say of the variable w . Take a point $w \in \mathbb{C}$ and consider its inverse image $P^{-1}(w) = \{z | P(z) = w\}$.

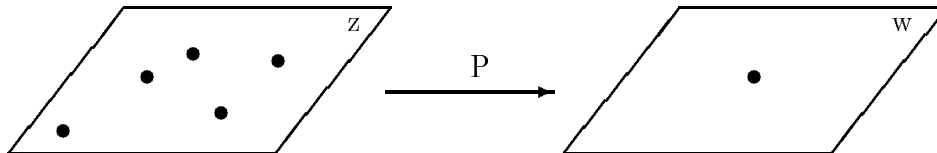


FIGURE 1. Inverse image of a point.

Usually, this set consists of n distinct points, i.e., solutions of the equation $P(z) = w$ for a given w (see Figure 1). But sometimes, for some specific values of w , this equation can have multiple roots. A multiple root of the equation $P(z) = w$ is, obviously, also one of the equation $P'(z) = 0$.

A point $z \in \mathbb{C}$ at which $P'(z) = 0$ is called a *critical point*. A value $w = P(z)$ of the polynomial at a critical point z is called a *critical value*. We say that critical point z has *order* k if at this point $P'(z) = 0$, $P''(z) = 0$, \dots , $P^{(k-1)}(z) = 0$, $P^{(k)}(z) \neq 0$. The least possible order of a critical point is thus equal to 2. However, in some situations by abuse of language we will call some *non-critical* points *critical points of order 1*.

Since $P'(z)$ is a polynomial of degree $n - 1$, the polynomial $P(z)$ usually has $n - 1$ critical points (of order 2) and $n - 1$ critical values. But in some degenerate cases one may observe critical points of higher order, and/or some critical values may be attached together into the same value.

What we are interested in is the most degenerate case, when the set of critical values of $P(z)$ is as small as possible.

EXAMPLE 1.1. The polynomial $P(z) = z^n$ has only one critical point of order n (namely, $z = 0$), and only one critical value $w = 0$.

Up to obvious changes of variables, this polynomial is the only one that has exactly one critical value. The next case, and a much more interesting one, is that of two critical values.

EXAMPLE 1.2. *Chebyshev polynomials*. It is well known that $\cos n\varphi$ can be expressed as a polynomial of degree n of $\cos \varphi$. For example,

$$\cos 2\varphi = 2 \cos^2 \varphi - 1, \quad \cos 3\varphi = 4 \cos^3 \varphi - 3 \cos \varphi, \quad \dots,$$

and in general

$$\cos n\varphi = T_n(\cos \varphi),$$

where $T_n(z)$ is the n -th *Chebyshev polynomial*,

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_2(z) = 2z^2 - 1, \quad T_3(z) = 4z^3 - 3z, \quad \dots,$$

and

$$T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z) \quad \text{for } n \geq 2.$$

The graph of the polynomial $T_n(z)$ on the segment $[-1, 1]$ resembles that of $\cos n\varphi$ on the segment $[-\pi, 0]$: all its maxima are equal to 1, and all its minima are equal to -1 . Thus, $T_n(z)$ has $n - 1$ critical points of order 2 (namely, $z = \cos \frac{k\pi}{n}$), but *only two critical values*: $w = \pm 1$.

Besides Chebyshev polynomials, there are many other polynomials that have only two critical values. Our main goal in this paper is to study the class of such polynomials.

DEFINITION 1.1. *We call a polynomial $P(z)$ a generalized Chebyshev polynomial if it has at most two critical values.*

In other words, for any such polynomial there exist two distinct complex numbers c_0, c_1 such that

$$P'(z) = 0 \quad \Rightarrow \quad P(z) \in \{c_0, c_1\}.$$

(For $P(z) = z^n$ one of the values (say, c_0) should be taken equal to $c_0 = 0$, and the other one may be chosen arbitrarily; by convention, we take $c_1 = 1$.)

REMARK 1.1. These polynomials were introduced in the paper [39]. In many publications (see for example [10], [11], [12] and some others) they are called *Shabat polynomials*.

1.2. Inverse image of a segment. Take two arbitrary complex numbers c_0, c_1 on the w -plane, and join them by a straight line segment. In what follows we will denote this segment by $[c_0, c_1]$, even in the case when one or both of its endpoints do not belong to \mathbb{R} , or when they are both real but $c_1 < c_0$. Suppose there are no critical values on this segment; then its inverse image $P^{-1}([c_0, c_1])$ is the a disjoint union of n separate sets on z -plane, each one being homeomorphic to a segment, but not necessarily straight (see Figure 2). We color the ends of $[c_0, c_1]$ black and white, in order to distinguish among their pre-images on the z -plane.

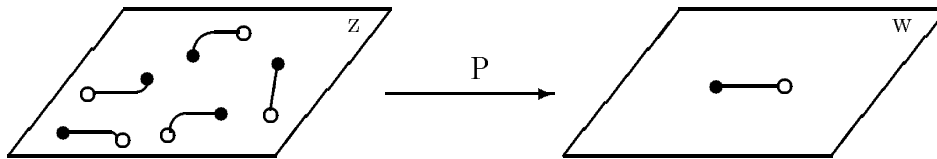


FIGURE 2. Inverse image of a segment.

Now suppose there are still no critical values *inside* the segment $[c_0, c_1]$, but one or both of its ends become critical. This leads to the fact that on the z -plane some “curvilinear segments” are glued one to another (see Figure 3).

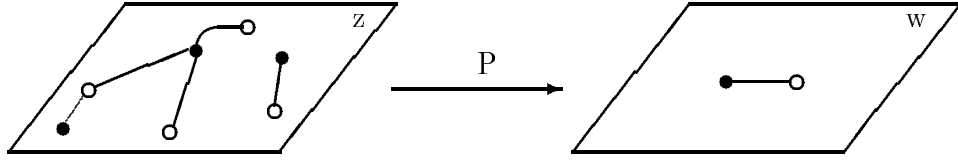


FIGURE 3. Ends of some segments become critical points.

Now, what happens if the polynomial $P(z)$ in question is a generalized Chebyshev polynomial, and c_1, c_2 are its (only) critical values? The answer is that we will see on the z -plane a *bicolored plane tree*.

EXAMPLE 1.3. For $P(z) = z^n$ the inverse image of the segment $[c_0, c_1] = [0, 1]$ is a “star-tree”; for $P(z) = T_n(z)$, a Chebyshev polynomial, the inverse image of the segment $[c_0, c_1] = [-1, 1]$ is a “chain-tree” (see Figure 4).

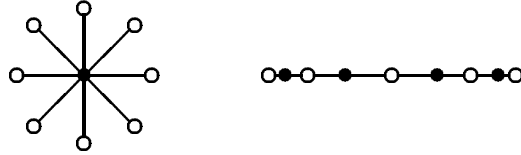


FIGURE 4. Trees that correspond to z^n and to $T_n(z)$.

Many other examples of generalized Chebyshev polynomials and of corresponding trees will be given later. But first we must introduce some definitions concerning trees.

1.3. Combinatorial bicolored plane trees. A *tree* is a connected graph without circuits. A *plane tree* is a tree which is drawn on (or embedded into) the plane. A “picture” of a tree on the plane determines, for any vertex of the tree, a cyclic permutation on the set of adjacent vertices. An isomorphism of plane trees is an isomorphism of trees as graphs that also preserves these cyclic permutations (and hence the orientation). An isomorphism class of plane trees is called a *combinatorial plane tree*. On Figure 5 one sees the same tree as an abstract (not embedded) graph, but two different plane trees.



FIGURE 5. One tree but two plane trees.

Any tree has a natural structure of a bipartite graph: its vertices can be colored in two colors (say, black and white) in such a way that adjacent vertices have different colors. Choosing one of the two possible colorings, we obtain a *bicolored plane tree*. An isomorphism of bicolored plane trees must preserve not only the structure of a plane tree, but also the colors of corresponding vertices. On Figure 6 one may see the same plane tree, but two different bicolored plane trees.

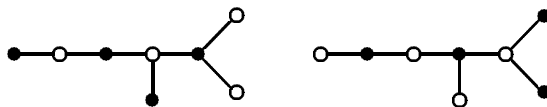


FIGURE 6. One plane tree, but two bicolored plane trees.

Let n be the number of edges, and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ (resp. $\beta = (\beta_1, \beta_2, \dots, \beta_q)$) be the sequence of valencies of black (resp. white) vertices in decreasing order. Every edge joins a black vertex to a white one; therefore,

$$\sum_{i=1}^p \alpha_i = \sum_{j=1}^q \beta_j = n.$$

On the other hand, the tree has p black vertices and q white vertices, hence

$$p + q = n + 1.$$

Thus, $\langle \alpha, \beta \rangle$ is just a pair of partitions of the number n , having (together) a total of $n + 1$ parts. We call it the *set of valencies* of the tree, and say also that the tree is of *type* $\langle \alpha, \beta \rangle$.

LEMMA 1.1. *For any pair $\langle \alpha, \beta \rangle$ of partitions of the number n having a total of $n + 1$ parts, there exists a bicolored tree of type $\langle \alpha, \beta \rangle$.*

PROOF. First note that at least one of the numbers α_i, β_j is equal to 1 (otherwise $\sum \alpha_i + \sum \beta_j \geq 2(n + 1)$). Let $\beta_q = 1$. Then at least one of the α_i is greater than 1; let $\alpha_1 > 1$. Therefore

$$\alpha' = (\alpha_1 - 1, \alpha_2, \dots, \alpha_p), \quad \beta' = (\beta_1, \beta_2, \dots, \beta_{q-1})$$

is a pair of partitions of the number $n - 1$ having together $p + (q - 1) = n$ parts. Suppose that the statement of the lemma is true for $n - 1$. Then, taking a tree with the set of valencies $\langle \alpha', \beta' \rangle$ and gluing an extra edge to a vertex of valency α_1 , we get a tree with the set of valencies $\langle \alpha, \beta \rangle$. \square

If we replace black by white and vice versa, the set of valencies $\langle \alpha, \beta \rangle$ will be replaced by $\langle \beta, \alpha \rangle$. Thus, we usually obtain a bicolored tree of another type. But in case when $\alpha = \beta$ the type remains the same. (Obviously, in such a case the number n of edges must be odd, because $p = q$ and $p + q = n + 1$.) Both trees shown in Figure 6 have the same type $\langle 3, 2, 1, 1; 3, 2, 1, 1 \rangle$.

Throughout the paper we will not consider any other species of trees but combinatorial bicolored plane trees. For the sake of brevity *from now on the term tree means combinatorial bicolored plane tree.*

1.4. Main theorem. It is obvious from the definition that linear transformations of the z -plane and w -plane do not destroy the property of a polynomial to be of “generalized Chebyshev” type. This fact is reflected in the following definition.

DEFINITION 1.2. *Let $P(z)$ and $Q(z)$ be two generalized Chebyshev polynomials, and let c_0, c_1 and d_0, d_1 be their critical values. We call the two pairs $(P, [c_0, c_1])$ and $(Q, [d_0, d_1])$ equivalent, if there exist constants $A, B, a, b \in \mathbb{C}$, $A, a \neq 0$, such that*

$$Q(z) = AP(az + b) + B, \quad \text{and} \quad d_0 = Ac_0 + B, \quad d_1 = Ac_1 + B.$$

By abuse of language, instead of saying that the pairs $(P, [c_0, c_1])$ and $(Q, [d_0, d_1])$ are equivalent, we shall say that the polynomials P and Q are equivalent.

THEOREM 1.1. *There is a bijection between the set of (combinatorial bicolored plane) trees and the set of equivalence classes of generalized Chebyshev polynomials.*

PROOF. In section 1.2 it was explained why the inverse image of the segment connecting two critical values of a generalized Chebyshev polynomial is a *graph*. Its black vertices are solutions of the equation $P(z) = c_0$, its white vertices are solutions of $P(z) = c_1$. Vertices of valency greater than 1 are critical points of order equal to the valency. Vertices of valency 1 are ordinary (non-critical) points, i.e. non-multiple roots of the equations $P(z) = c_0$ and $P(z) = c_1$.

The first question to answer is why this graph has no circuits. Suppose it has one (see Figure 7).

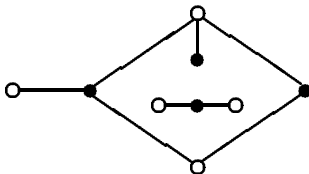


FIGURE 7. A graph with a circuit.

Using, if necessary, a linear transformation of the w -plane, assume the points c_0 and c_1 to be real. Then our polynomial takes only real values on the boundary of the domain bounded by our circuit. This means that its imaginary part $ImP(z)$, while being a harmonic function of variables x, y , is identically equal to zero on the boundary. Therefore, it is identically equal to zero inside, which is impossible.

