

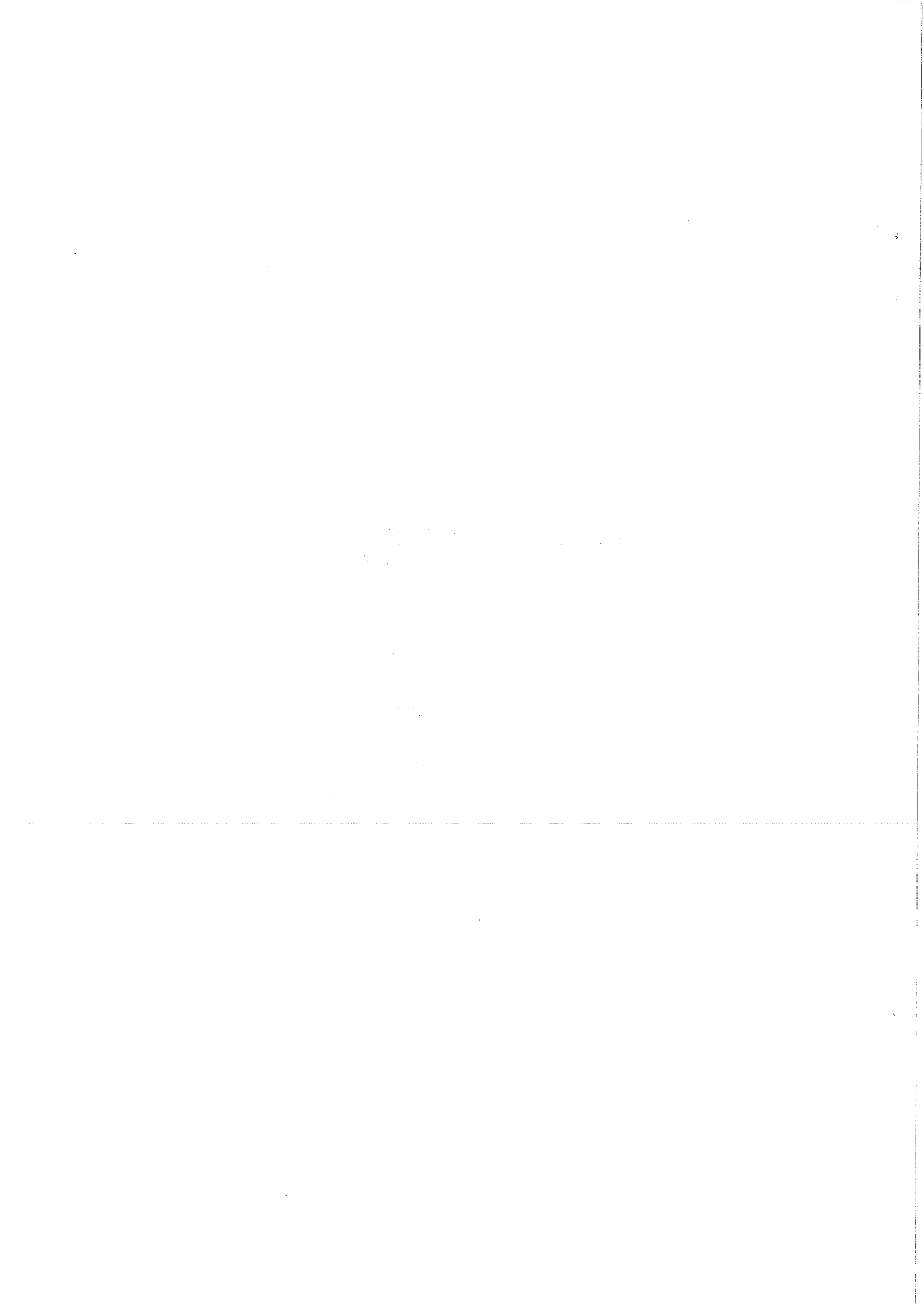
DECIDABILITY OF HOME SPACE PROPERTY

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DECIDABILITY OF HOME SPACE PROPERTY

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Abstract

We present a proof of the decidability of home space property for linear sets of markings, and extend this result to finite unions of linear sets having the same periods. We reduce this property to the same one for submarkings, and finite unions of submarkings having the same support. Decidability of home space property for this last class of sets is proved by applying a new result characterizing the set of reachable markings by means of a finite coinitial part of it.

Résumé

Nous présentons une preuve de la décidabilité de la propriété d'espace d'accueil pour les ensembles linéaires. Ensuite, nous étendons ce résultat à l'unions finies d'ensembles semi-linéaires ayant mêmes périodes. Pour cela, nous réduisons cette propriété à la même dans le cas de sous-marquages, et d'unions finies de sous-marquages ayant même support. La décidabilité de la propriété d'espace d'accueil pour cette dernière classe d'ensembles est prouvée en appliquant un nouveau résultat qui permet de caractériser l'ensemble des marquages accessibles en construisant une partie coinitiale finie à lui.

