

Memory efficient Self-Stabilizing k -Independent Dominating Set Construction*

Labri Technical Report RR-1473-13

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November 6, 2013

Abstract

We propose a memory efficient self-stabilizing protocol building k -independent dominating sets. A k -independent dominating set is a k -independent set and a k -dominating set. A set of nodes, I , is k -independent if the distance between any pair of nodes in I is at least $k + 1$. A set of nodes, D , is a k -dominating if every node is within distance k of a node of D .

Our algorithm, named *STD*, is silent; it converges under the unfair distributed scheduler (the weakest scheduling assumption).

We established that any k -independent sets contains at most $\lfloor (2n)/(k + 2) \rfloor$ nodes, n being the network size.

The protocol *STD* is memory efficient : it requires only $2\log((k + 1)n + 1) + 1$ bits per node.

The correctness and the termination of the protocol *STD* is proven.

The computation of the convergence time of the protocol *STD* is opened question.

keywords distributed computing, fault tolerance, self-stabilization, k -dominating set, k -independent set, k -independent dominating set, memory efficient

1 Introduction

The clustering of networks consists of partitioning network nodes into non-overlapping groups called clusters. Each cluster has a single head, called leader, that acts as local coordinator of the cluster, and eventually a set of standard nodes. In 1-hop clusters, the standard nodes are neighbor (at distance 1) of their leader. Clustering is found very attractive in infrastructure-less networks, like ad-hoc networks, since it limits the responsibility of network management only to leaders, and it allows the use of hierarchical routing. This is why numerous self-stabilizing 1-hop clustering protocols were proposed in the literature [9, 10, 8, 11, 13, 14, 15, 16, 19, 20].

*This work was partially supported by the ANR project *Displexity*.

Silent self-stabilizing protocols building k -hops clustering set are proposed for $k > 1$. In k -hop clusters, the distance between a standard node and its leader is at most k [1, 2, 3, 7]. The sets of cluster heads built by these protocols are not k -independent. The protocol of [1] is designed for $k = 2$. Routing tables are maintained by the cluster heads to store routing information to nodes both within and outside the cluster. The goal of the protocol in [2] is to build bounded size clusters (each cluster has at most *Cluster_Max* nodes). The protocol of [3] is designed for weighted edges networks; it requires $O(\log(k^4 \cdot \Delta^2 \cdot D^2 \cdot n^6))$ bits per node, where Δ is a bound on node degree and D is the network diameter. The protocol of [7] requires at least $2(k + 1)\log(n)$ bits per node.

In [17, 18], Larsson and Tsigas propose self-stabilizing (l, k) -clustering protocols under various assumptions. These protocols ensure, if possible, that each node has l cluster-heads at distance at most k .

Related Works. In [4], a silent self-stabilizing protocol extracting a minimal k -dominating set from any k -dominating set is proposed. A minimal k -dominating set has no proper subset which also a k -dominating set. The protocol requires at least $O(k \cdot \log(n))$ bits per node.

The paper [6] presents a silent self-stabilizing protocol building a small k -dominating set : the obtained dominating set contains at most $\lceil n/(k + 1) \rceil$ nodes. The protocol of [6] requires $O(\log(n) + k \cdot \log(n/k))$ bits per node. The protocol of [5] builds competitive k -dominating sets : the obtained dominating set contains at most $1 + \lfloor (n-1)/(k+1) \rfloor$ nodes. The protocol of [5] requires $O(\log(2k \cdot (\Delta + 1)^3 \cdot n^3))$ bits per node. These both protocols use the hierarchical collateral composition of several silent self-stabilizing protocols whose a leader election protocol and a spanning tree construction rooted to the elected leader. So they require more memory space than our protocol.

In [12], a fast silent self-stabilizing protocol building a k -independent dominating set is proposed the protocol converges in $4n + k$ rounds and it requires $(k + 1)\log(n + 1)$ bits per node. The computation of the convergence time of the protocol *SID* is an opened question.

