On the enumeration and generation of generalized Dyck words

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Résumé

Nous proposons une nouvelle grammaire algébrique engendrant les langages de Dyck généralisés introduits par Labelle et Yeh dans [7]. Leur étude suggère d'introduire les sous-langages des mots n'ayant aucun facteur propre dans le langage étudié, qui sont également algébriques. Nous proposons également une méthode de génération aléatoire des mots de ces langages, liée à des réultats asymptotiques d'énumération. Enfin, dans le cas des langages sur deux lettres ("mots de Dyck à pente rationnelle"), nous obtenons d'autres résultats d'énumération exacts ou asymptotiques, en particulier sur l'aire moyenne des chemins de Dyck à pente entière ou 3/2.

Abstract

We provide a new algebraic grammar for generalized Dyck languages as introduced by Labelle and Yeh ([7]). The study of these languages leads to the particular sublanguages of words without proper factors belonging to the studied language. A random generation scheme is shown for generalized Dyck languages, which leads to some asymptotic results. In the two-letter case, for which the words correspond to "rational slope Dyck paths", more exact and asymptotic enumerative results are obtained, including the asymptotic average area of integer or 3/2 slope Dyck paths.

1 Introduction

Dyck words, or well-parenthesed words, have been studied extensively, and there are well-known bijections between Dyck words and other combinatorial objects such as binary trees, parallelogram polyominoes, 2-lines standard tableaux and so on; the number of Dyck words of length $2n$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

To be well-parenthesed, a word needs to have exactly as many opening parentheses (here represented by the letter a) as it has closing parentheses (the letter b), with the added condition that a matching pair of parentheses has to appear in the right order (opening first). Giving value $+1$ to a and value -1 to b, this means a word is a Dyck word if its total value is 0, while none of its left factors have a negative value.

In [7], Labelle and Yeh introduced the notion of generalized Dyck languages, where a potentially larger set of letters are used, with any set of integer values. They proved that the resulting

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languages were algebraic, and exhibited unambiguous grammars. Slightly more general languages were studied by Sprugnoli et al. in [8]. Melançon and Jacquet [6] studied the 2-letter case from the point of view of Lyndon words. Petkovsek [9] independently proved that the generating function for a generalized Dyck language is algebraic.

In this paper, we first give bounds on the number of generalized Dyck words with a given composition, and show that random generation of these words is possible in linear expected time; then, new grammars are given for the generalized Dyck languages, which also make it possible to distinguish words with no proper factors that are also generalized Dyck words. The 2-letter case ("rational slope Dyck paths") is then studied more closely; the grammars and Lagrange inversion are used to obtain more precise asymptotic enumeration and statistics on these paths.

The general method we use for obtaining enumeration information on our languages is basically the Schützenberger methodology : an unambiguous algebraic grammar is found and the generating functions are solutions of the corresponding equations; for enumeration with respect to more complex parameters such as areas, we obtain q-analogs of algebraic equations ([1],[2]).

2 Definitions and notations

Let U be a finite alphabet, and h a *valuation* function from U to $\mathbf{Z}$. The valuation is extended additively to all words in U^*, i.e., a word has valuation equal to the sum of the valuations of its letters.

The letters in U will be written $x_1, \ldots, x_k$. U_+ will denote the set of letters with a positive valuation, and U_-, the set of letters with a negative valuation.

Definition 1 *The* generalized Dyck language *associated to U and h, denoted $D_{U,h}$, is the set of words w in U^* which satisfy the conditions that $h(w) = 0$, and for each left factor w_1 of w, $h(w_1) \geq 0$.*

Obviously, for $D_{U,h}$ to be non-trivial, U should have both letters with a positive and a negative valuation. The language $D_{U,h}$ does not change if all valuations are divided by a common value, so the gcd of the various valuations will be assumed to be 1.

The usual Dyck language corresponds to a 2-letter alphabet $\{a, b\}$ with $h(a) = 1$ and $h(b) = -1$; another classical example is the Motzkin language, which uses three letters $a, b, \overline{a}$ with respective valuations $1, 0, -1$.

Two frequently used representations of Dyck words as lattice paths starting at $(0, 0)$ are shown in figure 1. The Eastbound representation uses Northeast and Southeast steps for a and b respectively, and paths have to end on the $y = 0$ line without ever being lower. The Northeastbound representation uses East and North steps for a and b respectively, and paths have to end on the $y = x$ line without ever being higher.