1. Introduction

We discuss in this paper the notion of home spaces [Mem 83]. Home spaces i.e. sets of markings that can be reached from every reachable marking, are an important notion for net analysis. Some interesting properties, such as the possibility to always reach some special states (those where the process are idle, those where all the resources are free, ...) can be defined and studied by the notion of home space. On the other hand, some classical properties as liveness can be expressed as particular cases of home states. Hence, decidability of home space is a very interesting problem.

We will prove in this paper that we can decide if a finite union of linear sets having the same periods is a home space, as a consequence of the decidability of reachability [Mayr 81], [Kos 82],[Mayr 84], [Lam 86]. For that we will prove that the fact that a submarking is a home space is decidable. Then we will extend this result to finite unions of submarkings having the same support. Finally we establish that the property "a linear set is a home space", is reducible to the property "a submarking is a home space", and we extend this reduction to finite unions of linear sets and submarkings having same support.

Before presenting these proofs we define the required background notions. In section 2, some standard notions on nets are recalled ; in section 3, we remind some properties of linear sets and submarkings. In section 4, we focus on cointial parts of certain sets, and we present some algorithms to compute them. These algorithms are used in section 5 to prove the decidability of the fact a submarking is a home space, and then the proof is extended to finite unions of submarkings having the same support. And finally in section 6 we establish that the property "a linear set is a home space" is decidable, extending it afterwards to finite unions of linear sets having same periods.

2. Basic definitions and notations

In this section, we review some basic terminology on P/T systems ; see [BF 86] for more details if necessary.

Def. 1 : A P/T system Σ is a tuple $\Sigma = (S, T, F, W, M_0)$ where (S, T, F) is a net, $W : F \rightarrow \mathbf{N}$ is the weight function and $M_0 : S \rightarrow \mathbf{N}$ is the initial state.

Remark : Throughout this paper we will not impose finite capacities to places of S.

Def. 2 : The matrix $C : S \times T \rightarrow \mathbf{Z}$ with $C(p, t) = W(p, t) - W(t, p)$ for all p in S and t in T, is called the incidence matrix of the system.

Next we present dot notation witch will be extremely useful for us.

$$\begin{aligned} \forall x \in S \cap T \quad x^\bullet &= \{y \mid (x, y) \in F\} & \bullet x &= \{y \mid (y, x) \in F\} \\ \forall X \subseteq S \cap T \quad X^\bullet &= \sum_{x \in X} x^\bullet & \bullet X &= \sum_{x \in X} \bullet x \end{aligned}$$

Def. 3 : A marking of Σ , M, is a function $M : S \rightarrow \mathbf{N}$.

Def. 4 : We say that a transition $t \in T$ is enabled at a marking M if and only if for all p in $\bullet t$ $M(p) \geq W(p, t)$.

If t is enabled at M, we say that its firing produces M' , and we will denote it by $M[t > M']$, where M' is defined by $M'(p) = M(p) + C(p, t)$, for all p in S. This notation is extended in a natural way to arbitrary transition sequences.

We say that M is reachable from M_0 if and only if there exists a transition sequence s such

that $M_0[s] > M$. The set of reachable markings from M_0 is denoted by $[M_0 >$. We extend this notion to arbitrary markings M : $[M >$ denotes the set of markings reachable from M .

Def. 5 : Let HS be a set of markings. We say that HS is a home space of Σ [MV 87] if and only if $\forall M \in [M_0 >, \exists M' \in \text{HS} \mid M' \in [M >$
If HS is a singleton, we call its unique element a home state.

As we said liveness can be defined as a particular case of home spaces.

Def. 6 : A given transition $t \in T$ is live for Σ if and only if the set of markings where t is enabled is a home space of Σ . Σ is live if any transition of T is live for Σ .

A partial ordering for markings, that we will use extensively throughout the rest of the paper, can be defined in the following way.

Def. 7 :

$$\begin{aligned} M \geq M' & \text{ iff } \forall p \in S \ M(p) \geq M'(p) \\ M > M' & \text{ iff } \forall p \in S \ M(p) > M'(p) \end{aligned}$$

A very important property of this ordering for us will be the following :

Lemma 1 : All the infinite sequences of distinct elements of \mathbb{N}^S contain an infinite increasing subsequence.

Proof : See [Dic 13], [Rei 85]. \square

3. Linear sets

Linear sets and semi-linear sets were introduced in [Par 61] in order to study some problems from formal language theory.

Def. 8 : We say that $E \subseteq \mathbb{N}^S$ is a linear set if there exists some $M \in \mathbb{N}^S$ and a finite set $\{N_1, \dots, N_n\} \subseteq \mathbb{N}^S$ such that

$$E = \{M' \in \mathbb{N}^S \mid M' = M + \sum_{i=1}^{i=n} k_i \cdot N_i \text{ with } k_i \in \mathbb{N}\}$$

M is called the base of E , and N_1, \dots, N_n are said its periods. Semi-linear sets are the finite unions of linear sets.

The following proposition has been recently proved by J.L. Lambert ; as this proof has not been yet published we give it next.