A graph without circuits is either a tree, or a forest, i.e. a non-connected union of several trees. Let us prove that it is in fact connected. To do that, let us count the number of its vertices. Two equations $P(z) = c_0$ and $P(z) = c_1$ together have $2n$ roots (with multiplicity taken into account). But we need to know their number *without multiplicities*. The multiplicity of a root is equal to the valency $\text{Val}(v)$ of the corresponding vertex v . Thus

$$\#(\text{vertices}) = 2n - \sum_v (\text{Val}(v) - 1),$$

where the sum is taken over all the vertices. Recall that a vertex of valency k is also a root of $P'(z)$, its multiplicity being equal to $k - 1$. The number of roots of $P'(z)$ (with multiplicity taken into account) is equal to $n - 1$. Hence,

$$\sum_v (\text{Val}(v) - 1) = n - 1,$$

and the number of vertices is equal to $2n - (n - 1) = n + 1$.

We have proved that for any generalized Chebyshev polynomial the inverse image in question is a tree. (A more straightforward topological argument is as follows. The inverse image of the complement to $[c_0, c_1]$ in *Riemann sphere* is a disc: indeed, P maps it to this complement with just infinity as a ramification point of the maximal order. Hence, the inverse image of the segment itself cannot have either cycles, or more than one component.)

In the opposite direction, we need to prove two facts: (1) *For any (combinatorial bicolored plane) tree there does exist a generalized Chebyshev polynomial that produces exactly this tree*, and (2) *the polynomial in question is unique, up to linear transformations of the variables z and w* .

The proof actually consists in merely repeating several standard facts of the theory of Riemann surfaces. However, a careful exposition of the topic would demand quite a few pages. Therefore here we only give a sketch of the proof.

a) First we “kill” the linear transformation on the w -plane, moving the points c_0 and c_1 to their standard positions $c_0 = 0, c_1 = 1$.

b) Add an infinity to the z -plane, thus making it a sphere, and join it to *all* the vertices of the tree. We obtain a triangulation of the z -sphere. (On Figure 8 only two triangles are shown.)

In the same way we make a sphere out of the w -plane. Dividing it into two half-planes by the real axis we obtain a triangulation of the w -sphere: indeed, topologically a half-plane is a triangle, with vertices 0, 1 and ∞ .

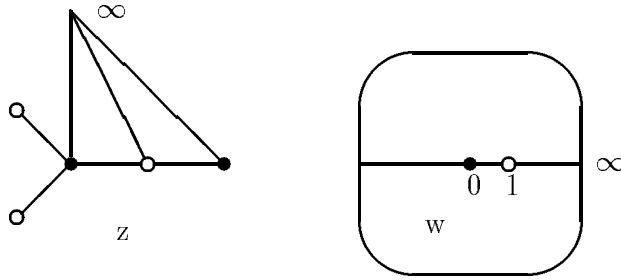


FIGURE 8. Triangulations of the Riemann spheres.

c) We need to “introduce a complex structure” on the triangulation of the z -sphere, i.e. introduce a complex coordinate on each triangle in such a way that these coordinates agree on the boundaries. In order to do that, let us *identify* each triangle with the upper or lower w -half-plane according to the orientation. This means that when we draw a triangle on the z -sphere, we have in mind a complex half-plane. (The coordinate in the half-planes is w , not z !)

d) Now glue together the half-planes in the way indicated by our triangulation. We will not describe the procedure of gluing here, but mention that it includes the “compactification” at the points $0, 1, \infty$. The result is a Riemann surface *by definition* of the latter. It is the Riemann sphere, because our triangulation topologically is a sphere. Being a sphere, it has its own natural coordinate z .

e) The “natural projection”, that assigns to each point z of our Riemann surface its “coordinate” w , is a meromorphic function from the Riemann sphere to the Riemann sphere, which takes ∞ to ∞ . Thus it is a polynomial.

f) The above function gives a covering of the sphere, unramified outside $0, 1$ and ∞ . Such a covering is uniquely determined up to an automorphism of the Riemann sphere. Automorphisms of Riemann sphere are fractional-linear transformations. But we need only those of them that fix infinity, hence, linear transformations.

The theorem is proved. \square

Details of the theory of Riemann surfaces can be found in many books (for example, [22], [33]).

1.5. Geometry of plane trees. Maybe the most surprising consequence of the main theorem is the following fact: **Every plane tree has a unique and canonical geometric form.** Any tree can be drawn on the plane in infinitely many different ways (cf. Figure 9). But one of its geometric forms is distinguished. Indeed, the linear transformation $z \rightarrow az + b$ that was mentioned in the previous section, may change the size of a tree and its position on the z -plane, but it does not change its geometric form.

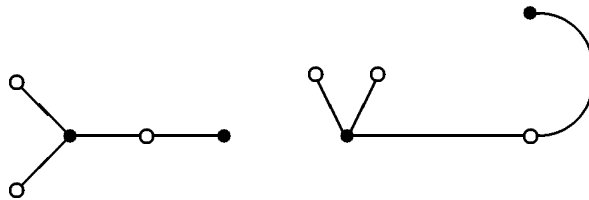


FIGURE 9. Two “geometric forms” of the same combinatorial tree.

In the catalog [10] the computer-made “true” geometric forms are given for all the plane trees up to 8 edges. On the next page we give an excerpt from this catalog.

1.6. Calculation of generalized Chebyshev polynomials. In subsequent sections we will discuss many concrete examples of the computation of generalized Chebyshev polynomials. Here we give only some generalities.

Let the type $\langle \alpha, \beta \rangle = \langle \alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q \rangle$ of a tree be given, and let us also fix $c_0 = 0, c_1 = 1$. Denote a_1, a_2, \dots, a_p and b_1, b_2, \dots, b_q coordinates of the black and white vertices respectively. Then we have

$$P(z) = \lambda(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} \dots (z - a_p)^{\alpha_p}$$

and

$$P(z) - 1 = \lambda(z - b_1)^{\beta_1}(z - b_2)^{\beta_2} \dots (z - b_q)^{\beta_q}.$$

The equalities between the coefficients provide us with n algebraic equations in $n + 2$ unknowns $\lambda, a_1, \dots, a_p, b_1, \dots, b_q$. Two additional “degrees of freedom” correspond to the possibility of making a linear transformation of the z -plane. We may use it in the way we find convenient. We may, for example, fix the positions of any two vertices, or make some other choice that places the tree in an unambiguous position on the complex plane. In addition to the above set of equations, we must impose the condition that all the coordinates a_i, b_j are distinct.

This method, taken literally, is far too complicated, even if we use a powerful system of symbolic computations, such as MAPLE. Many improvements of various levels of generality were proposed; see, for example, [10], [17], [34]. We may only remark that computing generalized Chebyshev polynomials is a very refreshing activity. While the usual enumerative approach considers the objects to be enumerated as a mass of more or less indistinguishable entities, the present approach demands to look at the objects in a more individual way, to make a “personal acquaintance” of sorts.

1.7. Generalizations. The reader is maybe interested to know if it is possible to extend this theory from the case of plane trees to a more general case, that of maps. In fact, it was the more general theory that was invented first, and we deal with its specialization. The theory, as it was proposed in [9], [23],

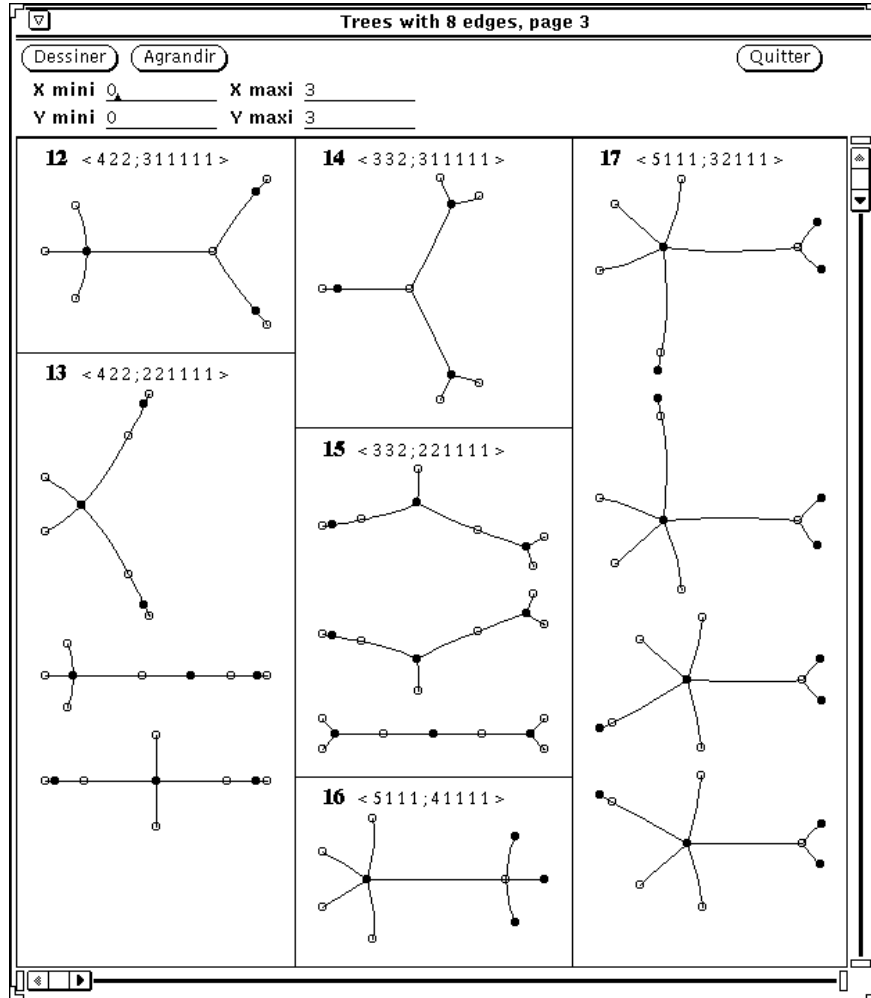


FIGURE 10. Excerpt from the catalog [10].

applies to arbitrary maps (and hypermaps) on arbitrary oriented surfaces. It was discovered later on that even the simplest case, that of plane trees, is rich enough to be worth considering more closely.

For an arbitrary plane map, instead of looking for a generalized Chebyshev polynomial, one must look for a *pure Belyi function*, i.e., a rational function $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ on the Riemann sphere that satisfies the following conditions:

- (1) f has only three critical values, namely, 0, 1 and ∞ ;
- (2) all the points of $f^{-1}(1)$ are critical points of degree exactly equal to 2.

The graph itself is the set $f^{-1}([1, \infty])$. Its vertices are the points of $f^{-1}(\infty)$, while the points of $f^{-1}(0)$ are the “centers” of faces, and the points of $f^{-1}(1)$ are the “midpoints” of the edges. The function f is unique, up to a fractional-linear transformation of $\overline{\mathbb{C}}$.

EXAMPLE 1.4. If a map in question is a tree, and $P(z)$ is its generalized Chebyshev polynomial *whose critical values are made equal to ± 1* then

$$f(z) = \frac{1}{1 - P^2(z)}$$

is the corresponding Belyi function.

For a map with n edges, the degrees of the numerator and denominator of f are less than or equal to $2n$, one of them being of degree exactly $2n$. Thus, from the computational point of view, computing Belyi function is “four times as hard” as computing a generalized Chebyshev polynomial.

When we pass to the surfaces of genus $g > 0$, we encounter a new and very serious difficulty. While from the topological point of view there is, up to homeomorphism, only one surface of genus g , there are infinitely many non-isomorphic Riemann surfaces of the same genus, depending on $3g - 3$ complex parameters (“moduli”) for $g \geq 2$, and on one complex parameter for $g = 1$. The problem is now to find a *Belyi pair*, i.e., a couple (X, f) , where X is a Riemann surface, and f is a Belyi function, $f : X \rightarrow \overline{\mathbb{C}}$. For $g = 1$, the Riemann surfaces in question are *elliptic curves* (topologically all of them are tori); several examples of the “dessins d’enfant” on elliptic curves are considered in [36], [28], [8]; in some cases the computation even for a 3-edged map becomes very difficult and involves the high-level machinery of algebraic geometry.

If the genus increases, the notion of moduli of Riemann surfaces acquires a more and more abstract form; starting from $g = 5$, it is in no way clear how to *write* the needed equations, to say nothing about solving them. Still, some examples are known, mainly for very regular maps. But theoretical results are of no less importance; to finish this section, we cite the fundamental Belyi theorem ([9]), which provides the foundation of the whole theory of “dessins d’enfant”.

THEOREM 1.2. *For a Riemann surface X , a meromorphic function $f : X \rightarrow \overline{\mathbb{C}}$ with only three critical values 0, 1 and ∞ exists if and only if the surface X is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.*

Thus, a very important and profound connection is established between the theory of maps and the theory of algebraic number fields.