Generalized Dyck paths can easily be associated to generalized Dyck words using the Eastbound representation : a letter u will become a $(1, h(u))$ step (if different letters have the same valuation, the path will have to be a colored path, using a different color for each letter with a given valuation), and the conditions on a path (i.e., end on the horizontal axis and never go below it) remain valid. This representation seems natural because the valuation of a word becomes the final y coordinate of the associated path. For this reason, the valuation function can also be called the height function.

There is no such natural version of the Northeastbound representation for the generalized Dyck words, except in the 2-letter case, as we will see in section 6.

Regardless of the path representation we use, an "up step" will always refer to a letter with positive height (a letter in U_+), while a "down step" will refer to a letter with negative height (a letter in U_-).

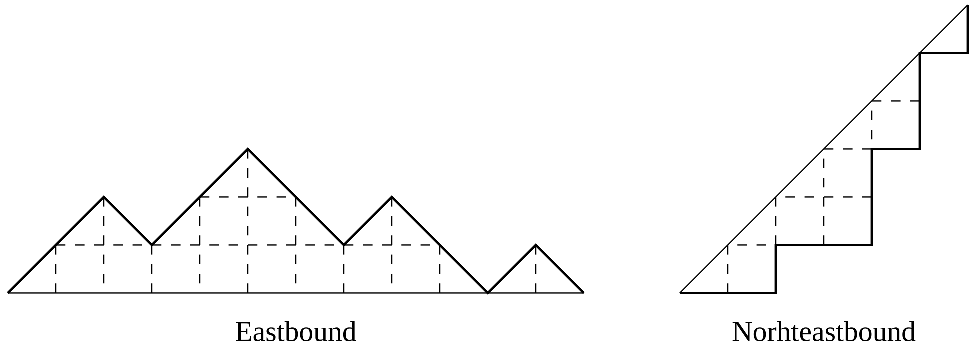

Figure 1: Representations of $aabaabbabbab$

For any word $w \in U^*$, its *composition* is the k-tuple $(|w|_{x_1}, \dots |w|_{x_k})$ of numbers of occurrences of each letter in w; the number of words in U^* with composition $N = (n_1, \dots, n_k)$ is the multinomial coefficient $M_N = \binom{n_1+\dots+n_k}{n_1,\dots,n_k}$.

A *factor* of a word w is any word w' such that $w = w_1 w' w_2$ for some words w_1 and w_2. If w_1 and w_2 are not both the empty word, w' is a *proper factor* of w. If w_1 (resp. w_2) is the empty word, then w' is called a left (resp. right) factor of w.

Following the usual group notation, whenever w_1 is a left factor of w, $(w_1)^{-1}w$ will denote the unique word w_2 such that $w = w_1 w_2$.

A word in a language L is said to be *factor-free* if it has no proper factor belonging to L; the set of factor-free words in L will be denoted $\tilde{L}$. When dealing with a given generalized Dyck language $D_{U,h}$ and an auxiliary language L, a factor-free word in L will in fact mean a word in L with no proper factors in $D_{U,h}$, and the set of such words will still be denoted $\tilde{L}$.

In the usual sense, a *conjugate* of a word w is any word w' such that $w = w_1 w_2$ and $w' = w_2 w_1$ for some words w_1 and w_2. This defines an equivalence relation.

If u is some letter in U, two words w and w' are said to be *u-conjugates* of each other if they are equal or $w = w_1 u w_2$ and $w' = w_2 u w_1$ for some words w_1 and w_2; this u-conjugation is also an equivalence relation. It can also be seen as adding a u at the end of w, then performing the usual conjugation with the added condition that the w_1 and w_2 factors must end with a u; then the tailing u is removed, and w' is obtained. In this sense, u-conjugation is a generalization of the Raney transform, which is one of the many ways to prove the Catalan enumeration formula for Dyck words; see [10] for the original Raney conjugation.

3 Random generation

We now turn to the question of randomly generating words in a generalized Dyck language. Random generation of Dyck words (or any other class of combinatorial objects in bijection with them) is easily done in linear time and space, thanks to the enumeration with Catalan numbers and the Raney conjugation principle [10]. In the general case, we only obtain bounds on the number d_N of words in $D_{U,h}$ with a given composition, but these bounds are tight enough for the random generation process.

In this section, random generation is always assumed to be uniform.

Let $B_{U,h}$ denote the languages of words $w \in U^*$ such that $h(w) = 0$ (*bilateral generalized Dyck words*). The number of words in $D_{U,h}$ with composition N will be noted d_N.