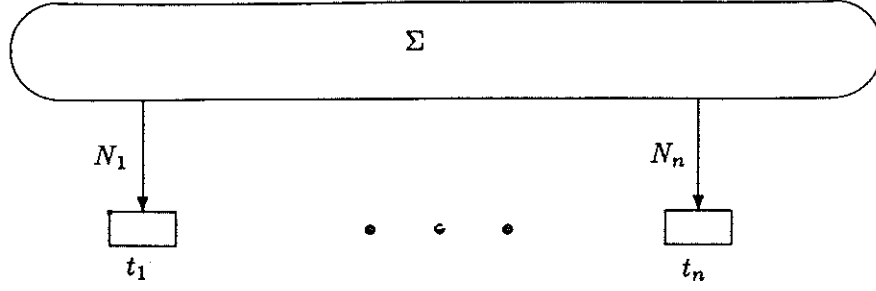
Prop. 1 : Let Σ a P/T system. Let $E \subseteq \mathbb{N}^S$ a linear set. We can decide if there is some reachable marking in E ($E \cap [M_0 > \neq \emptyset$).

Proof : We construct the system Σ_E from Σ and E (see figure 1), adding to Σ a new transition t_i for each period N_i that "takes away from a marking the period value", when this is possible :

$$\forall i \in \{1, \dots, n\} \quad \forall p \in S, \quad W(p, t_i) = N_i(p), \quad W(t_i, p) = 0.$$

Thus is very easy to check a marking of E is reachable in Σ if and only if the base of E is reachable in Σ_E . The proof is finished, the reachability being decidable [Mayr 81], [Kos 82], [Mayr 84], [Lam 86]. \square

Corollary 1 : Let Σ a P/T system and $E \subseteq \mathbb{N}^S$ a semi-linear set. The following properties are decidable :

Figure 1: Construction of Σ_E

a) $E \cap [M_0 > = \emptyset$

b) $[M_0 > \subset E$

Proof : a) If $E = \bigcup_{i=1}^{i=m} E_i$, with E_i a linear set,

we have $E \cap [M_0 > = \emptyset$ if and only if $\forall i \in \{1, \dots, m\} E_i \cap [M_0 > = \emptyset$.

b) We only have to use the fact that the complementary of a semi-linear set is a semi-linear set [Gin 66]. \square

Next we introduce submarkings, they are the sets of markings such that their elements have the same number of tokens over some set of places [Hack 74]. They are used for the local study of configurations.

Def. 9 : We say that $E \subseteq \mathbb{N}^S$ is a submarking if and only if there is some subset S' of S , and a family $\{c_p\}_{p \in S'}$ of natural numbers such that

$$E = \{M \in \mathbb{N}^S \mid \forall p \in S' \quad M(p) = c_p\}$$

We will call S' the support of E , and we will say also that E is a submarking over S' . Finally we say that two markings M and M' agree over S' if and only if $\forall p \in S' M(p) = M'(p)$, or equivalently they belong to the same submarking over S' .

Remark : Clearly submarkings are linear sets, and then reachability of submarkings is decidable by prop. 1.

Now we introduce sets $E_k(S')$ whose elements are the markings that have no more than k tokens in the set of places S' .

Def. 10 : Let S' a subset of S , and k a positive number. We define the set of markings $E_k(S')$ by

$$E_k(S') = \{M \in \mathbb{N}^S \mid \sum_{p \in S'} M(p) \leq k\}$$

Prop. 2 : For all $S' \subseteq S$ and $k \in \mathbb{N}$, $E_k(S')$ is a semi-linear set.

Proof :

$$C_k(S') = \{\{c_p\}_{p \in S'} \mid \sum_{p \in S'} c_p \leq k\}$$

For all $\{c_p\}_{p \in S'} \in C_k(S')$, we have $c_p \leq k$, it is obvious that $C_k(S')$ is a finite set ; and it is clear also that $E_k(S')$ is the union of the submarkings over S' corresponding to the elements of $C_k(S')$, and then it is a semi-linear set. \square

4. Computation of cointial parts

A cointial part of an ordered [Bou 77] is a part of the set such that any element of the set has a minor in the part.

Def. 11 : Let (E, \geq) a poset, Let $F \subseteq E$. We say that F is a cointial part of E if and only if $\forall x \in E \quad \exists y \in F \quad x \geq y$.

We can compute a finite cointial part of the set of reachable markings in the complementary of a given semi-linear set.

Prop. 3 : Given a system Σ and a semi-linear set $E \subseteq \mathbb{N}^S$, we can compute a finite cointial part of the set $[M_0 > - E$.

Proof : We present in figure 2 an algorithm to compute the desired cointial part.

```

{E is a semi-linear set of markings}

A ← E ;    F ← ∅ ;
while [M0 > ∉ A do
  begin
    Choice some M ∈ [M0 > - A ;
    A ← A ∪ {M' ∈ NS | M' ≥ M} ;
    F ← F ∪ {M} ;
  end
{F is a cointial part of [M0 > - E}

```

Figure 2: Algorithm to compute cointial part

In order to prove the correction of this algorithm please note the following facts :

- $F \subseteq [M_0 > - E$ is an invariant of while-loop.
- $A = E \bigcup_{m \in F} \{M' \in \mathbb{N}^S | M' \geq M\}$ is also an invariant of loop.