Contribution. In this paper, we consider the problem of computing a k -independent dominating set in a self-stabilizing manner in case where $k > 1$. A nodes set is k -independent dominating set (also called maximal k -independent set) if and only if this set is a k -independent set and a k -dominating set. A set of nodes, I is k -independent if the distance between any pair of I 's nodes is at least $k + 1$. A set of nodes D is k -dominating if every node is within distance k of a node of D .

The presented protocol, named *SID*, is simple : no use of the the hierarchical collateral composition, no need of leader election process, neither the building of spanning tree. The protocol *SID* converges under the unfair distributed scheduler (the weakest scheduling assumption). The algorithm *SID* is silent. The protocol *SID* is memory efficient : it requires only $2\log((k + 1)n + 1) + 1$ bits per node.

In section 2, we establish that any k -independent sets contain at most $\lfloor (2n)/(k+2) \rfloor$ nodes, n being the network size. So the protocol of [12] and the protocol SID have the same upper bound on the size of built k independent dominating sets : $\lfloor (2n)/(k+2) \rfloor$ nodes.

Paper outline. The rest of the paper is organised as follows. In section 2, communication and computation models are defined. The protocol SID is presented in section 3. In the section 4, the correctness of the SID protocol is proven. The termination of the SID protocol is established in the section 5.

2 Model and Concepts

A distributed system S is an undirected graph $G = (V, E)$ where vertex set V is the set of nodes and edge set E is the set of communication links. A link $(u, v) \in E$ if and only if u and v can directly communicate (links are bidirectional); so, u and v are neighbors. We denote by N_v the set of v 's neighbors: $N_v = \{u \in V \mid (u, v) \in E\}$. The distance between the nodes u and v is denoted $dist(u, v)$. k -neighborhood(v) = $\{u \in V \mid dist(u, v) \in [1, k]\}$.

Definition 1 (k -independent dominating set)

Let D be a subset of V ; D is a k -dominating set if and only if $\forall v \in V/D$ we have k -neighborhood(v) $\cap D \neq \emptyset$.

Let I be a subset of V ; I is a k -independent set if and only if $\forall u \in I$ we have k -neighborhood(u) $\cap I = \emptyset$.

A subset of V is a distance- k independent dominating set if this subset is a distance- k dominating set and a distance- k independent set.

Lemma 1 The size of a k -independent set is at most $\max(\lfloor (2n)/(k+2) \rfloor, 1)$.

Proof 1 Let I be a k -independent set such that $|I| > 1$. Let v be a node of I . We denote by $\mathbf{closer}(v)$ the set of nodes closer to v than any other node of I .

Notice that $\bigcup_{w \in I} \mathbf{closer}(w) \subset V$ and $\mathbf{closer}(v) \cap \mathbf{closer}(u) = \emptyset, \forall (u, v) \in I^2$. Let u be the closest node to v that belongs to I . Let x be node on the path from v to u such that $0 \leq dist(v, x) \leq \lfloor k/2 \rfloor$. Let w be a node of I other than v . We have $dist(w, x) > k - dist(v, x) \geq \lfloor k/2 \rfloor$ because $k < dist(w, v) \leq dist(v, x) + dist(x, w)$. So, $\mathbf{closer}(v)$ contains the first $\lfloor k/2 \rfloor + 1$ nodes in the path from v to u . We conclude that $|I| \leq \lfloor (2n)/(k+2) \rfloor$. ■

The protocol SID builds k -independent sets. So, the obtained k -independent dominating set contains at most $\lfloor (2n)/(k+2) \rfloor$ nodes.

Furthermore, at every node v in the network is assigned an identifier, denoted by id_v . Two distinct nodes have different identifier. It is possible to order the identifier values. The symbol \perp denotes a value smaller than any identifier value in the network.