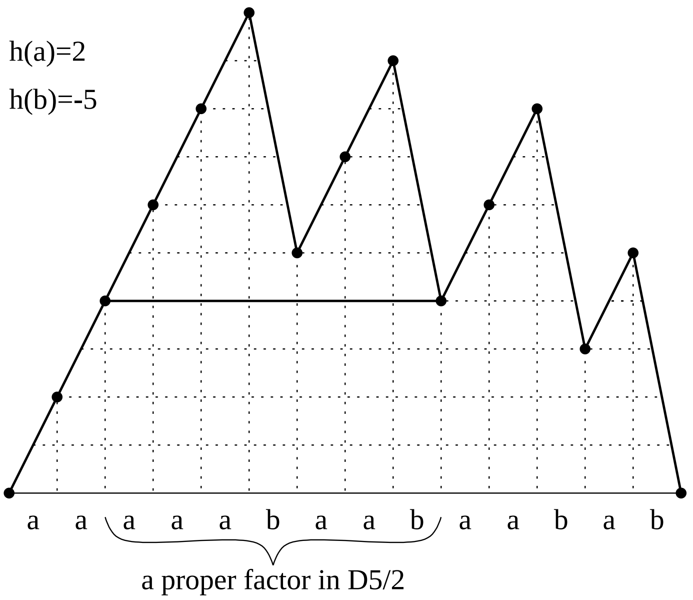

Figure 2: Eastbound representation of a generalized Dyck word

3.1 Upper bound for d_N

The key to our random generation pseudoalgorithm is the following proposition :

Proposition 2 *Let $u \in U$ have nonzero valuation. Then a u-conjugation class contains exactly zero or one word in $D_{U,h}$, and those classes which contain a word $w \in D_{U,h}$ have cardinality $1 + |w|_u$.*

Proof : We only have to prove that any word $w \in D_{U,h}$ has exactly $|w|_u$ distinct u-conjugates other than itself, and that none of those are in $D_{U,h}$. The second part is easy to prove by examining the left factors of a u-conjugate of u.

Now, w potentially has $|w|_u$ u-conjugates other than itself, so we have to prove that these u-conjugates are all distinct. Assume $h(u) < 0$; for two distinct left factors of w to lead to the same u-conjugate, wu would have to be of the form v^α with $\alpha > 1$, but this would imply $h(v) = h(wu)/k = h(u)/k < 0$, so v cannot be a left factor of w.

When $h(u) > 0$, the proof essentially works in the same way, except that right factors are examined instead of left factors. □

Since all d_N are separated into distinct classes, each of which contains exactly $1 + n_u$ words, we get the following corollary :

Proposition 3 *Let N be a composition such that $\sum n_i h(u_i) = 0$; then, for any letter u_i such that $h(u_i) \neq 0$,*

$$d_N \leq \frac{1}{1+n_i} M_N \tag{1}$$

Of course, the best upper bound is obtained when n_i is maximal; if the alphabet has no letter with a 0 valuation, then one can always have $n_i \geq n/k$, where $n = n_1 + \cdots + n_k$ is the total length of the words with composition N.

Note that when there is only one down-step u and it has valuation -1, as is the case in the original paper by Raney ([10]), each u-conjugation class can easily be proved to contain a word in $D_{U,h}$, so that inequality (1) becomes an equality. This is not true in the general case.

3.2 Lower bound for d_N

After giving an upper bound for d_N, we need a lower bound which is close enough to the upper bound; this is easily found using "normal" conjugation.

Proposition 4 *Let w be a word in $B_{U,h}$, i.e., $h(w) = 0$. Then w has at least one conjugate in $D_{U,h}$.*

Proof : This is obvious from the lattice path point of view. The left factor to be used is any left factor of minimal valuation. □

Since a word of length n has at most n distinct conjugates, the following corollary is immediate :

Proposition 5 *Let N be a composition such that $\sum n_i h(u_i) = 0$, and let $n = n_1 + \cdots n_k$. Then*

$$\frac{1}{n} M_N \leq d_N \tag{2}$$

3.3 The random generation pseudoalgorithm

We will first assume that U has no letters with zero valuation.

Let N be a composition for words in $D_{U,h}$, and u the most frequently used letter in N.

Proposition 3 ensures that generating words in $B_{U,h}$ with composition N, and repeating until one finds a word with a u-conjugate in $D_{U,h}$, will finally produce a random word in $D_{U,h}$ with composition N.

Since we are using a rejection pseudoalgorithm, the worst case complexity has no meaning. Generating a word in $B_{U,h}$, and looking for u-conjugates, can be done in linear time. Thus, the average complexity depends on the average number of words in $B_{U,h}$ that are generated, which is a geometric random variable, with its parameter λ being the probability that a word in $B_{U,h}$ with composition N, does *not* have a u-conjugate in $D_{U,h}$.