- "A is a linear set" is an invariant of algorithm ; then the condition of the loop can be effectively checked each time, as shown by the corollary 1.
- If $[M_0 > \not\subseteq A$, then we can effectively find some reachable marking M not in A by an exhaustive search along \mathbb{N}^S . Determine that a given marking is reachable and is not in A , is decidable.

Finally lemma 1 assures us that the algorithm terminates. Suppose that is not true, taking $F = \{M_n\}_{n \in \mathbb{N}}$, where M_n is the element inserted in F in the n -th iteration. We would have some $n_1 < n_2$ with $M_{n_2} > M_{n_1}$, but this not possible because after the insertion of M_{n_1} in F , we would have also M_{n_2} in A , and then it could not be selected as an element of $[M_0 > - A$.

And when the algorithm stops we have

$$F \subseteq [M_0 > - E, \quad \text{and} \quad [M_0 > \subset E \bigcup_{M \in F} \{M' \in \mathbb{N}^S | M' \geq M\}.$$

Then F is a finite cointial part of $[M_0 > - E$. \square

Corollary 2 :

a) Whatever $S' \subseteq S$ and $k \in \mathbb{N}$, a finite cointial part of the set $[M_0 > - E_k(S')$ can be computed.

b) for all $E \subseteq \mathbb{N}^S$ semi-linear set, we can compute a finite cointial part of the set $[M_0 > \cap E$.

Proof : Just note that both $E_k(S')$ and the complementary of E are semi-linear sets. \square

We need a cointial part of the set of reachable markings in $E_k(S')$ such that the minor of any marking agrees with it over S' . Such a cointial part is computable.

Prop. 4 : For all $S' \subseteq S$ and $k \in \mathbb{N}$, a finite cointial part of the set $[M_0 > \cap E_k(S')$ verifying that

$$\forall M \in [M_0 > \cap E_k(S') \quad \exists M' \in F \mid M \geq M' \text{ and } \forall p \in S' \quad M(p) = M'(p)$$

is computable.

Proof : We know that $E_k(S')$ is the finite union of a family of submarkings, whose support is S' . A finite cointial part of the set of reachable markings in each of the elements of this family can be computed. Then it is very easy to check that we can take as the requested set F , the union of these cointial parts. \square

5. Home spaces, submarkings

In this section we prove the decidability of the home space property for submarkings, and finite unions of submarkings having the same support.

Prop. 5 : Let $E \subseteq \mathbb{N}^S$ a submarking. A finite set of reachable markings F , such that E is a home space if and only if E is reachable from every marking in F , is computable.

Proof : First we will compute the desired set of markings F . Afterwards we will check that F is suitable. In order to compute F , we will study $E_k(S')$ where S' is the support of E and k is the number of tokens in S' that have the markings of E . We apply the prop. 4 in order to get a finite cointial part F' of the set $[M_0 > \cap E_k(S')$ having the property

$$\forall M \in [M_0 > \cap E_k(S') \quad \exists M' \in F' \mid M \geq M' \text{ and } \forall p \in S' \quad M(p) = M'(p).$$

We apply also the corollary 2 to compute a finite cointial part F'' of $[M_0 > - E_k(S')$. We will take as F the union of F' and F'' .

Obviously $F \subseteq [M_0 >$; in fact it is a finite cointial part of $[M_0 >$. Then we have that if E is a home space, it is must be reachable from any marking of F . To prove the converse result, we observe that

$$[M_0 > = \bigcup_{l \in \mathbb{N}} ([M_0 > \cap E_l(S')) ;$$

and we will see that for all M in $[M_0 > \cap E_l(S')$, E can be reached from M if E can be reached from any marking in F , by induction on $l \in \mathbb{N}$.

If $l \leq k$ we have $M \in E_k(S')$, and then we have some $M' \in F' \subseteq F$ such that $M \geq M'$ and M' agrees with M over S' . By hypothesis, there exists s , a transition sequence those its occurrence gives M'_1 in E at M' . We have that s is enabled at M and its firing gives M_1 that agrees with M'_1 over S' , as M and M' do too. As E is a submarking over S' we have $M'_1 \in E \iff M_1 \in E$, and so $M_1 \in E$. E is reachable from M .