For the applications to quantum field theory see [8], [40]. It is worth mentioning that the objects very similar to “dessins d’enfant” were introduced (under the name of “pictures”) as early as in 1970 [45].

2. Examples

2.1. A very simple example. The first tree which is not a star and not a chain is the tree with the set of valencies $\langle 3, 1; 2, 1, 1 \rangle$ (see Figure 11).

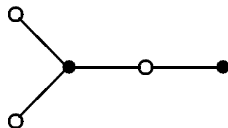


FIGURE 11.

Let us take $c_0 = 0$, so that the coordinates of the black vertices are the roots of the polynomial we are looking for, the vertex of valency 3 being the root of multiplicity 3. Let us place this vertex at the point $z = 0$, and the other black vertex, at the point $z = 1$. Then we may take $P(z) = z^3(z - 1)$, and we are finished.

Now to compute the coordinates of the other (white) vertices, we compute the derivative $P'(z) = z^2(4z - 3)$. The derivative has a double root at $z = 0$ (which is only natural), and one more root $z = \frac{3}{4}$; it is exactly the position of the white vertex of valency 2.

Now we may compute the other critical value of $P(z)$: namely,

$$c_1 = P\left(\frac{3}{4}\right) = -\frac{27}{256}.$$

Finally, to get the coordinates of two other white vertices, we solve the equation

$$z^3(z - 1) = -\frac{27}{256}.$$

It has a double root $z = \frac{3}{4}$, and two simple roots, $\frac{-1 \pm \sqrt{-2}}{4}$.

Looking at the coordinate $\frac{3}{4}$ of the white vertex and at the negative value of c_1 , we may conclude that it would have been more clever to take from the very beginning the position of the black vertex of valency 1 at $z = 4$, and to change the sign: $P(z) = z^3(4 - z)$. Then the white vertex of valency 2 finds itself at $z = 3$, the second critical value is $P(3) = 27$, and the coordinates of the other two white vertices become equal to $-1 \pm \sqrt{-2}$.

2.2. First example of conjugate trees. Consider the following plane tree of type $\langle 3, 2; 2, 2, 1, 1, 1 \rangle$, see Figure 12, and take, as usual, $c_0 = 0$. Let

us place the black vertex of valency 3 at the point $z = 0$. Place the two black vertices of valency 2 so that the midpoint of the segment joining them is $z = 1$, or, in other words, the sum of the corresponding complex numbers is equal to 2.

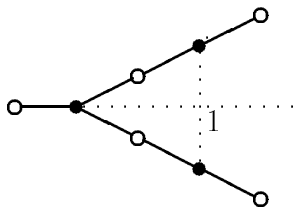


FIGURE 12.

Then these two points are the roots of a quadratic polynomial $z^2 - 2z + a$, with a yet unknown value of the parameter a . Thus, the corresponding generalized Chebyshev polynomial is equal to

$$P(z) = z^3(z^2 - 2z + a)^2.$$

What are the additional conditions that enable us to find a ? To answer the question, let us compute the derivative:

$$P'(z) = z^2(z^2 - 2z + a)(7z^2 - 10z + 3a).$$

The derivative has (as it should have) a double root at $z = 0$, two simple roots, those of $z^2 - 2z + a$, and two more simple roots, namely the roots of the polynomial

$$Q(z) = 7z^2 - 10z + 3a.$$

Obviously, these latter roots are the coordinates of the two white vertices of valency 2. Thus, we have two conditions to insure:

- (1) the roots of $Q(z)$ must be distinct, hence $21a - 25 \neq 0$;
- (2) (more important condition) *the values of $P(z)$ at the roots of $Q(z)$ must be equal to each other.*

Our first intention is to compute the roots of $Q(z)$ and to substitute them into $P(z)$. But this way is far too cumbersome. The following one is better: let us divide P by Q ,

$$P(z) = S(z)Q(z) + R(z),$$

where $R(z)$ is the remainder, and, hence, is linear (since Q is quadratic): $R(z) = Az + B$, with coefficients A and B depending on the parameter a . Whenever $Q = 0$, the values of P coincide with those of R , hence the values of $R(z)$ must be equal to each other at the roots of $Q(z)$. This means that we have $A = 0$.

The results of the computation are as follows:

$$A = -\frac{16}{7^6}(21a - 25)(49a^2 - 476a + 400),$$

$$B = -\frac{196}{7^6}a(28a - 25)(7a - 10).$$

Therefore the conditions $A = 0$ and $21a - 25 \neq 0$ lead to the equation

$$S(a) = 49a^2 - 476a + 400 = 0,$$

hence

$$a = \frac{1}{7}(34 \pm 6\sqrt{21}),$$

while the second critical value is equal to $c_1 = B = B(a)$. A priori it is not obvious which one of the two values of a we must take, so let us take both. Then, besides the tree in Figure 12, we also obtain the one of Figure 13.

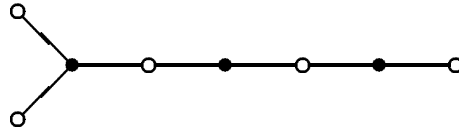


FIGURE 13. The tree conjugate to that of Figure 12.

What do these two trees have in common? The answer is, they are of the same type: $\langle 3, 2, 2; 2, 2, 1, 1, 1 \rangle$. If we look more closely at the process of computations, we will see that the set of valencies is the only information we used about the trees in question.

Proceeding from the above example, we would like to introduce a few terms. The number field $\mathbb{Q}(\sqrt{21})$ is the *field of definition* of the two trees. The polynomial $S(a) = 49a^2 - 476a + 400$, whose roots generate the field of definition, is called a *defining polynomial*. The trees themselves are called *conjugate* (to allude to the fact that the two values of the parameter a are algebraically conjugate). The set that consists of these two trees is a *Galois orbit*, i.e. an orbit under the action of the universal Galois group $\Gamma = \text{Gal}(\overline{\mathbb{Q}} | \mathbb{Q})$ on the set of bicolored plane trees. More precise meaning of each term will be explained later.

REMARK 2.1. Let us return to the case $21a - 25 = 0$. Why does the factor $21a - 25$ enter into the coefficient A ? Obviously, $a = \frac{25}{21}$ corresponds to a double root of $Q(z)$, which *does not contradict* the condition that “the values of P at the roots of Q are equal”. What it does contradict is the set of white valencies of the tree. Taking $a = \frac{25}{21}$ we obtain a tree in which instead of two white vertices of valency 2 we get one white vertex of valency 3, thus forming the tree with the set of white valencies $3, 1, 1, 1, 1$ instead of $2, 2, 1, 1, 1$. This tree is shown in Figure 14.

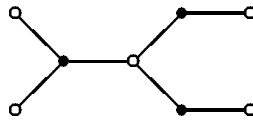


FIGURE 14.

2.3. A cubic orbit. In the same vein, we may consider a *cubic orbit*, i.e., a set of three conjugate trees with a cubic field of definition. Let us take the set of trees of the type $\langle 3, 2, 1; 2, 2, 1, 1 \rangle$ (see Figure 15).

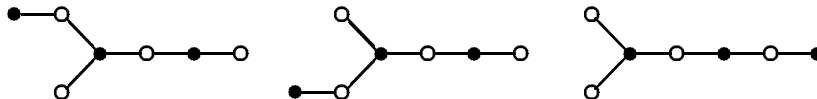


FIGURE 15. A cubic orbit.

Place the black vertex of valency 3 at $z = 0$, the black vertex of valency 2 at $z = 1$, and the black vertex of valency 1 at $z = a$. Then we get a generalized Chebyshev polynomial of the form

$$P(z) = z^3(z - 1)^2(z - a),$$

whose derivative is equal to

$$P'(z) = z^2(z - 1)Q(z), \quad \text{where } Q(z) = 6z^2 - (5a + 4)z + 3a.$$

The two white vertices of valency 2 are the roots of $Q(z)$. They must be different, hence the discriminant of $Q(z)$ must be non-zero: $25a^2 - 32a + 16 \neq 0$. The remainder after division of $P(z)$ by $Q(z)$ is $Az + B$, where

$$A = -\frac{1}{6^5}(25a^2 - 32a + 16)(25a^3 - 12a^2 - 24a - 16),$$

$$B = \frac{1}{2^5 3^4} a(5a - 8)(25a^3 - 6a^2 + 8).$$

Hence the condition $A = 0$ (together with the condition on the discriminant of $Q(z)$) gives us the cubic equation

$$S(a) = 25a^3 - 12a^2 - 24a - 16 = 0.$$

The roots of this cubic equation generate a cubic number field, which is the field of definition of this set of trees.

The solutions of the equation $25a^2 - 32a + 16 = 0$, which are equal to $a = \frac{4}{25}(4 \pm 3i)$, lead to identification of the roots of $Q(z)$; geometrically that means the gluing the two white vertices of valency 2 into one (white) vertex of valency 3. Thus we obtain type $\langle 3, 2, 1; 3, 1, 1, 1 \rangle$. The corresponding “quadratic orbit” is shown on Figure 16.

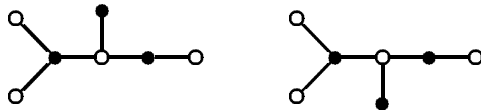


FIGURE 16.

For a cubic field we may ask what its **Galois group** is. For an irreducible polynomial of degree n , the corresponding Galois group is a certain group of permutations of its roots, which acts transitively on the set of roots. Thus, it is a transitive subgroup of the symmetric group S_n . For $n = 2$ there is only one possibility, namely, the group S_2 itself. But for $n = 3$ there are already two transitive subgroups of S_3 , namely, S_3 itself and the cyclic group C_3 . It is well known (see, for example, [14]) that the Galois group is equal to C_3 if and only if the discriminant of the defining polynomial $S(a)$ is a complete square. Computing the discriminant for $S(a) = 25a^3 - 12a^2 - 24a - 16$ we get $D = -5038848 = -2^8 3^9$, which is not a square. Therefore, the Galois group of this field is S_3 .

The question is, however, whether it is possible to “see” the Galois group geometrically (or combinatorially), without actually computing $S(a)$. This is not always the case, but sometimes we can give an affirmative answer. For instance, in this particular example, there are two trees that are symmetric to one another with respect to an axis, the third one being symmetric to itself. Therefore, complex conjugation transforms the first tree into the second one and vice versa, but does not touch the third one. This means that the Galois group contains a simple transposition and cannot therefore be cyclic.

By the way, the equation $S(a) = 0$ can be solved explicitly:

$$a_i = \frac{4}{25} + \frac{6}{25} \varepsilon_i (3\sqrt[3]{2} + \sqrt[3]{4}), \quad i = 1, 2, 3,$$

where ε_i are the cubic roots of 1. This gives a more explicit presentation of the field of definition: it is $\mathbb{Q}(\sqrt[3]{2})$, so the fact that its Galois group is S_3 is rather well-known.

2.4. Families vs. orbits. After considering the two previous examples, we may ask the following question: *Is it always the case that the set of all the trees of the same type forms an orbit of the Galois group action?*

More precisely, are the coefficients of the corresponding generalized Chebyshev polynomials always algebraically conjugate?

The answer is: very often but not always. Here is the simplest counter-example.

EXAMPLE 2.1. Take the set of valencies $\langle 4, 1, 1; 2, 2, 1, 1 \rangle$. There are two trees of that type (see Figure 17).

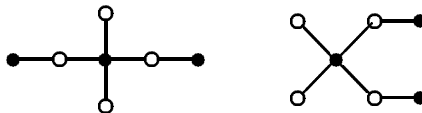


FIGURE 17. One family but two orbits.

On the basis of our previous experience we could suppose that they would form

a quadratic orbit, the coefficients of the corresponding generalized Chebyshev polynomials being conjugate quadratic irrationalities. However this is not true. Both polynomials have rational coefficients: for the first tree it is

$$P(z) = z^4(z^2 - 1),$$

and for the second one,

$$P(z) = z^4(z^2 - 2z + \frac{25}{9}).$$

Indeed, place the vertex of valency 4 at $z = 0$, and consider the quadratic polynomial $p(z)$ with the roots at the two other black vertices. The first tree is centrally symmetric; hence, the sum of the roots of $p(z)$ is equal to 0, and $p(z) = z^2 - a$; the parameter a can be chosen arbitrarily, for example, $a = 1$. On the other hand, the second tree is *not* symmetric, so the sum of the roots is *not* equal to 0; we may set this sum equal to an arbitrary non-zero number, for example, to 2. Then $p(z) = z^2 - 2z + a$, and we look for the value of a using the same procedure as in the examples above.