The upper and lower bounds for d_N combine into

$$\frac{1}{n} M_N \leq d_N \leq \frac{k}{n} M_N \tag{3}$$

This is enough to prove that the probability $1-\lambda$ that a word is accepted in the pseudoalgorithm is at least $1/k$. Since the expectancy for a geometric random variable with parameter λ is $1/(1-\lambda)$, the expected number of loops in the pseudoalgorithm is never more than k, which does not depend on the length of the words we are generating.

If the given composition has letters with zero valuation, the position of these letters has no influence on whether it is in $D_{U,h}$. The above pseudoalgorithm can be modified by generating a word without the zero valuation letters, and inserting these letters in randomly chosen positions at the end of the process, which can obviously be done in linear time.

Thus, we get the following theorem :

Theorem 6 *The rejection pseudoalgorithm has an average execution time which is linear with respect to the length of the generated word.*

4 Factor-free words

In this section, U and h are fixed and $D_{U,h}$ will simply be written as D.

Factor-free words in the Dyck or Motzkin languages are limited to very short words, and so are in finite numbers. This is true whenever all down steps in U have valuation -1 (or when all up steps have valuation $+1$); in such a case, $\tilde{D}$ is always finite and limited to words with at most one up step.

In simple cases like the 2-letter alphabet $\{a, b\}$ with $h(a) = 3$ and $h(b) = -2$, though, words like $a.a^n.(bbab)^n.babb$ are factor-free.

The reason for introducing this notion will be made clear by the following theorem :

Theorem 7 *1. For each word $w \in D$, there is a unique word $w' \in \tilde{D}$, $w' = u_1 \dots u_n$ ($u_i \in U$), and n unique words $w_1, \dots, w_n \in D$, such that $w = u_1 w_1 u_2 \dots u_n w_n$;*

2. Any word $w = u_1 w_1 u_2 \dots u_n w_n$, where $w' = u_1 u_2 \dots u_n$ is a (factor-free or not) word in D, and $w_1, \dots, w_n$ are n words in D, belongs to D.

What this theorem means is that generalized Dyck words can be obtained uniquely by "inserting", after each letter in a factor-free generalized Dyck word of the same language, other words of the same language. Note that this insertion operation, when using the empty word ϵ as the factor-free word, will always leave it unchanged.

Proof : (2) is obvious from the definition of generalized Dyck words. It is just as obvious that deleting a factor in $D_{U,h}$ from a word in $D_{U,h}$ leaves a word in $D_{U,h}$ (that is, if $w \in D_{U,h}$, $w' \in D_{U,h}$, and $w = w_1 w' w_2$, then $w_1 w_2 \in D_{U,h}$).

Existence in (1) easily follows : to obtain the given factorization, simply remove non-left factors in $D_{U,h}$ from w, keeping track of which words were removed immediately after each remaining letter, and repeat until the remaining letters form a word in $\tilde{D}_{U,h}$ (since the length of the word will strictly decrease after each deletion, this will happen after a finite number of steps). The remaining letters give the word w'.

The factorization in (1) can be characterized as follows :

- u_1 is the first letter in w, and, for any $k > 1$, u_k is the first letter in $(u_1 w_1 \dots u_{k-1} w_{k-1})^{-1} w$;
- w_k is the longest left factor of $(u_1 w_1 \dots w_{k-1} u_k)^{-1} w$ that is in $D_{U,h}$.

Thus, uniqueness in (1) is also proved. □

5 Grammars for generalized Dyck words

In this section, we simultaneously write algebraic grammars for the languages D and $\tilde{D}$.

Let $m = \max_{u \in U_+} h(u)$ and $n = -\min_{u \in U_-} h(u)$; we will use auxiliary languages $(L_i)_{1 \leq i \leq m}$ and $(R_j)_{1 \leq j \leq n}$ for D, and the $\tilde{L}_i$ and $\tilde{R}_j$ versions for $\tilde{D}$.

Definition 8 • *$\tilde{L}_i$ is the set of factor-free words $w \in U^*$ with total valuation i, such that each non-empty left factor w_1 of w has $h(w_1) > i$;*

- *$\tilde{R}_j$ is the set of factor-free words $w \in U^*$ with total valuation $-j$, such that each left factor w_1 of w has $h(w_1) > 0$.*

In terms of lattice paths, $\tilde{L}_i$ is the set of paths that begin in $(0, 0)$ and end on the $y = i$ line without having a step end on or under this line, with the added factor-free condition. $\tilde{R}_j$ is the set of paths that *end* on the $y=0$ line and start on the $y=j$ line, and never go on or below the $y=j$ line before the last step, with the added factor-free condition.