Let us take now $M \in [M_0 > \cap E_{l+1}(S')$ with $l \geq k$, and assume that for all $l' \leq l$, E can be reached from any $M' \in [M_0 > \cap E_{l'}(S')$. Of course, we assume $M \notin E_l(S')$, and then $M \notin E_k(S')$. So we will have some $M' \in F'' \subseteq F$, such that $M \geq M'$. Then, as before, we take s whose firing at M' gives M'_1 in E , and we observe that s is enabled at M , and if M_1 is reached marking after its firing, we have for all p in S :

$$M_1(p) = M(p) + M'_1(p) - M'(p)$$

and then in particular

$$\sum_{p \in S'} M_1(p) = \sum_{p \in S'} M(p) + \sum_{p \in S'} M'_1(p) - \sum_{p \in S'} M'(p)$$

but as $M'_1 \in E$, $E \subseteq E_k(S')$ and $M' \notin E_k(S')$, we have

$$\sum_{p \in S'} M'_1(p) < \sum_{p \in S'} M'(p) \quad \text{and then} \quad \sum_{p \in S'} M_1(p) < \sum_{p \in S'} M(p) = l + 1$$

So we can apply the induction hypothesis to M_1 , reaching E , and as $M_1 \in [M >$, we conclude that E is reachable from M . \square

Prop. 6 : The property "a submarking is a home space" is decidable.

Proof : We only have to compute F as said in prop. 5, and then to check if E is reachable from any marking in F , applying the decidability of the reachability of a submarking. \square

Next we extend these results to finite unions of submarkings with the same support.

Prop. 7 : Let \mathcal{U} a finite union of submarkings having the same support. We can compute a finite set of reachable markings F such that \mathcal{U} is a home space if and only if \mathcal{U} is reachable from every marking of F .

Proof : We only will construct F . You can check following similar arguments to those used in the proof of prop. 5, that it is a suitable set.

$$\text{Let } \mathcal{U} = \bigcup_{i=1}^{i=m} E^i \quad S' \text{ is the common support of } E^i.$$

$$\text{We will take } k = \max \left\{ \sum_{p \in S'} E^i(p) \mid i \in \{1, \dots, n\} \right\}$$

and consider the set $E_k(s')$ exactly as in the proof of the prop. 5. So F' will be a finite coinital part of the set $[M_0 > \cap E_k(S')$ verifying the property

$$\forall M \in [M_0 > \cap E_k(S') \quad \exists M' \in F' \mid M \geq M' \quad \text{and} \quad \forall p \in S' \quad M(p) = M'(p)$$

and F'' will be a finite coinital part of the set $[M_0 > - E_k(S')$. Finally F will be the union of F' and F'' . \square

Theorem 1 : The property "a finite union of submarkings having the same support is a home space" is decidable.

Proof : It is an immediate consequence of the preceding proposition. \square

6. Home spaces, linear sets

In this section we will see that it is possible to decide if a given linear set is a home space of a P/T system. For we will prove that this property is recursively equivalent to the studied in the previous section "a submarking is a home space".

Let us remind the definition of recursive equivalence of two properties.

Def. 12 : Let P_1 and P_2 two properties over universes \mathcal{U}_1 and \mathcal{U}_2 . We say that P_1 is reducible to P_2 if and only if there is some algorithm \mathcal{A} transforming the objects of \mathcal{U}_1 in objects of \mathcal{U}_2 so that $e_1 \in \mathcal{U}_1$ verifies P_1 if and only if $\mathcal{A}(e_1)$ verifies P_2 .

P_1 and P_2 are recursively equivalent if and only if P_1 is reducible to P_2 and P_2 is reducible to P_1 .

Prop. 8 : If P_1 is reducible to P_2 and P_2 is decidable, then P_1 is decidable too.

Proof : Obvious. \square

Given P/T system Σ , and a linear set $E \in \mathbb{N}^S$, we will construct another system Σ_E and a submarking E' over it, such that E is a home space of Σ if and only if E' is a home space of Σ_E .

Def. 13 : Let $\Sigma = (S, T, F, W, M_0)$ a P/T system, and $E \in \mathbb{N}^S$ a linear set with base M_b and periods N_1, \dots, N_n . We construct $\Sigma = (S_E, T_E, F_E, W_E, M_{E0})$ and E' as described in figure 3.

$\Sigma_E = (S \cup S'_E, T \cup T'_E, F \cup F'_E, W \cup W'_E)$ $E = \{M \in \mathbb{N}^S \mid \forall p \in S \quad M(p) = M_b(p)\}$		
$T'_E = C \cup B$	$S'_E = \{p_1, \dots, p_n\}$ $C = \{c_1, \dots, c_n\}$	$B = \{b_1, \dots, b_n\}$
$\forall i \in [1, n] \quad \forall p \in P$ $\forall i \in [1, n]$ $\forall i, j \in [1, n]$ $\forall i \in [1, n] \quad \forall p \in P$ $\forall i \in [1, n]$ $\forall i, j \in [1, n]$	$W'_E(p, b_i) = N_i(p)$ $W'_E(p_i, b_i) = 0$ $W'_E(p_j, b_i) = 0$ $W'_E(p, c_i) = 0$ $W'_E(p_i, c_i) = 1$ $W'_E(p_j, c_i) = 0$	$W'_E(b_i, p) = 0$ $W'_E(b_i, p_i) = 1$ $W'_E(b_i, p_j) = 0$ $W'_E(c_i, p) = N_i(p)$ $W'_E(c_i, p_i) = 0$ $W'_E(c_i, p_j) = 0$
$\forall p \in S \quad M_{E0}(p) = M_0(p)$	$\forall p \in S'_E \quad M_{E0}(p) = 0$	

Figure 3: definition of Σ_E

With the help of figure 4 you will understand the idea of this construction. We add a transition b_i for each period N_i , that takes away the period value from the places in S . Firing these transitions we can reach M_b from any marking in E . But those transitions could be fired in some other moment removing perhaps our possibility to reach E in the future. To avoid this fact we introduce the places of S'_E and the transitions of C . We put

a token over p_i each time b_i is fired, and this will enable to fire c_i one time in any moment in the future. The firing of c_i will turn round the effect of b_i , making possible to recover a reachable marking of $[M_0 >_{\Sigma}$.

