Proof Complexity Meets Algebra

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ICALP'17, Warsaw
11th July 2017

(CSP problem) $\mathcal{P}$ $\mathcal{S}$ (proof system)

Proofs in $\mathcal{S}$ of the fact that an instance of $\mathcal{P}$ is unsatisfiable.

(CSP problem) $\mathcal{P}$ 3-COL

$\mathcal{S}$ (proof system) resolution

Proofs in $\mathcal{S}$ of the fact that an instance of $\mathcal{P}$ is unsatisfiable.

Resolution proofs of a graph being not 3-colorable.

(CSP problem) $\mathcal{P}$ $\mathcal{S}$ (proof system)

Proofs in $\mathcal{S}$ of the fact that an instance of $\mathcal{P}$ is unsatisfiable.

Standard CSP reductions.

Proof Systems

- propositional
- algebraic / semi-algebraic

Propositional Proof Systems

$\mathcal{C}$ - a set of propositional formulas
E - a propositional formula

A proof of E from the set $\mathcal{C}$ is a sequence of formulas:

- from $\mathcal{C}$ or
- obtained from previous formulas using some rules.

Resolution

$\mathcal{C}$ - a set of clauses (disjunctions of literals, e.g. $p \vee q \vee r$)
E - a clause

A resolution proof of E from the set $\mathcal{C}$ is a sequence of clauses:

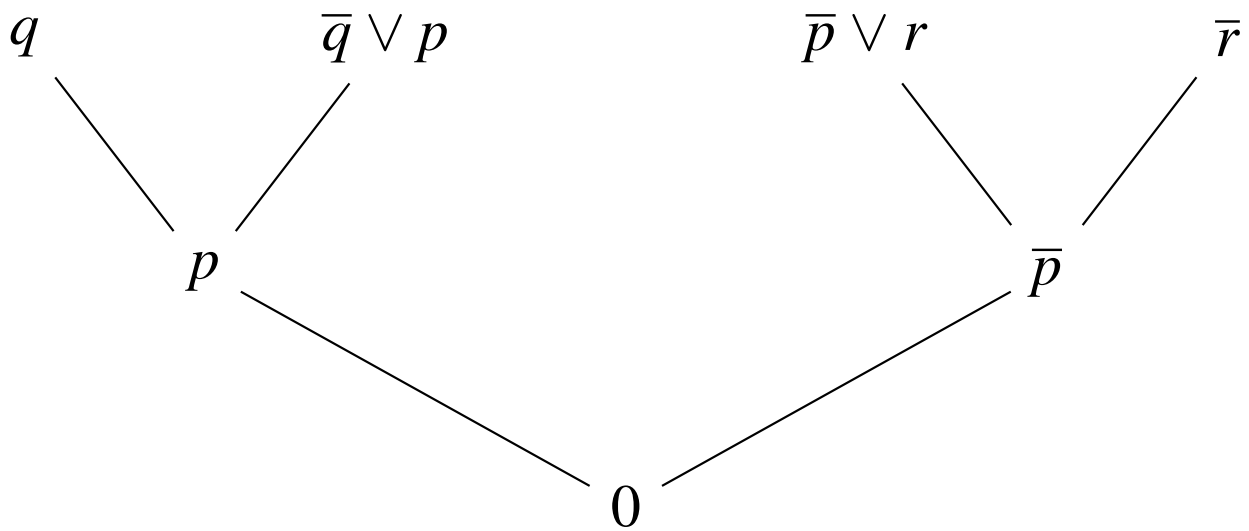
- from $\mathcal{C}$ or
- obtained from previous formulas using the rules:

$$\frac{C \vee p \quad D \vee \overline{p}}{C \vee D} \qquad \frac{C}{C \vee p}$$

Fact. Resolution is sound and implicationally complete.

Example

$$\mathcal{C} = \{q,\ \overline{q} \vee p,\ \overline{p} \vee r,\ \overline{r}\}$$



refutation - ends with an empty formula (proof of unsatisfiability)

Constraint Satisfaction Problems

CSP is a class of problems which contains:

- k-satisfiability,
- k-colorability,
- solving linear equations over finite fields,
- etc.

Goal: Understand proof complexity of problems in this class.

Proof Complexity of CSP

$\mathcal{P}$ - problem from the CSP class

- Size of proofs that an instance of $\mathcal{P}$ is unsatisfiable? Polynomial? Exponential?
- Size of proofs using only some kind of formulas?

Goal: Systematic approach via theory of reductions.

2-SAT

$$(q \vee r) \wedge (\overline{r} \vee p) \wedge (\overline{p}) \quad \leftrightsquigarrow \quad \{q \vee r,\ \overline{r} \vee p,\ \overline{p}\}$$

Fact. 2-SAT formulas have resolution refutations using only 2-clauses (clauses with at most 2 literals).

- polynomial size refutations
- polynomial time algorithm

3-SAT

Theorem [Beame et al.]. 3-SAT formulas do not have bounded depth Frege refutations of subexponential size.

bounded depth Frege - the maximum number of alternations between conjunctions and disjunctions in a formula is bounded

Efficient Algorithms for CSP

resolution using k-clauses

Datalog

local consistency

Sum-of-Squares proof system

semidefinite programming relaxations

Reductions

$\mathcal{P} \leq_{CSP} \mathcal{P}'$ - "classical" reduction preserving the complexity of CSP

Theorem. If $\mathcal{P} \leq_{CSP} \mathcal{P}'$ then "small" refutations for $\mathcal{P}'$ imply "small" refutations for $\mathcal{P}$.

Gap Theorem

Theorem. Exactly one of the following holds for a CSP problem $\mathcal{P}$:

- either $\mathcal{P}$ has resolution refutations using k-clauses,
- or $\mathcal{P}$ has no Frege refutations of bounded depth and subexponential size.

Lemma. Unsatisfiable systems of linear equations over $\mathbb{Z}_q$ have no Frege refutations of bounded depth and subexponential size.

If $\mathrm{LIN}(\mathbb{Z}_q) \leq_{CSP} \mathcal{P}$ then $\mathcal{P}$ has no Frege refutations of bounded depth and subexponential size.

Otherwise $\mathcal{P}$ has resolution refutations using k-clauses.

Efficient proofs for LIN($\mathbb{Z}_q$)

Lemma. Unsatisfiable systems of linear equations over $\mathbb{Z}_q$ are hard for many proof systems.

Proof system which is well-behaved with respect to CSP reductions and has efficient unsatisfiability proofs for LIN($\mathbb{Z}_q$)?

Theorem. Bounded degree Lovász-Schrijver is such a proof system.

Question. Characterise CSPs with efficient proofs in bounded degree Lovász-Schrijver.

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Standard CSP reductions.