Each word (except one-letter words) w in $\tilde{D}$, $\tilde{L}_i$ or $\tilde{R}_j$ has a proper left factor with minimal valuation. This factor is unique thanks to the factor-free condition. If k is the valuation of this factor, $w_1 \in \tilde{L}_k$ and the corresponding right factor w_2 is in $\tilde{R}_k$ if $w \in \tilde{D}$, or in $\tilde{R}_{k-i}$ if $w \in \tilde{L}_i$, or in $\tilde{R}_{j+k}$ if $w \in \tilde{R}_j$.

This unique decomposition (and uniqueness of k) allows us to write the unambiguous grammar for $\tilde{D}$ (in additive form) :

$$\begin{aligned}
\tilde{D} &= \epsilon + \sum_{h(u)=0} u + \sum_k \tilde{L}_k \tilde{R}_k \\
\tilde{L}_i &= \sum_{h(u)=i} u + \sum_k \tilde{L}_k \tilde{R}_{k-i} \\
\tilde{R}_j &= \sum_{h(u)=-j} u + \sum_k \tilde{L}_k \tilde{R}_{j+k}
\end{aligned}$$

where the sums are over all suitable values of k : indices must be between 1 and m for L languages, and between 1 and n for R languages.

Theorem 7 lets us transform this grammar into a grammar for D simply by adding a D symbol after each letter in the right members of derivation rules (and dropping all tildes), thus getting the following :

$$\begin{aligned}
D &= \epsilon + \sum_{h(u)=0} uD + \sum_k L_k R_k \\
L_i &= \sum_{h(u)=i} uD + \sum_k L_k R_{k-i} \\
R_j &= \sum_{h(u)=-j} uD + \sum_k L_k R_{j+k}
\end{aligned}$$

If $n=1$ or $m=1$, it is easy to see that the grammar for $\tilde{D}$ generates only a finite language, so that the grammar for D can be rewritten using only D as a symbol. This is not true in any other case; the grammar for $\tilde{D}$ generates an infinite language as soon as the derivation rules for $\tilde{L}_i$ or $\tilde{R}_j$ are not trivial.

6 The 2-letter case: rational slope Dyck paths

6.1 Rational slope Dyck paths

When alphabet U has only two letters a and b, the language is of course much simpler than the general case, but can still prove interesting. We will assume $h(a)=\alpha$ and $h(b)=-\beta$, with α and β having gcd 1, and write $D_{\alpha/\beta}$ for the resulting generalized Dyck language. Since all words in $D_{\alpha/\beta}$ must have $|w|_a = \beta n$ and $|w|_b = \alpha n$ for some n, the composition of such words will simply be described with integer n; d_n will denote the number of words in $D_{\alpha/\beta}$ of length $n(\alpha+\beta)$.

A very natural path representation for these words is the Northeastbound representation : the paths have steps $(1,0)$ for a, or $(0,1)$ for b, must begin in $(0,0)$ and end on the $y=\frac{\alpha}{\beta}x$ line while never stepping higher than this line; we will call such paths *α/β-Dyck paths*. For simplicity's sake, we will assume that $\alpha \geq \beta$; otherwise, α and β can be switched.

When $\beta = 1$ (Dyck paths with an integer slope), we have an exact expression for d_n (see proposition 3 and the associated remark) :

$$d_n = \frac{1}{\alpha n + 1} \binom{(\alpha+1)n}{n} \tag{4}$$

Dyck paths with slope α and length $(\alpha+1)n$ are thus in bijection with $\alpha+1$-ary trees with αn leaves; it is easy to see that the word is nothing but a prefix traversal of the corresponding tree[1], internal nodes being read as a and leaves as b.

In the more general α/β case, inequalities (2) and (1) translate into

$$\frac{1}{(\alpha+\beta)n} \binom{(\alpha+\beta)n}{\beta n} \leq d_n \leq \frac{1}{\alpha n + 1} \binom{(\alpha+\beta)n}{\beta n} \tag{5}$$

Since the Stirling formula gives

$$\binom{(\alpha+\beta)n}{\beta n} \sim \sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta n}} \left(\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha \beta^\beta} \right)^n \tag{6}$$

we get, up to a multiplicative constant, a good aymptotic estimate for d_n.

6.2 Generating functions

Since the composition of a rational slope Dyck path is fully described by an integer n, we will use as the generating function for these paths (or words) the formal power series $D_{\alpha/\beta}(t) = \sum_n d_n t^n$; this amounts to setting $t = a^\beta b^\alpha$ in the appropriate component $D(a,b)$ of the solution to the algebraic equations system given by the grammar defined in section 5 (choosing the branches that are analytic around the origin).