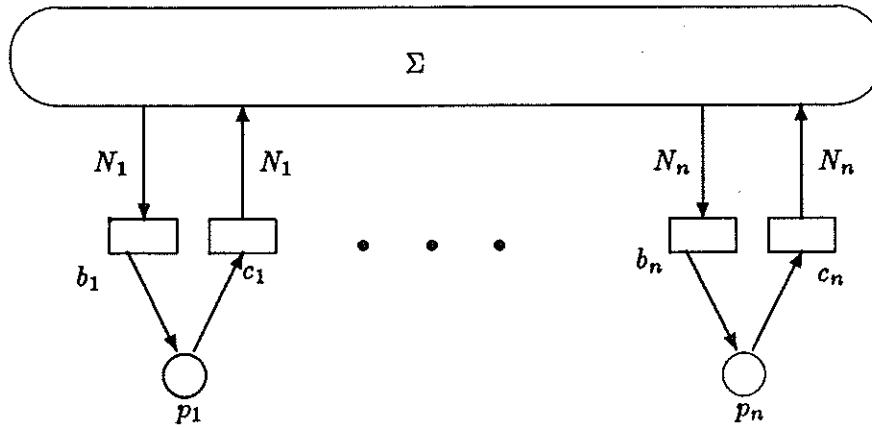


Figure 4: Presentation of Σ_E

Probably if you have understood how Σ_E works, you will see that the reduction $(\Sigma, E) \rightarrow (\Sigma_E, E')$ transforms E a linear set in E' a submarking conserving the home space property. Nevertheless we will give next a formal proof of this result. For we will prove before a sequence of properties of our transformation.

Property 1 : Let M be a marking of Σ_E such that $M(p_i) = 0$ for some $p_i \in S'_E$. Let s be a transition sequence enabled at M . Then if we denote by $\text{occ}(s, t)$ the number of occurrences of t in s , we have that $\text{occ}(s, c_i) \leq \text{occ}(s, b_i)$.

Proof : As b_i and c_i are the only transitions that take or put tokens over p_i , more exactly we have $W'_E(b_i, p_i) = 1 = W'_E(p_i, c_i)$ and $W'_E(p_i, b_i) = 0 = W'_E(c_i, p_i)$, you can check by an straight forward induction that if s verifies $M[s > M']$, we have $\text{occ}(s, b_i) = \text{occ}(s, c_i) + M'(p_i)$.
□

Remark : Each occurrence of a transition c_i of C is preceded by a different occurrence of b_i .

Property 2 : Let M be a marking of Σ_E such that for all $p \in S'_E$ we have $M(p) = 0$, and let s such that $M[s > M']$. Then we have two transition sequences s_T and s_B such that $s_T \in T^*$ and $s_B \in B^*$ verifying $M[s_T s_B > M']$.

Proof : By definition of Σ_E , it is very easy to conclude that $\forall b_i, b_j \in B, \forall c_i, c_j \in C, \forall s_1, s_2 \in (T \cap T'_E)^*$ and $\forall t \in T$

if	$s = s_1 b_i t s_2$	with	$M[s > M']$	then we have	$s' = s_1 t b_i s_2$	with	$M[s' > M]$
if	$s = s_1 b_i b_j s_2$	with	$M[s > M']$	then we have	$s' = s_1 b_j b_i s_2$	with	$M[s' > M]$
if	$s = s_1 b_i c_j s_2$	with	$M[s > M']$	then we have	$s' = s_1 c_j b_i s_2$	with	$M[s' > M]$
if	$s = s_1 b_i c_i s_2$	with	$M[s > M']$	then we have	$s' = s_1 s_2$	with	$M[s' > M]$

You can check by an induction that the transitions of s can be reordered getting s' that is enabled at M and its occurrence gives M' with $s' = s'_0 s'_1$ $s'_0 \in (T \cap C)^*$ and $s'_1 \in B^*$. By the property 1, we have $s'_0 \in T^*$.
□

Property 3 : If M is a reachable marking of Σ , M' given by $M'(p) = M(p)$ for all p in S ,

and $M'(p) = 0$ for all p in S'_E , is a reachable marking of Σ_E .

Proof : Let s a transition sequence allowing to reach M from M_0 in Σ . s is enabled at M_{E0} in Σ_E and its occurrence gives M' . \square

Property 4 : If M' is a reachable marking of Σ_E such that for all p in S'_E $M'(p) = 0$ then M given by $M(p) = M'(p)$ for all p in S , is a reachable marking of Σ .