Thus, instead of having one (quadratic) orbit of the Galois group action, we have in this example two distinct orbits, and for both trees the field of definition is \mathbb{Q} . We use the following terminology: the trees of the same type are called *combinatorially conjugate*. Usually, but not always, they are also algebraically conjugate. The set of all the trees of the same type is called a *family* of combinatorially conjugate trees. Every family is either an orbit of the Galois group action, or a union of several different orbits.

The phenomenon of the splitting of families into several orbits is very interesting, and an important problem is to find the combinatorial reasons for such a splitting. One of the reasons can be obtained by generalizing from the last example: if a family contains a tree that is centrally symmetric and another that is not, then these trees will certainly belong to different orbits. But this is not the only possible reason, as is demonstrated by the following example.

EXAMPLE 2.2. Take the set of valencies $\langle 4, 2, 1; 2, 2, 1, 1, 1 \rangle$. The family of trees of that type contains 4 trees, see Figure 18.

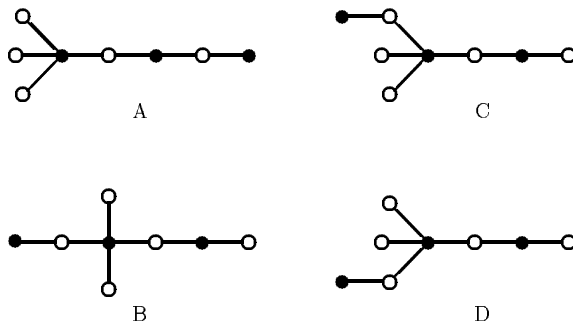


FIGURE 18. Two quadratic orbits.

If this were a generic case, we would have a field of definition of degree 4, and all the trees would belong to a single orbit. Computations show, however, that in fact the family splits into two quadratic orbits. One of them consists of the trees (A) and (B), with the field of definition $\mathbb{Q}(\sqrt{21})$, and the second one, of the trees (C) and (D), with the field of definition $\mathbb{Q}(\sqrt{-7})$. We don't see any apparent reason for this splitting.

REMARK 2.2. The following observation may lead to a plausible conjecture. The usual method of combinatorial representation of maps on oriented surfaces is via an *effective cartographic group*, namely, the group of permutations on the set of the *ends of edges*, generated by two permutations: the first one permutes the ends of each edge (thus, it is an involution without fixed points); the second one is the circular rotation of the ends adjacent to each vertex. For a map with n edges, this group is a transitive subgroup of \mathfrak{S}_{2n} , its transitivity meaning that the graph itself is connected (see [16]). We have observed that for the trees (A) and (B) the order of the effective cartographic group is $12700800 = 2^7 3^4 5^2 7^2$, while for the trees (C) and (D) the order of the group is $56448 = 2^7 3^2 7^2$. But we don't know whether such a "splitting" of the order of the cartographic group always implies the splitting of the family into separate orbits.

We add a few additional remarks concerning the last example. We admit that, by merely looking at the picture, it is impossible to guess that the family splits into two orbits. Still, we may eliminate at least certain possibilities on the basis of purely geometric information. We may reason as follows: "Suppose that this family forms an orbit of degree 4. What could have been its Galois group? We see that the complex conjugation takes the trees (C) and (D) into each other, but does not touch the trees (A) and (B). Therefore, its Galois group contains a simple transposition. Then it cannot be a subgroup of the group A_4 (obvious); and it cannot be commutative (exercise)."

A following very interesting example of the same kind is given in [34].

EXAMPLE 2.3. Take the family of trees of type $\langle 5, 1, 1, \dots, 1; 6, 5, 4, 3, 2 \rangle$ (20 edges). There are 24 trees in the family, all of the diameter 4: they have a black "center" of valency 5, with 5 "branches" going to the white vertices of valencies 6, 5, 4, 3, 2; and the latter vertices, besides the center, are adjacent to 5, 4, 3, 2, 1 black "leaves". Each tree is uniquely determined by a cyclic permutation of branches around the center. If we put the center into the point $z = 0$, and put, for example, the white vertex of valency 6 into the point $z = 1$, then 24 trees correspond to 24 permutations of the branches of valencies 5, 4, 3, 2.

Instead of having one orbit of degree 24, we obtain in this example two separate orbits of degree 12 each. It turns out that one of them corresponds to the set of even permutations, i.e., to the elements of the group A_4 , while the set of odd permutations forms another orbit. This description of the orbits looks more than natural, but, as far as we know, there is no convincing explanation of the phenomenon.

An infinite series of examples of splitting families, constructed by Adrianov,

will be given in Example 5.2. One more “combinatorial reason” for the splitting will be explained in section 6.

2.5. Unambiguous positioning. In all previous examples we took the liberty of placing some vertices at certain points of complex plane. This procedure is very simple when applied to the so-called “bachelors”: a vertex is a *bachelor* if it is the only vertex of given color and valency. For vertices that are not unique, i.e., when there are several of them of the same color and valency, we may encounter certain difficulties.

Consider the family of type $\langle 3, 3, 2; 2, 2, 1, 1, 1, 1 \rangle$. There are three trees of that type (see Figure 19), one of them is centrally symmetric.

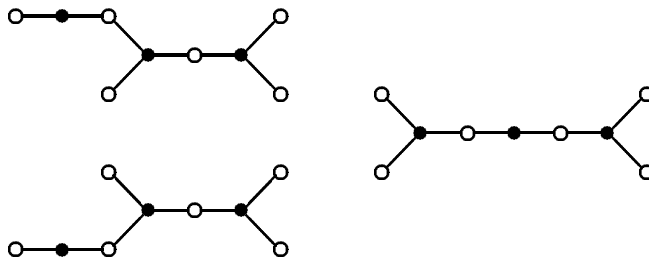


FIGURE 19.

There is no problem with the symmetric one: we place its center at $z = 0$, and two other black vertices, at the points $z = \pm 1$, thus getting generalized Chebyshev polynomial $P(z) = z^2(z^2 - 1)^3$.

Now let us turn to the other two trees. First trial: put two black vertices of valency 3 at the points $z = \pm 1$, and the other black vertex, at the point $z = a$. We obtain a polynomial of the form $P(z) = (z^2 - 1)^3(z - a)^2$. Proceeding in the same manner as in all previous examples, we get, quite unexpectedly, a field of degree 4, with defining polynomial $S(a) = a^4 - 8a^2 + 32$. Making a picture with the help of a computer, we get each of the trees twice: once with the “long branches” on the left, the second time on the right. From the theoretical point of view, we have not obtained the field of definition but some bigger field, which includes the field of definition as its proper subfield.

Let us make another trial; in order to “break the symmetry” put one of the black vertices at $z = 0$, and the other one at $z = 1$. The result is even worse. We still have 4 different values of a , though this time the symmetry is less obvious: they are symmetric with respect to the axis $x = \frac{1}{2}$ instead of the axis $x = 0$.

Several other possibilities lead to similar results. We may put, for example, the black vertex of valency 2 at $z = 0$, and one of the black vertices of valency 3, at $z = 1$, and we still get a field of degree 4 instead of 2. The point is the black vertex of valency 2 is a bachelor, and we may put it anywhere we like, for example, at $z = 0$. But the two black vertices of valency 3 form a “couple”;

there is no unambiguous way of positioning only one of them, because we cannot algebraically distinguish among them. Our only possibility is to “position” the quadratic polynomial that has these two vertices as its roots. And, indeed, taking this polynomial equal to $p(z) = z^2 - 2z + a$, that is, the generalized Chebyshev polynomial of the form $P(z) = z^2(z^2 - 2z + a)^3$, we get the quadratic field $\mathbb{Q}(\sqrt{-1})$ we were looking for, and $a = \frac{1}{8}(7 \pm i)$. If we now ask a computer to make a picture, both trees will appear on it only once.

The notion of an unambiguous positioning of a tree is not yet fully formalized. We believe that there exists a (yet unknown) group that acts on the set of vertices by permuting those of the same color and valency. But the nature of this group is not yet clarified. The above recipe, however, is of a general nature. Every time we encounter a tree with several vertices of the same color and valency, we must consider the polynomial having all these vertices as roots. See [17] for details.

Note that in the general case of *maps* the situation described above is sometimes inevitable. There are plane maps with a certain field of definition, such that the coefficients of the corresponding Belyi function belong to a bigger field, and it is impossible to find them inside the field of definition itself. This phenomenon was discovered independently by V. Filimonenkov and J.-M. Couveignes (see [21], [17]); it is related to the theory of Galois cohomology.

An inquisitive reader may ask one more question. Suppose that there are 6 trees in a family, we did obtain an equation of degree 6, and found 6 solutions. *Why do all the solutions correspond to different trees?* Maybe two of them correspond to the same tree, and one of the trees does not have any representation at all? The answer is that this contradicts the main theorem (section 1.4). According to this theorem, for any tree there exists at least one solution of our equation. We see that the main theorem is very powerful: the methods of proof are not algebraic in nature, but they have very strong algebraic consequences.

2.6. Mistakes promote understanding. Sometimes making a mistake, and then trying to understand what the matter is, considerably helps to understand the subject. To finish this long section containing a lot of examples, we describe here three mistakes made by the *second author* in the process of understanding the algebraic structures related to plane trees.

EXAMPLE 2.4. (Mistake 1) Consider the trees of type $\langle 4, 2, 1; 3, 1, 1, 1, 1 \rangle$ (see Figure 20).

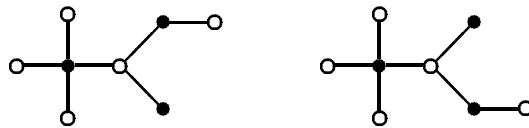


FIGURE 20.

Positioning the black vertex of valency 4 at $z = 0$ and the white vertex of valency 3 at $z = 1$, we get a generalized Chebyshev polynomial $P(z)$ whose derivative is equal to $P'(z) = z^3(z-1)^2(z-a)$. Computing $P(z)$ as an integral, and adding the condition $P(a) = 0$, we get $a = \frac{1}{10}(14 \pm \sqrt{-14})$. The appearance of complex conjugation is quite natural, since the trees are axially symmetric to each other. But the square root $\sqrt{-14}$ seems to be a kind of exaggeration, and we started to look for a possible simplification of the field of definition, hoping to reduce it to $\mathbb{Q}(\sqrt{-1})$.

Alas, all our efforts were in vain. A dozen of different approaches to writing a system of algebraic equations corresponding to this orbit all lead to the same field $\mathbb{Q}(\sqrt{-14})$. This is what made us understand that the field of definition is an *objective* characteristic of the orbit due to the nature of things, and cannot be subjected to arbitrary change.

For exact definition of this notion see section 4.

EXAMPLE 2.5. (Mistake 2) Consider the trees of type $\langle 3, 2, 1, 1; 3, 2, 1, 1 \rangle$ (we leave it to the reader to make a picture). There are six plane trees of that type; three of them are centrally symmetric, the other three are not. It is rather simple to obtain the field of definition for the orbit consisting of three symmetric trees: it is $\mathbb{Q}(\sqrt[3]{28})$. But the three asymmetric ones gave us enormous trouble. All our efforts to find a cubic field proved to be futile: we kept getting a field of degree 6. Finally we gave up and drew a computer-made picture. And we saw immediately the source of our mistake: yes, indeed, there are three *plane* trees; but there are six *bicolored plane* trees (two of them are shown on Figure 6). Thus we learned that the proper combinatorial structure related to the number fields is that of a bicolored plane tree, and not that of a plane tree as such.

Obviously, such a situation may occur only when the black and white partitions α and β coincide.

EXAMPLE 2.6. (Mistake 3) While preparing the catalog [10], the computer produced a picture of an orbit containing two trees of type $\langle 3, 2, 2; 2, 2, 1, 1, 1 \rangle$ (see section 2.2). The picture looked awkward: one of the trees was very narrow and horizontal, while the other was also very narrow but vertical, and thus the sheet was almost empty. We decided to change the position of the second tree, making it horizontal. The result should have been easy to predict: the first one immediately became vertical. Thus we learned that an orbit is an “indivisible” entity; we can only work with the whole orbit, but not with its parts.

3. Enumeration of bicolored plane trees

Plane trees have always been one of the classical objects of enumerative studies. Various species of plane trees were enumerated in [25] using the Pólya theorem. More explicit formulas for the number of plane trees are given in [29].

As one can easily see from the examples given above, enumeration of bicolored plane trees according to their sets of valencies plays a central role in the theory

presented in our paper. The first and very important result in this direction was obtained in [42]; for later developments see [2], [3], [24], [30].

In this section we give some of the relevant results, mainly without proofs.