Now, interpreting theorem 7 in terms of generating functions, the original (two-variables) generating functions $D_{\alpha/\beta}(a,b)$ and $\tilde{D}_{\alpha/\beta}(a,b)$ statisfy the (non-algebraic) equation

$$D_{\alpha/\beta}(a,b) = \tilde{D}_{\alpha/\beta} \left(a D_{\alpha/\beta}(a,b), b D_{\alpha/\beta}(a,b) \right) \tag{7}$$

By setting $t = a^\beta b^\alpha$ and using the Lagrange inversion formula, we get the following theorem :

Theorem 9

$$[t^n] D_{\alpha/\beta} = \frac{1}{1 + n(\alpha+\beta)} [t^n] \tilde{D}_{\alpha/\beta}^{1+n(\alpha+\beta)}(t) \tag{8}$$

The equations for $\tilde{D}_{\alpha/\beta}$ are much simpler than those for $D_{\alpha/\beta}$, though we now have to examine powers of the generating function.

6.3 Enumeration of 3/2-Dyck paths

We now give an example where equation (8) can be used to obtain an exact enumeration result.

Since the enumeration of Dyck paths with an integer slope was easily done, the next value of interest for the slope is $3/2$. In this case, the grammar for $\tilde{D}_{3/2}$ is (dropping the tildes for readability)

$$\left\{ \begin{array}{lllllll} D & = & \epsilon + L_1 R_1 + L_2 R_2 & & L_1 & = & L_2 R_1 + L_3 R_2 \\ R_1 & = & L_1 R_2 & & L_2 & = & L_3 R_1 \\ R_2 & = & b & & L_3 & = & a \end{array} \right.$$

[1] A similar bijection holds for a generalized Dyck language with more than two letters if the only down step has valuation -1; then the bijection is with trees whose internal nodes have a prescribed set of possible numbers of sons : a letter with valuation k means nodes with $k+1$ sons are allowed.

Eliminating L_1, L_2, L_3 and R_2, and factoring $R_1 = aUbb$, yields the noncommutative system

$$\begin{aligned} D &= \epsilon + aaUbbb + abaUbb + aaUbbaUbbaUbb \quad (9)\\ U &= \epsilon + aUbbaUb \quad (10) \end{aligned}$$

Equation (10) clearly shows that words in U with length $5n$ are enumerated by the Catalan numbers (since this equation is essentially the classic equation $D = \epsilon + aDbD$ satisfied by the Dyck language). Switching to one variable generating functions (setting $t = a^2b^3$ and having letters commute) yields :

$$\begin{aligned} D(t) &= 1 + 2tU(t) + t^2U^3(t) \quad (11)\\ U(t) &= 1 + tU^2(t) \quad (12) \end{aligned}$$

Eliminating $U(t)$ in this system yields an algebraic equation for $D(t)$:

$$D(t) = 1 + 2tD^5(t) - tD^6(t) + tD^7(t) + t^2D^{10}(t) \quad (13)$$

which is somewhat impractical for exact resolution purposes.

However, $U(t)$ is known, and, using (12) to change (11), we get

$$D(t) = (1 + t)U(t) \quad (14)$$

(The last transformations have no direct noncommutative equivalent on the languages.) We get the following enumeration result :

Proposition 10 *The number of factor-free Dyck words with slope $3/2$ and length $5n$, is $C_n + C_{n-1}$, where C_n is the nth Catalan number.*

Now, remember we want an expression for $[t^n]D^k(t)$, so we simply expand $D^k(t) = (1+t)^kU^k(t)$:

$$[t^n]D^k(t) = \sum_i \binom{k}{n-i} [t^i]U^k(t) \quad (15)$$

Getting $[t^i]U^k(t)$ once more involves a simple application of the Lagrange inversion formula, and yields

$$[t^i]U^k(t) = \frac{k}{k+i}\binom{2i+k-1}{i}$$

Now going back to enumerating Dyck paths with slope $3/2$, we set $k = 1 + 5n$ to get from equation (8) :