Proof : It is a consequence of property 2. Let s' be a transition sequence of Σ_E such that $M_{E0}[s' > M'$; there must be some other sequence s'' such that $M_{E0}[s'' > M'$ and $s'' = s'_1 s''_2$ with $s'_1 \in T^*$ and $s''_2 \in B^*$. But then s''_2 must be empty, because otherwise there would be some token over the place of S'_E in the reached marking M' . But then if we consider s'_1 as a transition sequence of Σ , it is clear that $M_0[s'_1 > M$. \square

Lemma 2 : If E' is a home space of Σ_E , then E is a home space of Σ .

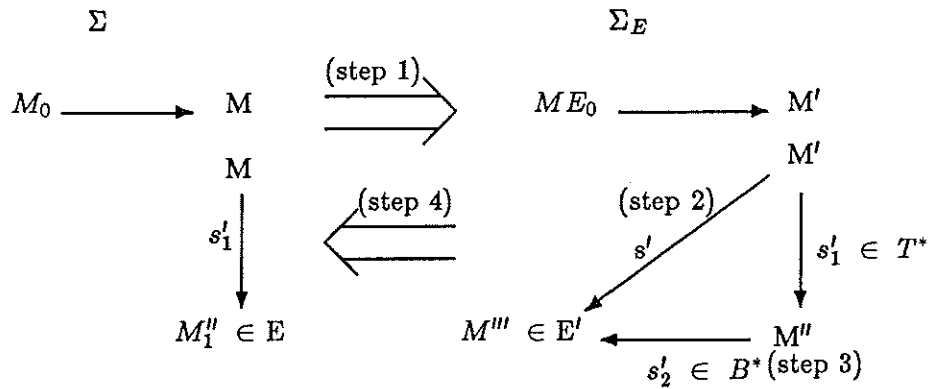


Figure 5: E' is a home space of $\Sigma_E \implies E$ is a home space of Σ

Proof : (synthesized in figure 5)

Let M be a reachable marking of Σ . Then by property 3 we have that M' given by $M'(p) = M(p)$ for all p in S , and $M'(p) = 0$ for any other place, is a reachable marking of Σ_E . Then according to the hypothesis, we will have some transition sequence s' whose occurrence gives M''' at M' , in Σ_E . Applying the property 2 we can take $s' = s'_1 s'_2$ with $s'_1 \in T^*$ and $s'_2 \in B^*$. Let M'' the marking reached after the firing of s'_1 over M' . We have for all p in S

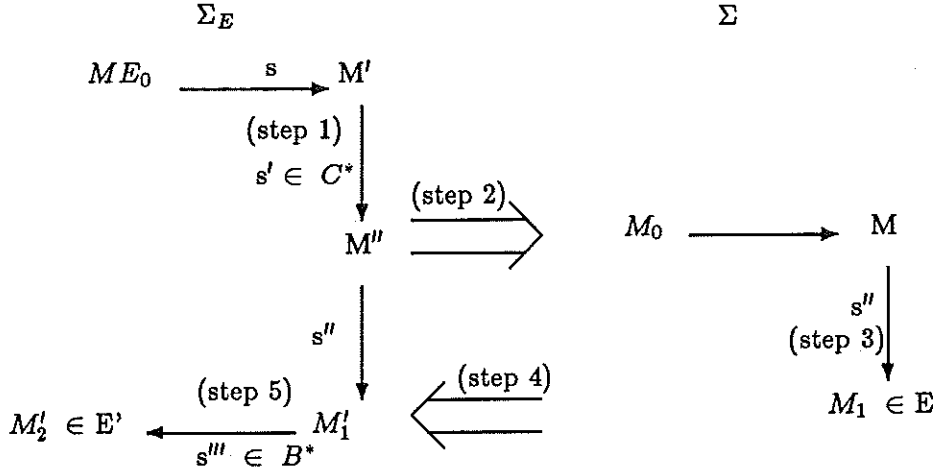
$$M''(p) = M'''(p) + \sum_{i=1}^{i=n} \text{occ}(s'_2, b_i) \cdot N_i(p).$$

Now if we consider s'_1 as a transition sequence for Σ , we have $M[s'_1 > M''$ with $M''(p) = M''(p)$ for any place p in S , and since M''' is an element of E' we conclude

$$M''(p) = M_b(p) + \sum_{i=1}^{i=n} \text{occ}(s'_2, b_i) \cdot N_i(p) \text{ for all } p \text{ in } S.$$

M'' is an element of E , and consequently E is home space of Σ . \square

Lemma 3 : If E is a home space of Σ , then E' is a home space of Σ_E .