3.1. Some enumeration formulas. The following formula and numerical data are borrowed from [29]. The number of plane trees with n edges, $n \geq 1$ is given by

$$\frac{1}{2n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) \binom{2d}{d} - \frac{1}{2}C_n + \frac{1}{2}\chi_{\text{odd}}(n)C_{\frac{n-1}{2}},$$

where φ is the Euler function, χ_{odd} is the characteristic function of odd numbers, and C_n are the Catalan numbers,

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

The first 24 values of this sequence are given in the table:

| | | | | | | | |
|---|----|----|------|----|----------|----|-------------|
| 1 | 1 | 7 | 34 | 13 | 28640 | 19 | 46509358 |
| 2 | 1 | 8 | 95 | 14 | 95640 | 20 | 164107650 |
| 3 | 2 | 9 | 280 | 15 | 323396 | 21 | 582538732 |
| 4 | 3 | 10 | 854 | 16 | 1105335 | 22 | 2079165208 |
| 5 | 6 | 11 | 2694 | 17 | 3813798 | 23 | 7457847082 |
| 6 | 14 | 12 | 8714 | 18 | 13269146 | 24 | 26873059986 |

If we replace the Euler function φ by the Möbius function μ , we get a formula for the number of (centrally) asymmetric plane trees. The following table gives the corresponding numbers.

| | | | | | | | |
|---|---|----|------|----|----------|----|-------------|
| 1 | 0 | 7 | 28 | 13 | 28507 | 19 | 46504495 |
| 2 | 0 | 8 | 85 | 14 | 95393 | 20 | 164098390 |
| 3 | 0 | 9 | 262 | 15 | 322938 | 21 | 582521687 |
| 4 | 1 | 10 | 827 | 16 | 1104525 | 22 | 2079133141 |
| 5 | 3 | 11 | 2651 | 17 | 3812367 | 23 | 7457788295 |
| 6 | 9 | 12 | 8626 | 18 | 13266366 | 24 | 26872946466 |

Examples given in section 2 show quite clearly that the most important problem for our theory is that of the enumeration of the bicolored plane trees of a given type, i.e., with prescribed set of black and white valencies. The main result in this direction, which is in fact equivalent to the theorem given below, was obtained by Tutte (see [42]). A reformulation more convenient for our goal (and a new proof) is given in [3]. A partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$ of the number n may also be represented as $1^{p_1}2^{p_2} \dots n^{p_n}$, which means that k enters in the partition p_k times, $k = 1, 2, \dots, n$ (hence $\sum_{k=1}^n p_k = p$ and $\sum_{k=1}^n kp_k = n$). We

denote by $N(\alpha)$ the following number (rational in general):

$$N(\alpha) = \frac{(p-1)!}{p_1! p_2! \dots p_n!} = \frac{1}{p} \binom{p}{p_1 \ p_2 \ \dots \ p_n}.$$

Let us assign to each bicolored plane tree T a “weight” equal to $\frac{1}{\#(\text{Aut}(T))}$, where $\#(\text{Aut}(T))$ is the number of its automorphisms (i.e. rotational symmetries). (Note that an automorphism must preserve not only the structure of a plane tree, but also the colors of its vertices.)

THEOREM 3.1. *For a given type $\langle \alpha, \beta \rangle$, where α represents the set of black valencies, and β that of the white ones, the following formula is valid:*

$$\sum \frac{1}{\#(\text{Aut}(T))} = N(\alpha)N(\beta),$$

where the sum is taken over all the trees of the type $\langle \alpha, \beta \rangle$.

EXAMPLE 3.1. (a) For $\langle \alpha, \beta \rangle = \langle 4, 2, 1, 1; 3, 2, 1, 1, 1 \rangle$ the above formula gives $\frac{3!}{2!1!1!} \times \frac{4!}{3!1!1!} = 12$, and there are, in fact, 12 trees of that type, all of them asymmetric.

(b) For $\langle \alpha, \beta \rangle = \langle n; 1, 1, \dots, 1 \rangle$ the formula gives $\frac{1}{n}$, and there is in fact one tree of that type, with the symmetry of order n .

(c) For $\langle \alpha, \beta \rangle = \langle 3, 3, 1, 1; 2, 2, 2, 1, 1 \rangle$ the formula gives 3; there are, in fact, 4 trees of that type, and two of them are symmetric, with the symmetry of order 2; thus we have indeed $\frac{1}{2} + \frac{1}{2} + 1 + 1 = 3$.

(d) For $\langle \alpha, \beta \rangle = \langle 3, 2, 1, 1; 3, 2, 1, 1 \rangle$ the formula gives 9. There are 6 asymmetric bicolored plane trees of that type, see the discussion in section 2.6 (“Mistake 2”), and our formula agrees with that, i.e., it does not make the same mistake as we did. There are also 3 symmetric plane trees of that type, with symmetry of order 2. But this symmetry changes the colors, and therefore the trees must be counted with weight 1.

A similar question for maps would be to count the number of maps with a given set of valencies of vertices and faces. One may find some information on the subject in [19].

We saw in many examples, and it is true in general, that if a family of the trees of the same type contains N trees, we arrive at a system of algebraic equations of degree N . But the algebraic and/or combinatorial mechanism of this correspondence is yet to be understood. One may hope that in case of maps it is maybe possible to proceed in the opposite direction, namely, to enumerate maps by a skillful computation of the degree of the system of algebraic equations in the process of computing the corresponding Belyi functions.

We saw as well that if a family contains symmetric trees, then they form a separate orbit of the Galois group action. In fact, if there are trees of different orders of symmetry, they must be treated separately. Therefore it is interesting

to know the number of trees of prescribed type and prescribed order of symmetry. This problem is solved in [30].

We do not know if anybody ever considered the problem of enumerating the number of *pairs of partitions* of a number n verifying the additional condition that the total number of parts is equal to $n+1$. Though it is rather easy to write a generating function, we do not have a ready asymptotic result. The following are the numerical data up to $n = 24$ (computations of D. Péré); for each pair of “reciprocal” couples $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ we count only one of them.

| | | | | | | | |
|---|----|----|-----|----|------|----|-------|
| 1 | 1 | 7 | 18 | 13 | 343 | 19 | 3941 |
| 2 | 1 | 8 | 30 | 14 | 529 | 20 | 5719 |
| 3 | 2 | 9 | 52 | 15 | 810 | 21 | 8225 |
| 4 | 3 | 10 | 84 | 16 | 1221 | 22 | 11742 |
| 5 | 6 | 11 | 137 | 17 | 1823 | 23 | 16626 |
| 6 | 10 | 12 | 218 | 18 | 2695 | 24 | 23411 |

One may see that for $n = 24$ there are, on the average, more than a million of trees for one type.

3.2. Inverse problem of enumerative combinatorics. The last remark leads to the following problem: *For a given k , enumerate all the types $\langle \alpha, \beta \rangle$ such that the family of the trees of that type have exactly k trees.* We call this problem “an inverse problem of enumerative combinatorics” (this time referring to bicolored plane trees).

For $k = 1$ this problem was solved by N. Adrianov in 1989 (see [2]). Here is the list of the corresponding types:

- (i) $\langle n; 1, 1, \dots, 1 \rangle$ (the “star-trees”);
- (ii) $\langle 2, 2, \dots, 2; 2, 2, \dots, 2, 1, 1 \rangle$ or $\langle 2, 2, \dots, 2, 1; 2, 2, \dots, 2, 1 \rangle$ (the “chain-trees”);
- (iii) $\langle r, 1, 1, \dots, 1; s, s, \dots, s, t \rangle$;
- (iv) $\langle r, s, 1, 1, \dots, 1; 2, 2, \dots, 2 \rangle$;
- (v) $\langle r, r, 1, 1, \dots, 1; 3, 3, \dots, 3 \rangle$;
- (vi) $\langle 3, 3, 3, 1, 1, 1, 1, 1; 2, 2, 2, 2, 2, 2 \rangle$.

We leave it to the reader to make the corresponding pictures. We see that the five first examples are, in fact, infinite series of types, while the sixth one is a single type which does not enter into any series (one may call it a “sporadic” tree). The uniqueness of the tree of given type implies that the corresponding field of definition is the field \mathbb{Q} of rational numbers. All the corresponding generalized Chebyshev polynomials with rational coefficients are found in [2].

This work was continued by D. Péré [32], who gives the complete list of types having exactly 2 trees:

- (i) $\langle r, s, t; 3, 1, 1, \dots, 1 \rangle$
- (ii) $\langle r, r, s, s; 4, 1, 1, \dots, 1 \rangle$
- (iii) $\langle r, r, r, s, s; 5, 1, 1, \dots, 1 \rangle$
- (iv) $\langle r, s, 1, 1, \dots, 1; 3, 3, \dots, 3 \rangle$
- (v) $\langle r, r, 1, 1, \dots, 1; 4, 4, \dots, 4 \rangle$

- (vi) $\langle r, r, 1, 1, \dots, 1; 5, 5, \dots, 5 \rangle$
- (vii) $\langle 3, 3, 1; 2, 2, 1, 1, 1 \rangle$
- (viii) $\langle 3, 2, 2; 2, 2, 1, 1, 1 \rangle$
- (ix) $\langle 3, 3, 1, 1, 1; 2, 2, 2, 2, 1 \rangle$
- (x) $\langle 3, 2, 2, 1, 1, 1; 2, 2, 2, 2, 2 \rangle$
- (xi) $\langle 4, 4, 4, 1, 1, 1, 1, 1, 1, 1; 2, 2, 2, 2, 2, 2, 2, 2, 2 \rangle$
- (xii) $\langle 5, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 \rangle$

We see in this list (a) three infinite series of trees (or, more exactly, families consisting of two trees) of diameter 4 (cases i to iii); (b) three infinite series of trees of diameter 6 (cases iv to vi); (c) six “sporadic” families (cases vii to xii). This pattern is generalized in [3]: *For any $k \geq 2$ the families of trees containing exactly k elements each are classified as follows: there are several infinite series of diameter 4, several infinite series of diameter 6, and a finite set of “sporadic” families, with number of edges $n \leq 16k + 1$.*

It is interesting to note that to compute the generalized Chebyshev polynomial for a family of diameter 6 is rather an easy task, and all the corresponding fields of definition are cyclotomic (see [38]). On the contrary, the case of diameter 4, as well as that of sporadic families, is very rich in all sorts of examples.

We have never heard of any results of “inverse enumerative” type concerning maps.

3.3. Trees of diameter 4. For trees of diameter 4, all the enumerative problems become very simple. Indeed, the “black” partition α is of the type $\alpha = (q, 1, 1, \dots, 1)$, hence $N(\alpha) = 1$, and everything depends on the “white” partition β . The proof of the enumeration formula of section 3.1 becomes trivial: we merely need to compute the number of cyclic orders of q branches around the center. The computation of the number of trees of a given order of symmetry is also easy: all we need to know is the number of *asymmetric* trees, because a tree with order of symmetry exactly k is a smaller *asymmetric* tree with $\frac{n}{k}$ edges, “repeated k times”.

Let D be the greatest common divisor of the numbers q_1, q_2, \dots, q_n . Denote by $\bar{\beta}$ the partition of the number $\frac{n}{D}$ corresponding to $\frac{q_1}{D}, \frac{q_2}{D}, \dots, \frac{q_n}{D}$. Then the above considerations lead to the following formula: the number of asymmetric trees with the white partition β is equal to

$$\frac{1}{D} \sum_{d|D} \mu \left(\frac{D}{d} \right) dN(d \times \bar{\beta}),$$

where $d \times \bar{\beta}$ is the partition $\bar{\beta}$ “repeated” d times.

4. Fields of definition and the Galois group action

We have mentioned the *fields of definition* of the trees many times, but have so far given no precise definition of this notion. This will be done below.

4.1. Galois theory: a computer science point of view. We will try to explain, very briefly, what Galois theory is about. We have a kind of a descriptive formal language, with the following entities:

- integers;
- variables for complex numbers;
- four arithmetic operations (and parentheses);
- logical operations \vee , $\&$ and \neg , and quantifiers \exists and \forall ;¹
- a unique predicate, that of equality $=$ (but by means of the negation we may also use \neq).

What we are interested in is the expressive power of this language. Is it possible, for example, to say that $a = \sqrt{17}$? We may well write that $a^2 = 17$, but there is no way to distinguish between $\sqrt{17}$ and $-\sqrt{17}$, because there is no way to express the fact that something is “positive” (as well as there is no way to express the conditions of the kind “with negative real part”, or “less than 1 by absolute value”, etc.). The same holds for any polynomial $S(a)$ with rational coefficients, irreducible over \mathbb{Q} : its roots are indistinguishable (it is, of course, not a proof but an “explanation”). In algebra they usually say instead of “indistinguishable” the “(algebraically) conjugate”.