Theorem 11 *The number of words in $D_{3/2}$ with length $5n$ or, equivalently, the number of generalized Dyck paths with slope $3/2$ and length $5n$, is*

$$d_n = \sum_i \frac{1}{5n+i+1}\binom{5n+1}{n-i}\binom{5n+2i}{i} \quad (16)$$

Deriving precise asymptotics from the expression of d_n is not easy (though we already know, from the bounds on d_n, that the exponential part will be $\left(\frac{5^5}{2^2 3^3}\right)^n$). Singularity analysis on the generating functions (see [5]) is probably the best method for this. Using system (11-12), we get as the corresponding system for the generating functions of the $D_{3/2}$ language :

$$D = 1 + 2tD^5U + t^2D^{10}U^3$$
$$U = 1 + tD^5U^2$$

Using a computer algebra system such as Maple, the dominant singularity at $t_0 = 108/3125$ of function $D(t)$ is found to be of type $D(t) = D_0 - K\sqrt{t_0 - t} + o(\sqrt{t_0 - t})$, where $K \simeq 1.67$ and $D_0 \simeq 1.20$ are algebraic numbers satisfying

$$\begin{cases} 0 = 18D_0^3 - 60D_0^2 + 150D_0 - 125, \\ 0 = 2^8.3^{16}(K^2/5^6)^3 + 2^7.3^{10}(K^2/5^6)^2 + 2^4.3^5(K^2/5^6) - 1. \end{cases}$$

This in turn yields the following asymptotic expansion for d_n :

$$d_n \sim \frac{K}{2}\sqrt{\frac{t_0}{\pi}}n^{-3/2}t_0^{-n} \simeq 0.0887n^{-3/2}28.935^n. \tag{17}$$

7 Area enumeration and asymptotics

7.1 General case

When facing a combinatorial class of polygons, it is a natural problem to try to enumerate them according to their area. The area of a Dyck word with slope α/β can be defined as the area $A(w)$ between the corresponding lattice path and the line $y = \frac{\alpha}{\beta}x$, or as the area $I(w)$ "under" the lattice path, that is, between the path, the $y = 0$ line, and the $x = |w|_a$ line. It is easy to see that $I(w)$ is also the number of inversions of w (i.e., the number of pairs of occurrences of b and a appearing in this order in word w).

Since the two areas $I(w)$ and $A(w)$, put together, are the area of a triangle whose vertices are at $(0, 0$, $(\beta n, 0)$ and $(\beta n, \alpha n)$ (the length of w being $n(\alpha + \beta)$, the following relationship holds :

$$I(w) + A(w) = \frac{n^2\alpha\beta}{2} \tag{18}$$

Figure 3 shows the same word from $D_{2/5}$ as figure 2, in Northeastbound representation; this word has area 11 and 9 inversions.

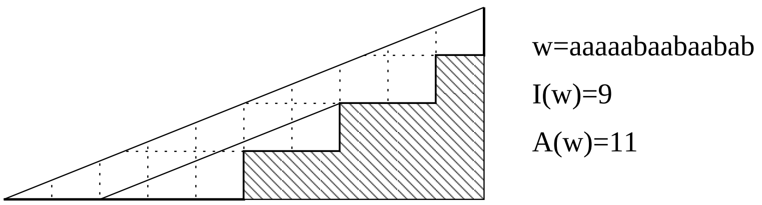

Figure 3: Northeastbound representation of a word in $D_{2/5}$

Obviously, $I(w)$ is always an integer, so when α and β are odd $A(w)$ may not be an integer; this seems like a good enough reason to pick $I(w)$ as our "area" parameter. However, closer examination of the experimental mean values of both parameters, shows $I(w)$ to be dominant and to have mean value close to $n^2\alpha\beta/2$, so that the interesting asymptotic behavior is that of $A(w)$.

The area $A(w)$ of a word w can be seen as a sum of contributions from individual letters in the word : each step contributes the area of a trapezium that corresponds to its projection on the $y = \frac{\alpha}{\beta}x$ line along a given direction. We will use a vertical projection, so that only a steps contribute, and an a step ending in (x,y) (at height $h = \alpha x - \beta y$) contributes $h/\beta - \alpha/2\beta$. Other possible projection directions include horizontal (where only b steps contribute) or orthogonal (where all steps contribute to the area).

This projection lets us compute the "area" parameter of a word obtained from one of the derivation rules of the grammar, as a linear combination of parameters "area", "length in a" and "length in b" of the words appearing in the right side of the rule. This is exactly the kind of situation where a q-grammar [3] can be used.

$$\begin{aligned} D(a,b;q) &= 1 + \sum_k L_k(a,b;q)R_k(a,b;q) && (19)\\ L_i(a,b;q) &= \sum_k L_k(a,b;q)R_{k-i}(aq^{2i},b;q) && (20)\\ L_\alpha(a,b;q) &= aD(aq^{2\alpha},b;q)q^\alpha && (21)\\ R_j(a,b;q) &= \sum_k L_k(aq^{2j},b;q)R_{j+k}(a,b;q) && (22)\\ R_\beta(a,b;q) &= bD(a,b;q) && (23) \end{aligned}$$

This system is no longer algebraic; solving it exactly is difficult, but information regarding the mean area of generalized Dyck paths with a given slope and length can still be obtained through asymptotic methods.