Figure 6: E is a home space of $\Sigma \implies E'$ is a home space of Σ_E

Proof : (syntesized in figure 6)

Let M' be a reachable marking of Σ_E . Let us consider the transition sequence $s' = s'_1, \dots, s'_n$ where $s'_i \in \{c_i\}^*$ and $|s'_i| = M'(p_i)$. we can fire it at M' , to reach some marking M'' with $M''(p_i) = 0$ for all p_i in S'_E . Then, applying property 4, we have $M \in [M_0 >$, in Σ such that $M(p) = M''(p)$ for all p in S . Now since E is a home space of Σ , we have some s'' such that $M[s'' > M_1$ with $M_1 \in E$. Obviously s'' is enabled at M' in Σ_E , and if M'_1 is the reached marking we have $M'_1(p) = M_1(p)$ for all p in S . As $M_1 \in E$, we have a tuple of natural numbers k_i such that

$$\text{for all } p \text{ in } S, \quad M'_1(p) = M_b(p) + \sum_{i=1}^{i=n} k_i \cdot N_i(p).$$

Finally the transition sequence $s''' = s'''_1, \dots, s'''_n$ with $s'''_i \in \{b_i\}^*$ and $|s'''_i| = k_i$ is enabled at M'_1 , and it is clear that its occurrence gives M'_2 with $M'_2(p) = M_b(p)$ for all p in S , then $M'_2 \in E'$. \square

Theorem 2 : We can decide if a given linear set is a home space of a P/T system.

Proof : It follows from the two preceding lemmas that the property "a linear set is a home space" is reducible to the property "a submarking is a home space", and then the searched result is immediate consequence of prop. 6 and prop. 8. \square

Corollary 3 : Liveness of a transition of a P/T system is a decidable property.

Proof : Let $t \in T$. We consider the linear set $E_t \subseteq \mathbb{N}^S$ defined as follows

$$E_t = \{M \in \mathbb{N}^S \mid \forall p \in S \quad M(p) \geq W(p, t)\}$$

It is clear that t is enabled exactly at any marking in E_t , and then t is live if and only if E_t is a home space, what is decidable by theorem 2. \square

This corollary was first proved by M. Hack [Hack 74] (more exactly he proved that liveness and reachability are recursively equivalent, some years before decidability of reachability was proved). Nevertheless, it seems to us his proof is much more complicated than ours.

We conclude this section extending decidability of the problem “is a given linear set a home space ?” to finite unions of linear sets having same periods (a restricted class of semi-linear sets). To prove this fact we will reduce this property to the property “a finite union of submarkings having the same support is a home space”. The reduction is a straight forward generalization to that we have proved for linear sets, this is why we will only present the propositions without giving proof.

Def. 14 : let Σ be a P/T system. Let E a finite union of linear sets having the same periods :

$$E = \bigcup_{j=1}^{j=m} E_j \text{ with } E_j = \{M \in \mathbf{N}^S \mid M = M_j + \sum_{i=1}^{i=n} k_i \cdot N_i \text{ with } k_i \in \mathbf{N}\}$$

We consider the system Σ_E , equal to any of the systems Σ_{E_j} defined by the def. 13. Indeed they all are equal, as in that definition the dependence from the given linear set comes from its periods, but there is not any dependence from the base of the set. Then we define E' by

$$E' = \bigcup_{j=1}^{j=m} E'_j \text{ with } E'_j = \{M \in \mathbf{N}^S \mid \forall p \in S \ M(p) = M_j(p)\},$$

That obviously is a finite union of submarkings having the same support.

Lemma 4 : E is a home space of Σ if and only if E' is a home space of Σ_E .

Theorem 3 : The property “a finite union of linear sets having the same periods is a home space” is decidable.

Corollary 4 : The property “the set of integer solutions of a system of integer linear equations is a home space” is decidable.

Proof : The set of integer solutions of a system of integer linear equations admits an integer Hilbert base [Gor 73]. More precisely, this set is a finite union of linear sets having the same periods. \square

7. Conclusion

In this paper we have proved that the property of home space is decidable for linear set, and even for finite unions of linear sets having the same periods. For we presented an algorithm to decide this property for submarkings, and extended it to cover finite unions of submarkings having the same support, and we proved that the property for linear sets (resp. finite unions of linear sets having the same periods) is reducible to the property for submarkings (resp. finite unions of submarkings having the same support).

The decision procedure is too unefficient to be used in practice, because it uses intensively the known (but very complex !) algorithm to decide reachability. One of the authors became interested on this problem looking for efficient algorithms to decide home spaces in some particular, but in our opinion very interested for applications [Joh 88].

This not the last word on the decidability of home spaces. A rather natural extension of our results would concern arbitrary semi-linear sets. You can see that our restriction fixing the same set of periods for all the linear sets generating the elements of the particular class of semi-linear sets that we have studied is capital in order to get an effective algorithm to decide home spaces, following our method. More exactly we could reduce the problem to

the decidability of home spaces for arbitrary finite unions of submarkings. But we can not extend the reasonings supporting the proofs of prop. 5 and prop. 7, to these finite unions of submarkings because we only know that from any reachable marking we approach to one of these submarkings. Unfortunately now we can not iterate this process in order to finally reach one of them : it is possible that any step approaching us to one of them will move us away from other. So the closest submarking could be always changing, making impossible any conclusion on the reachability of the union of the submarkings.

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