On the other hand, it is easy to express the fact that a_1, a_2, \dots, a_n are the (different) roots of $S(a)$:

$$(S(a_1) = 0) \& \dots \& (S(a_n) = 0) \& (a_1 \neq a_2) \& \dots \& (a_{n-1} \neq a_n).$$

Now another question arises: if we re-enumerate the roots in a different manner, i.e., permute their indices, will we be able to distinguish among these two enumerations?

Let us consider an example. The *discriminant* of the polynomial S is defined as

$$\text{Discrim}(S) = \prod_{i < j} (a_i - a_j)^2.$$

It is a symmetric function of a_1, \dots, a_n and thus it can be expressed via the coefficients of S . For example,

$$S(a) = a^3 + 2 \implies \text{Discrim}(S) = -108,$$

while

$$T(a) = a^3 + a^2 - 2a - 1 \implies \text{Discrim}(T) = 49.$$

In the second case the discriminant of T is a complete square in \mathbb{Q} , and we may therefore compute the product $\prod (a_i - a_j)$, which will be equal to 7 or to -7 , depending on the numeration of the roots. Now, all even permutations of roots

¹The formal theory we are presenting admits the *elimination of quantifiers*: any proposition with quantifiers is equivalent to some proposition without quantifiers. For example, “ $\exists z(P(z) = 0) \& (Q(z) = 0)$ ” is equivalent to “ $\text{Resultant}(P, Q) = 0$ ”. Elimination of quantifiers is a very efficient tool for certain algebraic problems. We believe that it may also be useful for the theory of dessins d’enfant.

do not change the sign of the last product, and we cannot distinguish among the corresponding numerations of roots, while the odd permutations change the sign, and we can distinguish among the sequence of roots and the same sequence oddly permuted. As to the polynomial $S(a) = a^3 + 2$, the fact that its discriminant is equal to -108 does not provide us with any new information. In other words, this means that the Galois group of the polynomial $a^3 + 2$ is the group S_3 of all permutations of three elements, while the Galois group of the polynomial $a^3 + a^2 - 2a - 1$ is the alternating group $A_3 = C_3$ (which is cyclic in this case).

In general, the Galois group of an irreducible polynomial of degree n is a transitive subgroup of S_n which permutes the roots of the polynomial. The transitivity of the group corresponds to the intuitive notion of indistinguishability of roots. If the discriminant of the polynomial is a complete square, the Galois group is a subgroup of A_n . The notion of indistinguishability may be described in more algebraic terms: each admissible permutation generates an automorphism of the field $K = K_S$, an extension of the field \mathbb{Q} by the roots of the polynomial S . Thus, the Galois group is the group of automorphisms of the field K . The field K is called the *splitting field* of S . We also call the polynomial S a *defining polynomial* of the field K . (Defining polynomial, of course, is not in any way unique.)

We mention very briefly, not going into detail, that a *finite extension* k of \mathbb{Q} may be obtained by adding to \mathbb{Q} *some* roots of S . Then the field K which is obtained by adding to \mathbb{Q} *all* the roots of S is called the *Galois extension* of k . Usually it is much more convenient to work with the fields like K ; this is why they are called *normal*.

The *universal* (or *absolute*) *Galois group* $\Gamma = \text{Gal}(\overline{\mathbb{Q}} | \mathbb{Q})$ is the group of automorphisms of the field $\overline{\mathbb{Q}}$ of algebraic numbers. It is one of the most important (and most intractable) objects in mathematics. Actually, this single group embodies the whole of classical Galois theory over \mathbb{Q} . For any finite extension k of \mathbb{Q} there is a subgroup G_k of finite index in Γ consisting of all the automorphisms of $\overline{\mathbb{Q}}$ that leave the elements of k fixed. Taking the maximal normal subgroup H of Γ contained in G_k , and then the corresponding (finite) factor $G = \Gamma/H$, we again obtain the Galois group of the field K .

Can *any* finite group be a Galois group of a finite extension of \mathbb{Q} ? This question is the famous *Inverse Problem of Galois Theory*. There are numerous works concerning this subject (the paper of Belyi [9] among them), but the complete answer is yet unknown.

A terminological remark: the term *number field* means a subfield of $\overline{\mathbb{Q}}$.

4.2. Galois group action on trees. In many examples that we have seen in previous sections, we used our “two degrees of freedom” by putting some points into $z = 0$, or $z = 1$, or in some other ways that may look rather arbitrary. Why 1 and not $\sqrt[3]{17}$, or even π ? OK, π is a kind of exaggeration, and we may agree that *wherever we are to chose an arbitrary number, we chose an algebraic*

one. Then, by solving the system of algebraic equations for a given tree, we get a *collection of algebraic numbers*, namely: (a) the coordinates of black and white vertices; (b) the coefficients of the generalized Chebyshev polynomials (which are functions of the coordinates); (c) two critical values.

The universal Galois group Γ acts on algebraic numbers, replacing them by the *conjugate* numbers. Conjugate means verifying the same algebraic relations with integer coefficients (see above). In particular, this means that

- if we replace the coefficients of a generalized Chebyshev polynomial by the conjugate algebraic numbers, we again obtain a polynomial of the “generalized Chebyshev” type;
- the above polynomial corresponds to a tree of the same type.

To prove these two assertions one has to express the statement that “ $P(z)$ is a generalized Chebyshev polynomial corresponding to a tree of type $\langle \alpha, \beta \rangle$ ” in the “algebraic language” described above, an easy but rather cumbersome task that we leave to the reader.

Thus, the group Γ acts on the set of generalized Chebyshev polynomials by replacing their coefficients with conjugate algebraic numbers. In fact, Γ acts also on the set of (bicolored plane) trees. Why?

Let us formulate the question more accurately. Let P_1 and Q_1 be two generalized Chebyshev polynomials corresponding to the same tree T_1 . Let $\varphi \in \Gamma$ be an automorphism of $\overline{\mathbb{Q}}$, and let φ take P_1 to P_2 and Q_1 to Q_2 . Why the polynomials P_2 and Q_2 correspond to the same tree T_2 ? Because the fact that P_1 and Q_1 correspond to the same tree (i.e., Q_1 is equivalent to P_1) implies that

$$Q_1(z) = AP_1(az + b) + B, \quad A \neq 0, a \neq 0.$$

Therefore

$$Q_2(z) = \varphi(A)P_2(\varphi(a)z + \varphi(b)) + \varphi(B),$$

which gives a polynomial equivalent to P_2 .

There is a finite number of trees of any given type. Therefore, the Galois orbit of any tree is finite. Hence, the subgroup of Γ that fixes this tree is a *subgroup of finite index*.

Now, the *Main Theorem of Galois Theory* (see any appropriate reference in Galois theory) affirms that *the subgroups of Γ of finite index are in one-to-one correspondence with the finite extensions of \mathbb{Q}* , namely, to each group there corresponds the field of all algebraic numbers that are left in place by the action of the corresponding subgroup.

DEFINITION 4.1. *The field of definition of a bicolored plane tree T is the number field K , the Galois extension of the field k that corresponds to the subgroup of Γ that fixes the tree T .*

The approach based on the descriptive language helps to clarify many things. The question whether two given trees belong to the same orbit of the universal Galois group action may turn out to be too difficult for a beginner. Instead

we ask ourselves if it is possible to express some quality of a tree in terms of our language. For example, it is clear why the centrally symmetric trees form separate Galois orbits. The statement that a certain tree is centrally symmetric is easily expressible in our formal language (roughly as follows: “the center lies at $z = 0$; the sum of the coordinates of all the other black vertices is equal to 0; the sum of the coordinates of all the white vertices is equal to 0”). On the contrary, the statement that a tree has an axial symmetry is not expressible: to express an axial symmetry would mean to express the fact that the sum of certain coordinates is *real*, but this predicate does not exist in the language.

Note that from the point of view of the theory of maps an axial symmetry is *not* an automorphism of a plane tree, because it changes the orientation. The group of automorphisms of a plane tree is always cyclic (while any finite group may serve as group of automorphisms of a certain *map*, see [15]).

4.3. Two fundamental theorems. The theorems that follow clarify important aspects of the present theory.

THEOREM 4.1. *For any bicolored plane tree there exists a generalized Chebyshev polynomial whose coefficients belong to the field of definition of the tree.*

This theorem was first proved in [35] using the theory of Galois cohomology. An elementary proof was independently found in [17]; below we give a sketch of this proof. Note that, as we have mentioned above, a similar statement is not valid in the case of maps and Belyi functions. V. Filimonenkov ([21]) and J.-M. Couveignes ([17]) have found plane maps with field of definition \mathbb{Q} such that there is no corresponding Belyi function with rational coefficients.

PROOF. Let $p_k(z)$ (resp. $q_k(z)$) be a monic polynomial whose roots are black (resp. white) vertices of valency k . The coefficients of these polynomials will be our unknowns. These coefficients may be expressed as elementary symmetric functions of the coordinates of the vertices of the same color and valency. For brevity we will call these symmetric functions *vertex combinations*. Repeat once more that any vertex combination includes only vertices of the same color and valency.

We look for a generalized Chebyshev polynomial of the form

$$P(z) = \lambda \prod p_k(z)^k,$$

such that

$$P(z) - 1 = \lambda \prod q_k(z)^k.$$

The latter equalities give us the required number of equations but two. We must add two additional equations in such a way as to obtain the simplest possible field.

Let σ be an arbitrary *sum* of vertices of the same color and valency, i.e., an arbitrary vertex combination of *degree* 1. The first equation we add is

$$\sigma = 0.$$

Now the position of the tree is determined up to a transformation $z \mapsto Az$ with arbitrary $A \in \mathbb{C}$.

Let Σ be a vertex combination of the smallest degree m which is not equal to 0. In fact, m is the order of symmetry of the tree. The second equation we add is

$$\Sigma = 1.$$

Now the position of the tree is determined up to $z \mapsto Az$ with A verifying $A^m = 1$.

Let \overline{P} be a *conjugate polynomial*, i.e., another solution of the same system of equations, and *suppose that \overline{P} gives the same tree as P itself*. This means that

$$\overline{P}(z) = P(Az).$$

Now the fact that the tree is symmetric of order m implies that $P(z)$ is in fact a polynomial in z^m , and hence $P(Az) \equiv P(z)$. The contradiction thus obtained proves the theorem. \square

On the practical level, if one has obtained a field and would like to know if it is the field of definition or some bigger field, the best way to proceed is to draw a computer-made picture. If every tree of the orbit is repeated only once, it is the field of definition; if more than once, it is a bigger field.

THEOREM 4.2. (H. W. Lenstra, L. Schneps [34]) *The action of Γ on the set of trees is faithful.*

This means that for any non-identity element of the group Γ there is a bicolored plane tree which is transformed into another tree by the action of this element. The class of fields of definition of the trees is not a specific class of fields: all the number fields belong to this class.

In more philosophical terms, the last theorem means that the world of bicolored plane trees is as rich as that of algebraic numbers. Any phenomenon we encounter in the algebraic number theory has its counterpart in the theory of bicolored plane trees. One may consider the present theory as another (more combinatorial and geometric in nature) approach to algebraic number theory.

We give a sketch of the proof, following [34].

LEMMA 4.1. *Let f be a polynomial of degree n and let $d \mid n$. Suppose there exists a polynomial h such that $h(0) = 0$, h is monic, $\deg h = d$ and for some polynomial g , $f = g(h)$. Then h is unique.*

Indeed, $f = a_m h^m + a_{m-1} h^{m-1} + \dots + a_0$, and the terms of degrees $n, n-1, \dots, n-d+1$ all come from the leading term $a_m h^m$, which provides us with enough information to find all the coefficients of h .

LEMMA 4.2. *Let g, h, \bar{g} and \bar{h} be polynomials such that $g(h) = \bar{g}(\bar{h})$ and $\deg h = \deg \bar{h}$. Then there exist constants c and d such that $\bar{h} = ch + d$.*

It suffices to turn h and \bar{h} into monic polynomials with the free terms equal to 0, and then apply Lemma 4.1, as $g(h) = g_1(h/c_1 - d_1) = \bar{g}(\bar{h}) = g_2(\bar{h}/c_2 - d_2)$.

LEMMA 4.3. *For any polynomial p with algebraic coefficients there exists a polynomial f with rational coefficients such that $f(p)$ is a generalized Chebyshev polynomial.*

This lemma constitutes the main part of the proof of the Belyi theorem, and we refer the reader to his original paper [9]. We now turn to the proof of Theorem 4.2.

PROOF. Let K be an arbitrary number field, and let α be its primitive element, i.e., an algebraic number that generates the field. Let us take a polynomial $p_\alpha(z)$ such that

$$p'_\alpha(z) = z^3(z-1)^2(z-\alpha),$$

and find a generalized Chebyshev polynomial $P_\alpha(z) = f(p_\alpha(z))$, $f \in \mathbb{Q}[z]$. Denote by T_α the corresponding tree. Let β be a number conjugate to α , and suppose that $\beta \neq \alpha$. In the same way as before, we construct the polynomials $p_\beta, P_\beta = f(p_\beta)$ and the tree T_β .