7.2 Area of Dyck paths with an integer slope

When $\beta = 1$, the $D_{\alpha/\beta}$ language has only 2 factor-free words, ϵ and ab^α. As a consequence, it is generated by the grammar $\{D \to \epsilon, D \to aD(bD)^\alpha\}$.

Using the same kind of decomposition as above, if $w = aw_1bw_2\dots bw_{\alpha+1}$ (with $w_i \in D_\alpha$), then

$$A(w) = A(w_1) + \cdots + A(w_{\alpha+1}) + \frac{\alpha}{2} + \frac{1}{\alpha}\left(\alpha|w_1|_b + (\alpha-1)|w_2|_b + \cdots + |w_\alpha|_b\right) \qquad (24)$$

If we set $l(w) = |w|_b/\alpha = |w|/(\alpha+1)$ and $A'(w) = A(w) - l(w)\alpha/2$ (both parameters being integers), we get the following recurrence relation :

$$A'(w) = \sum_{i=1}^{\alpha+1} A'(w_i) + \sum_{i=1}^{\alpha}(\alpha+1-i)l(w_i) \qquad (25)$$

Now, using as generating function

$$D(x;q) = \sum_{w\in D_\alpha} x^{l(w)}q^{A'(w)} \qquad (26)$$

we get the following q-equation :

$$D(x;q) = 1 + xD(xq^\alpha;q)D(xq^{\alpha-1};q)\cdots D(xq;q)D(x;q) \qquad (27)$$

Searching for solutions to (27) in the form $D(x;q) = F(xq;q)/F(x;q)$ yields $F(xq;q) = F(x;q) + xF(xq^{\alpha+1};q)$, so that writing $F(x;q) = \sum_n a_n(q)x^n$ yields in turn

$$q^n a_n(q) = a_n(q) + q^{(\alpha+1)(n-1)}a_{n-1}(q) \qquad (28)$$

for which the solution with $a_0(q) = 1$ is

$$a_n(q) = \frac{q^{(\alpha+1)\binom{n}{2}}}{(q^n - 1)\cdots(q-1)} \tag{29}$$

Thus we have the following theorem :

Theorem 12 *The bivariate generating function $D_\alpha(x;q) = \sum_{w \in D_\alpha} x^{l(w)} q^{A'(w)}$, is*

$$D_\alpha(x;q) = \frac{\sum_{n\geq 0} \frac{q^{n+(\alpha+1)\binom{n}{2}} x^n}{(q^n-1)\cdots(q-1)}}{\sum_{n\geq 0} \frac{q^{(\alpha+1)\binom{n}{2}} x^n}{(q^n-1)\cdots(q-1)}} \tag{30}$$

7.3 Mean area evaluation

We now give the outline of a method which can be used to determine the asymptotics of the mean area of all generalized Dyck paths with a given α/β slope and $n(\alpha+\beta)$ length, when n tends to infinity.

First, the algebraic system given by the grammar is transformed into a system of q-equations (19-23). After factoring the appropriate powers of a and b, and setting $x = a^\beta b^\alpha$, the generating functions become $D(x;q) = \sum_{w\in D} x^{l(w)} q^{2A(w)}$, where $l(w)$ represents the length of words (for words in $D_{\alpha/\beta}$, $l(w) = |w|/(\alpha+\beta)$). The equations are then q-analogs of the original algebraic system, involving series like $\sigma(U)(x;q) = U(xq;q)$.

By differentiating with respect to variable q, then setting $q = 1$, we can obtain equations for the generating function $\sum_{w\in D} 2A(w)x^{l(w)}$, whose n-th coefficient is twice the sum of areas of all paths of length $(\alpha+\beta)n$ (this series is also algebraic, and can be expressed as a rational function of $D(x,1)$; see [4] for details). By using singularity analysis, asymptotics for the coefficients are obtained; dividing the n-th coefficient of $D^*(x)$ by that of $D(x)$ yields the mean area of all paths of length $(\alpha+\beta)n$.

This can be done for paths with an integer slope ($\beta = 1$), and yields the following result :

Proposition 13 *The mean area of generalized Dyck paths with slope α and length $(\alpha+1)n$ is*

$$\overline{A}_{n,\alpha} = \frac{[t^n]D^*_\alpha(t)}{[t^n]D_\alpha(t)} \sim n^{3/2}\frac{\sqrt{2\pi\alpha(\alpha+1)}}{4} \tag{31}$$

The computations for non-integer slopes are much harder. Even the relatively simple $3/2$ case involves long computations; the mean area is asymptotic to $Kn^{3/2}$, but the constant K can only be approximated to 3.43.

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