Each node v maintains a set of shared variables such that v can read its own variables and those of its neighbors, but it can modify only its variables. The *state* of a node is defined by the values of its local variables. The cartesian product of states of all nodes determines the *configuration* of the system. The *program* of each node is a set of *rules*. Each rule has the form: $Rule_i : \langle Guard_i \rangle \rightarrow \langle Action_i \rangle$. The *guard* of a v 's rule is a boolean expression involving the state of the node v , and those of its neighbors. The *action* of a v 's rule updates v 's state. A rule can be executed only if it is *enabled*, i.e., its guard evaluates to true. A node is said to be enabled if at least one of its rules is enabled. In a *terminal configuration*, no node is enabled.

During a *computation step* $c_i \rightarrow c_{i+1}$, one or several enabled nodes perform an enabled action and the system reaches the configuration c_{i+1} from c_i . A *computation* e is a sequence of configurations $e = c_0, c_1, \dots, c_i, \dots$, where c_{i+1} is reached from c_i by one computation step: $\forall i \geq 0, c_i \rightarrow c_{i+1}$. We say that a computation e is *maximal* if it is infinite, or if it reaches a terminal configuration. We note by \mathcal{C} the set of all possible configurations, and by \mathcal{E} the set of all maximal computations. The set of maximal computations starting from a particular configuration $c \in \mathcal{C}$ is denoted \mathcal{E}_c . \mathcal{E}_A denotes the set of all maximal computations where the initial configuration belongs to the set of configurations $A \subset \mathcal{C}$.

Definition 2 (Attractor) Let B_1 and B_2 be subsets of \mathcal{C} . B_2 is an attractor from B_1 , if and only if the following conditions hold:

- **Convergence:** $\forall c \in B_1$, If $(\mathcal{E}_c = \emptyset)$ then $c \in B_2$ otherwise $\forall e \in \mathcal{E}_{B_1}(e = c_1, c_2, \dots), \exists i \geq 1, c_i \in B_2$;
- **Closure:** $\forall e \in \mathcal{E}_{B_2}(e = c_1, \dots), \forall i \geq 1 : c_i \in B_2$.

Definition 3 (Self-stabilization) A distributed system S is self-stabilizing for the specification \mathcal{SP} (a predicate on configurations) if and only if there exists a non-empty set $\mathcal{L} \subseteq \mathcal{C}$, called set of legitimate configurations, such that the following conditions hold:

- \mathcal{L} is an attractor from \mathcal{C} ;
- Configurations of \mathcal{L} satisfied \mathcal{SP} .

A self-stabilizing protocol is *silent* if all maximal computations are finite.

Stabilization time. We use the *round* notion to measure the time complexity. The first round of a computation $e = c_1, \dots, c_j, \dots$ is the minimal prefix $e_1 = c_1, \dots, c_j$, such that every enabled node in c_1 either executes a rule or it is neutralized during a computation step of e_1 . A node v is *neutralized* during a computation step cs $c_i \rightarrow c_{i+1}$, if v is enabled in c_i and disabled in c_{i+1} , but it did not execute any action during cs .

Let e_2 be the suffix of e such that $e = e_1e_2$. The second round of e is the first round of e_2 , and so on.

The stabilization time is the number of rounds of a computation reaching a legitimate configuration from any initial one.

3 The protocol *STD*

In the following subsection, we give the notation used by the protocol *STD*.

3.1 *k*-augmentedID type

Definition 4 *k*-augmentedID type An *k*-augmentedID value, a , is \perp or an n -tuple (d, x) such that d is integer with $0 \leq d \leq k$, and x is a node identifier. Let $a = (d, x)$ be *k*-augmentedID value. We use the following notation $a.dist = d$ and $a.id = x$. Let v be a node of V , id_v^+ is the following *k*-augmentedID value: $(0, id_v)$.