We must show that the trees T_α and T_β are distinct. Suppose the converse. This would mean that we would have

$$P_\beta(z) = P_\alpha(az + b) \quad \text{for some } a, b.$$

Now, applying Lemma 2 to $g = \bar{g} = f$, $h = p_\alpha(az + b)$ and $\bar{h} = p_\beta(z)$, we see that there exist constants c and d such that

$$p_\alpha(az + b) = cp_\beta(z) + d.$$

The right-hand side has three critical points, namely, 0 (of order 4), 1 (of order 3) and β (of order 2). The left-hand side has three critical points z_0, z_1 and z_α , of orders 4, 3, 2 respectively, such that

$$az_0 + b = 0, \quad az_1 + b = 1, \quad az_\alpha + b = \alpha.$$

Since critical points must match each other (respecting their orders), we must have

$$z_0 = 0, \quad z_1 = 1, \quad z_\alpha = \beta,$$

which leads to $a = 1$, $b = 0$ and $\beta = \alpha$. The last equality contradicts our supposition.

The theorem is proved. \square

More constructive results concerning various types of the fields of definition will be presented in the next section.

5. Factorization of discriminants

In this section we will only consider trees of diameter 4. This class of trees is especially rich in *infinite series* of examples (see section 3). Each tree of diameter 4 has a (black) center, from which several branches go outside to the white vertices of certain valencies; and these white vertices are connected only with (black) leaves.

5.1. Several interesting examples. We will temporarily change our usual convention and will turn the *white* vertices into the roots of the corresponding generalized Chebyshev polynomials (in other words, we will take $c_1 = 0$).

EXAMPLE 5.1. Consider the trees with 3 branches, and with white vertices of valencies m, n, k . We will suppose that all the three numbers m, n and k are different, and will call the expression $m + n + k$ the *symbol* of the tree.

There are, obviously, exactly two plane trees of that type: they correspond to two cyclic orders of the letters m, n, k , namely,

$$(m, n, k) \quad \text{and} \quad (m, k, n).$$

We may suppose that we deal with a quadratic Galois orbit. Even more than that: the trees are symmetric to each other with respect to an axis, so we may guess that the field of definition will be an *imaginary* quadratic field, i.e., it will be generated by the square root of a negative number.

Let us put the center at $z = 0$, and the vertices of valencies m, n, k , at the points $1, a, b$ respectively. We obtain a generalized Chebyshev polynomial of the following form:

$$P(z) = (z - 1)^m (z - a)^n (z - b)^k.$$

Its derivative is

$$P'(z) = (z - 1)^{m-1} (z - a)^{n-1} (z - b)^{k-1} Q(z),$$

where $Q(z)$ is equal to

$$Q(z) = (m + n + k)z^2 - ((m + k)a + (m + n)b + (n + k))z + (mab + nb + ka).$$

The point $z = 0$ being a critical point of order 3, the polynomial $Q(z)$ must be proportional to z^2 , hence $Q(z) = (m + n + k)z^2$, and we obtain the following system of two equations for the unknowns a and b :

$$(m + k)a + (m + n)b + (n + k) = 0, \quad mab + nb + ka = 0.$$

This system is easily reduced to a quadratic equation with discriminant

$$D = -mnk(m + n + k),$$

and the field of definition is $\mathcal{Q}(\sqrt{D})$.

Thus our guess that we will have a negative number under the square root sign is confirmed. One may ask *whether any imaginary quadratic field can be obtained in that way?* The answer was found by Adrianov: for any given d , let us take

$$m = 1, \quad n = 16d, \quad k = 64d^2.$$

Then m, n, k are all different, and we have

$$D = -mnk(m + n + k) = -1024d^3(1 + 16d + 64d^2) = -1024d^3(1 + 8d)^2,$$

so that

$$\sqrt{D} = 32d(1 + 8d)\sqrt{-d}.$$

In that way we obtain, in constructive terms, all the imaginary quadratic fields.

EXAMPLE 5.2. Our next step is to obtain all the real quadratic fields. Let us take a family of two trees with 5 branches, and with the symbol $2m + 3n$: this means that there are 2 white vertices of valency m and 3 white vertices of valency n , the numbers m and n being different. Two trees of this family correspond to two cyclic orders of white valencies:

$$(m, m, n, n, n) \quad \text{and} \quad (m, n, m, n, n).$$

Note that this time the trees are not symmetric to each other with respect to an axis.

Suppose that the vertices of valency m are the roots of the quadratic polynomial $z^2 - 2z + a$, while those of valency n are the roots of the cubic polynomial $z^3 + b_2z^2 + b_1z + b_0$. This means that we take a generalized Chebyshev polynomial of the form

$$P(z) = (z^2 - 2z + a)^m (z^3 + b_2z^2 + b_1z + b_0)^n.$$

Making the same type of calculations as in the previous example, we get a quadratic equation with discriminant

$$D = 3(m + 2n)(2m + 3n).$$

In order to obtain an arbitrary real quadratic field, i.e., a field of the form $\mathbb{Q}(\sqrt{d})$ for arbitrary positive d , we may take, say,

$$m = 6ds^2 - 3t^2, \quad n = 2t^2 - 3ds^2,$$

where t and s are arbitrary positive integers verifying the inequalities

$$\sqrt{\frac{3}{2}d} < \frac{t}{s} < \sqrt{2d}.$$

Then we get

$$m + 2n = t^2, \quad 2m + 3n = 3ds^2,$$

so \sqrt{D} reduces to \sqrt{d} .

The following observation is very important:

REMARK 5.1. If d itself is a complete square, we *do not* obtain a quadratic orbit! In that case the family splits into two separate orbits, each containing a single tree with the field of definition \mathcal{Q} . We have already given several examples of that kind in section 2.4. Now we have an infinite series of examples. We do not see any apparent combinatorial reason for this splitting.

EXAMPLE 5.3. Let us now consider a cubic family, that with the 7 branches and with the symbol $2m + 5n$ (2 white vertices of valency m , 5 white vertices of valency n , and $m \neq n$).

As usual, we take $P(z) = p_1(z)^m p_2(z)^n$, where $\deg p_1 = 2$ and $\deg p_2 = 5$, we compute the derivative $P'(z) = p_1(z)^{m-1} p_2(z)^{n-1} Q(z)$, where $\deg Q = 6$, we set $Q(z) \sim z^6$, we eliminate from the system of equations thus obtained all unknowns but one, and finally we get the following cubic equation for the last variable:

$$15n^3 a^3 - 90n^2(m+3n)a^2 + 60n(m+3n)(m+4n)a - 8(m+3n)(m+4n)(m+5n) = 0.$$

When we compute the discriminant of the above polynomial, it miraculously splits into linear factors:

$$D = 518400n^6(m+3n)^2(m+4n)(2m+3n)(2m+5n)^2.$$

A cubic field is called *cyclic*, if its Galois group is cyclic, i.e., C_3 . Recall that a cubic field is cyclic if and only if the discriminant of a cubic polynomial whose roots generate the field is a complete square. Noting that $518400 = 720^2$, we see that the only possibly non-square factor of D is $(m+4n)(2m+3n)$. To make it a square, let us take (say)

$$m = t^2 + 6t - 3, \quad n = t^2 - 4t + 2 \quad (t \geq 4);$$

then

$$m + 4n = 5(t-1)^2 \quad \text{and} \quad 2m + 3n = 5t^2,$$

and D becomes a complete square. For example, for $t = 4$ we have $m = 37$ and $n = 2$, the trees thus having $2m + 5n = 84$ edges. We don't know any "smaller" example of a cyclic cubic orbit.

In this way we obtain an infinite series of cyclic cubic fields of definition of trees. To be sure that we deal in fact with a cubic orbit, we must also verify that the cubic polynomial in a is irreducible. Substituting the above values $m = m(t)$ and $n = n(t)$ into its coefficients, we obtain a polynomial in a with coefficients depending on t . Considered as a polynomial in two variables a and t , it is irreducible. Hence, because of Hilbert irreducibility theorem, there exist infinitely many values of t such that the corresponding polynomial is irreducible as a polynomial in a .

EXAMPLE 5.4. Consider once more a cubic family, that with 4 branches and with symbol $2m + n + k$.

Among the three trees of that family, two are symmetric to each other with respect to an axis, and the third one is symmetric to itself. Therefore, without any computations, we may assert (see section 2.3) that the field of definition cannot be cyclic. Let us make the computations nevertheless. We do not enter into the details, but just give the resulting cubic equation:

$$\begin{aligned} S(a) &= 2m^2(n-k)(2m+k)a^3 + 3mn(4m^2 + 4mn + 2mk + nk + k^2)a^2 \\ &+ 12mn(m+n)(m+n+k)a + 4n(m+n)(m+n+k)^2 = 0. \end{aligned}$$

Returning to the previous example, the splitting of the discriminant into linear factors is not so much of a miracle. First, the coefficients of the polynomial in question themselves split into linear factors. Second, the discriminant is a homogeneous polynomial in two variables (m and n), and therefore it *must* split into linear factors (we owe this remark to M. Zaidenberg). The only unexpected thing is that the coefficients of the linear factors are integers (and rather small integers).

Now, in the present example, the situation seems to be quite hopeless: this time the discriminant depends on three variables (m , n and k), so the fact that it is homogeneous yields nothing; and one of the coefficients of the cubic polynomial we are dealing with does not split into linear factors. So this time it is a full-fledged miracle when we see that the discriminant still splits into linear factors:

$$D = -432m^3n^2k^2(m+n)(m+k)(m+n+k)^3(2m+n)^2(2m+n+k)^2.$$

The minus sign in front of D confirms our opinion that the field cannot be cyclic (since D cannot be a square). But we may ask another question: when is the field of definition a *purely cubic field*, i.e., a field of the form $\mathbb{Q}(\sqrt[3]{c})$ with rational c ? The general criterion is well known (see, for example, [14]): a cubic field is purely cubic if and only if the discriminant of the corresponding cubic polynomial is

$$D = -3 \times \text{complete square.}$$

Taking into consideration the fact that $432 = 3 \times 12^2$, the only thing we must provide for is that the product

$$m(m+n)(m+k)(m+n+k)$$

be a square. For this, it is sufficient (though not necessary) to set

$$m = x^2, \quad m+n = y^2, \quad m+k = z^2, \quad m+n+k = t^2.$$

These equations are not independent: we have the relation

$$x^2 + t^2 = y^2 + z^2 = 2m + n + k.$$

Incidentally, $2m+n+k$ is the number of edges of the tree. Thus, we come to the following condition: *the number of edges must have two different representations as a sum of two squares.*

The problem of representation of a number as a sum of two squares is one of the most classical problems of elementary number theory. The following statement was first announced by P. Fermat (1640) and proved 100 years later by L. Euler (see [44]): *any prime number of the form $4N + 1$ has a unique representation as a sum of two squares; no prime of the form $4N + 3$ does.* The statement to follow was already known to Diophantus: *the product of two numbers representable as sums of two squares has a non-unique representation as a sum of two squares.* Indeed,

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + bc)^2,$$

and these two representations are different if $a \neq b$ and $c \neq d$. Let us take, for example, the number of edges equal to $5 \times 13 = 65$. Then we have $65 = 1 + 64 = 16 + 49$, so we may take $m = 1$, $n = 15$, $k = 48$.

In this way we may construct an infinite series of examples of purely cubic fields.

5.2. General conjecture concerning discriminants. The observations of the previous section can be generalized to the following conjecture.

CONJECTURE 5.1. *Suppose a tree of diameter 4 has the symbol*

$$k_1 m_1 + \dots + k_r m_r,$$

i.e., it has k_1 white vertices of valency m_1 , ..., k_r vertices of valency m_r , the number of branches thus being $K = k_1 + \dots + k_r$, all the numbers m_1, \dots, m_r being different. Then the discriminant in question splits into linear factors of the form

$$a_1 m_1 + \dots + a_r m_r, \quad 0 \leq a_i \leq k_i, \quad i = 1, \dots, r.$$

Besides, the discriminant has a numeric factor all of whose prime divisors are less than K .

Note that in all linear combinations the coefficient a_i of m_i is less than or equal to k_i , the number of “branches” of valency m_i . If this conjecture is true, it implies that all prime factors of the discriminant are not greater than $k_1 m_1 + \dots + k_r m_r$, which is the number of the edges of the tree, the fact that could be important for “reduction” theory, i.e. for the study of the same objects over the fields of finite characteristics.

A similar statement should also be valid for general trees, but this time it is more difficult to form an infinite series, and we will not formulate any precise hypothesis.

The above conjecture was verified by a computer for several series of trees of diameter 4. We give two more examples.

EXAMPLE 5.5. 5 branches, symbol $m + n + 3k$; field of definition is of degree 4. The discriminant is

$$D = 2^{10}3^3m^3n^3k^6(m+k)^2(n+k)^2(m+2k)(n+2k)(m+n+k)(m+n+2k)^2(m+n+3k)^3.$$

EXAMPLE 5.6. 19 branches, symbol $2m + 17n$; field of definition is of degree 9. The discriminant is

$$\begin{aligned} D &= 2^{112}3^{43}5^{14}7^{10}11^513^617^8n^{72}(m+9n)^8(m+10n)^7(m+11n)^6(m+12n)^5 \\ &\times (m+13n)^4(m+14n)^3(m+15n)^2(m+16n)(2m+3n)(2m+5n)^2 \\ &\times (2m+7n)^3(2m+9n)^4(2m+11n)^5(2m+13n)^6(2m+15n)^7(2m+17n)^8. \end{aligned}$$

The first example in which MAPLE-5 was able to compute the discriminant, but was unable to factorize it, is for the family with 4 branches and with the symbol $m + n + k + l$, i.e., all the 4 white valencies are different (the field is of degree 6). The expression for the discriminant is very complicated: it runs along the screen for about 40 seconds. Hence we cannot say whether we have found a counterexample to our conjecture, or the task is simply too difficult for a computer.

6. Composition of polynomials (and trees)

Throughout this section, we fix the image segment $[c_0, c_1] = [0, 1]$.

Let us take two generalized Chebyshev polynomials $P(u)$ and $Q(z)$ and make a simple exercise in undergraduate calculus: compute the derivative of their composition $R(z) = P(Q(z))$:

$$R'(z) = P'(u)Q'(z), \quad \text{where } u = Q(z).$$

Therefore

$$R'(z) = 0 \iff P'(u) = 0 \text{ or } Q'(z) = 0.$$

In the first case

$$P'(u) = 0 \implies P(u) \in \{0, 1\} \implies R(z) \in \{0, 1\}.$$

In the second case

$$Q'(z) = 0 \implies Q(z) = u \in \{0, 1\} \implies R(z) \in \{P(0), P(1)\}.$$

We see that it is sufficient to add the condition

$$P(0), P(1) \in \{0, 1\}$$

for $R(z)$ to become a generalized Chebyshev polynomial.

We sum up this observation as a theorem:

THEOREM 6.1. *If P and Q are generalized Chebyshev polynomials, and*

$$P(0), P(1) \in \{0, 1\},$$

then $R(z) = P(Q(z))$ is a generalized Chebyshev polynomial.

Denote by T_P , T_Q and T_R the trees that correspond to the polynomials P , Q and R . Our goal is to reconstruct the tree T_R combinatorially from the trees T_P and T_Q , or, in other words, to define an operation of the *composition of trees*.

First note that the geometrical meaning of the condition $P(0), P(1) \in \{0, 1\}$ is that there are two vertices of T_P (colors are irrelevant) that lie at the points $u = 0$ and $u = 1$. Thus, T_P is not simply a tree but *a tree with two distinguished vertices*. Let us mark the vertex which is at $u = 0$, by a little square, and that at $u = 1$, by a little triangle (see Figure 21).

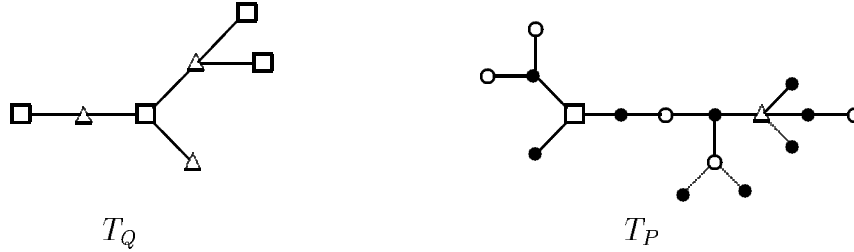


FIGURE 21.

The vertices of T_P are black and white, as usual. As for the vertices of T_Q , they are pre-images of the “square” and “triangular” points of the plane of variable u . Therefore we mark them not in black and white, but we mark them as square and triangular vertices (see Figure 21).

In general terms, the procedure of obtaining the tree T_R from T_P and T_Q consists in *substituting the tree T_P for of every edge of the tree T_Q* . We would like to draw the reader’s attention to the fact that for the polynomials it is the polynomial Q which is substituted into P , but for the trees it is T_P which is substituted into T_Q .

A more detailed description of the procedure will be as follows. We decompose the tree T_P into the following parts (see Figure 22):

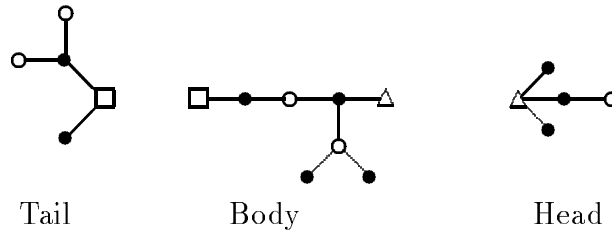


FIGURE 22. Morphology of a plane tree.

- *spine*, i.e., the path from the square to the triangle;
- *body*, which consists of the spine and all the branches attached to it;

- *head*, which consists of all branches (except the spine) attached to the triangle;
- *tail*, which consists of all branches (except the spine) attached to the square.

Now, the composition procedure goes as follows. Every edge of T_Q has a distinguished direction, that from square to triangle. Instead of each edge we put in the “body” of the tree T_P , each time respecting this direction. Then we attach to each triangular vertex of T_Q a number of “heads” of T_P equal to the valency of that vertex, and then we do similar thing with the square vertices and with the “tails”. The result is the tree T_R . (For the example given above the resulting tree is rather complicated, so its *very schematic* representation is shown in Figure 23.)

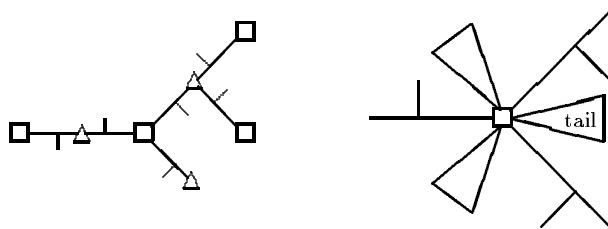


FIGURE 23. Schematic representation of the composition procedure.

EXAMPLE 6.1. The well-known situation of a centrally symmetric tree is a particular case of the composition, with $Q(z) = z^n$. Thus, borrowing a terminology from theoretical physics, we may say that the composition of trees is a manifestation of the *hidden symmetries* in the world of plane trees.

Sometimes these symmetries are not very deeply hidden, as the following example shows.

EXAMPLE 6.2. Let us take for the polynomial P a Chebyshev polynomial, but renormalized in such a way as to have critical values 0 and 1 instead of ± 1 , and the ends of the “chain”-tree at points $u = 0$ and $u = 1$:

$$P(u) = \frac{1}{2}(T_n(2u - 1) + 1).$$

The composition with such a P consists in the subdivision of every edge of the tree T_Q into n parts.

REMARK 6.1. The following fact was proved by J. Bernstein in 1979, but not published ([13]): *if a sequence P_n of polynomials verifies the following conditions: (a) $\deg P_n = n$; (b) $P_n(P_m(z)) = P_m(P_n(z))$, then, up to linear transformations, this sequence is either $P_n(z) = z^n$, or $P_n(z) = T_n(z)$.*

Contrary to the previous examples, sometimes by looking at a tree it is not at all easy to guess that it is in fact the composition of simpler trees, as one may see below.

EXAMPLE 6.3. See Figure 24.

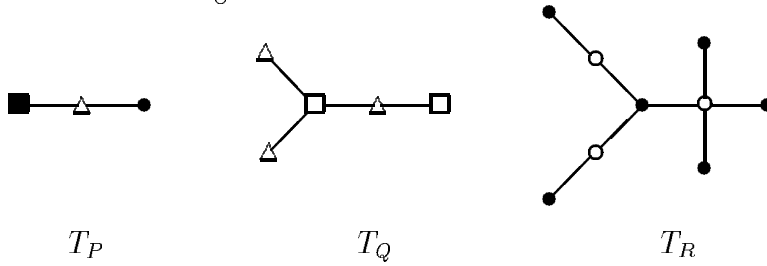


FIGURE 24. Example of composition.

Finally, one more example.

EXAMPLE 6.4. Consider the family of trees with the following set of valencies: $\langle 2, 2, 2, 1, 1, 1; 4, 2, 2, 1 \rangle$. This family contains 10 trees, and none of them is centrally symmetric (we leave it to the reader to make a picture). One may suppose that this is a Galois orbit with field of definition of degree 10. However, this is not the case: the defining polynomial splits into two factors, of degrees 1 and 9. Thus, among these 10 trees there is an “odd man out”, while the other 9 guys form a Galois orbit of degree 9. How does one find out which one is “out”? The regular way would be to make the pictures on a computer. But this is “cheating”, because what we are looking for is a “combinatorial reason” that could explain the splitting of the family. And, in fact, after some search we find that the tree shown on Figure 25 is the composition of simpler trees, and thus it is rational (i.e., with the field of definition \mathbb{Q}), since both polynomials P and Q have rational coefficients.

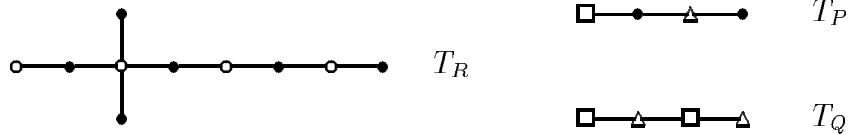


FIGURE 25.

REMARK 6.2. Belyi functions are also closed under composition, providing some simple additional conditions. But in this case we must inevitably pass from maps to *hypermaps*, see [16]. This topic is developed in [20].

7. Pell equation for polynomials

We hope that by now the reader is sufficiently convinced that generalized Chebyshev polynomials form a remarkable class of polynomials. The goal of this short section is, nevertheless, to add one more argument to this statement.

The *Pell equation* is a diophantine equation of the form

$$x^2 - dy^2 = 1$$

for a given positive square-free integer d , well-known in number theory. Abel [1] considered a similar problem for polynomials (we are indebted to E. Rakhmanov who attracted our attention to Abel's work): for a given square-free polynomial $D(z)$ find the polynomials $P = P(z)$ and $Q = Q(z)$ verifying

$$P^2 - D(z)Q^2 = 1.$$

EXAMPLE 7.1. For $D(z) = z^2 - 1$ there are infinitely many solutions of the Pell-Abel equation, namely, the sequence of the Chebyshev polynomials of the first and of the second kind.

Abel proved the following theorem.

THEOREM 7.1. ([1]) *The following three statements concerning any square free polynomial $D(z)$ are equivalent:*

- (i) *The Pell-Abel equation has a solution (in this case it has infinitely many solutions).*
- (ii) *The square root $\sqrt{D(z)}$ may be represented as a periodic continued fraction.*
- (iii) *An integral*

$$\int \frac{p(z)}{\sqrt{D(z)}} dz$$

for some $p(z)$ of $\deg p \leq \deg D - 2$ can be computed in elementary functions.

Integrals of this sort (for $\deg D = 4$) are called *quasi-elliptic*. The search of quasi-elliptic integrals was in fact the main motivation of Abel's work. In the 19-th century finding a particular example of a quasi-elliptic integral used to be a sufficient reason for publishing a paper (two papers of this sort belong to Chebyshev). This subject continues to interest mathematicians in our days as well; see, for example, [18].

Therefore it is interesting to note that to any plane tree it corresponds a polynomial $D(z)$ verifying the conditions of Abel theorem. Namely,

$$D(z) = \prod (z - d_i),$$

where the product is taken over the coordinates of *all the vertices of odd valencies* (regardless of their colors).

In order to prove that, let us take $c_0 = -1$ and $c_1 = 1$. Then we have

$$P(z) + 1 = \lambda \prod_{i=1}^p (z - a_i)^{\alpha_i}$$

and

$$P(z) - 1 = \lambda \prod_{j=1}^q (z - b_j)^{\beta_j},$$

therefore

$$P^2 - 1 = \lambda^2 \prod_{i=1}^p (z - a_i)^{\alpha_i} \prod_{j=1}^q (z - b_j)^{\beta_j} = D(z)Q^2,$$

where $D(z)$ just collects the square-free part of the whole product.

Our first example, that of $D(z) = z^2 - 1$, is obviously related to the chain-tree. The next interesting class of trees to be considered is that of trees having a “center” of valency 3 (or 4), from which go out 3 (resp. 4) simple chains of arbitrary length.

This subject deserves a more thorough study.

It is amazing to see how rich in different mathematical structures is such a seemingly simple combinatorial object as a plane tree.

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