Definition 5 The total order relation *dom* on *k*-augmentedID

- $dom(a, b) = a$ if $b = \perp$, $a.id < b.id$ or $a.id = b.id \wedge a.dist < b.dist$, otherwise $dom(a, b) = b$.
- The *k*-augmented value $a1$ dominates the *k*-augmented value $a2$ if and only if $dom(a1, a2) = a1$.
- Let X be a finite set of *k*-augmentedID values. $dom(X)$ is the *k*-augmentedID value belonging to X such that any value of X is dominated by $dom(X)$ (i.e. $\forall y \in X$ we have $dom(dom(X), y) = dom(X)$).

Definition 6 The total order relation *min* on *k*-augmentedID

- $min(a, b) = a$ if $b = \perp$, $a.dist < b.dist$ or $a.dist = b.dist \wedge a.id < b.id$ otherwise $min(a, b) = b$.
- The *k*-augmented value $a1$ is larger than the *k*-augmented value $a2$ if and only if $min(a1, a2) = a2$.
- Let X be a finite set of *k*-augmentedID values. $min(X)$ is the *k*-augmentedID value belonging to X such that any value of X is larger than $min(X)$ (i.e. $\forall y \in X$ we have $min(min(X), y) = min(X)$).

The node $u1$ is closer to the node v than the node $u2$ iff $dist(u1, v) < dist(u2, v)$ or $id_{u1} < id_{u2}$. We have $min((dist(u1, v), id_{u1}), (dist(u2, v), id_{u2})) = (dist(u1, v), id_{u1})$.

Definition 7 The operation $+1$ on *k*-augmentedID is defined as follow : $a + 1 = a$ if $a = \perp$ or if $a.dist = k$ otherwise $a + 1 = (a.dist + 1, a.id)$

Protocol 1 : Variables, Predicates and Rules of the Protocol *STD* on the node v

Shared variables

- $\mathbf{firstHead}(v)$ and $\mathbf{secondHead}(v)$. They take value in k -augmentedID.

Internal variable

- \mathbf{beReal} is a boolean variable used by some macros.

Notation

- $\mathbf{firstAugmentedIdSet}(v) = \{a + 1 \in k\text{-augmentedID} \mid a = \mathbf{firstHead}(u) \vee a = \mathbf{secondHead}(u) \text{ with } u \in N_v \wedge a.\mathit{dist} < k \wedge a.\mathit{id} \neq \mathit{id}_v\}$
- $\mathbf{secondAugmentedIdSet}(v) = \{a \in \mathbf{firstAugmentedIdSet} \mid a.\mathit{id} \neq \mathbf{firstHead}(v).\mathit{id}\}$

Macros

- $\mathbf{isDefended}(v)$ returns true iff $\mathbf{firstAugmentedIdSet}(v) \neq \emptyset$.
- $\mathbf{isDominated}(v)$ returns true iff $\mathit{id}_v^+ \neq \mathit{dom}(\mathbf{firstAugmentedIdSet}(v) \cup \mathit{id}_v^+)$.
- $\mathbf{correctFirstHead}(v)$ returns true iff
$$\mathbf{firstHead}(v) == \mathit{min}(\mathbf{firstAugmentedIdSet}(v)).$$
- $\mathbf{computingFirstHead}(v)$ returns the value of $\mathit{min}(\mathbf{firstAugmentedIdSet}(v))$.
- $\mathbf{correctSecondHead}(v)$ returns true iff
$$\mathbf{secondHead}(v) == \mathit{min}(\mathbf{secondAugmentedIdSet}(v) \cup \perp).$$
- $\mathbf{computingsSecondHead}(v)$ returns the value of $\mathit{min}(\mathbf{secondAugmentedIdSet}(v) \cup \perp)$.

Predicates

- $\mathbf{Head}(v) \equiv \mathbf{firstHead}(v) == (0, \mathit{id}_v)$
- $\mathbf{toResign}(v) \equiv \mathbf{isDominated}(v)$
- $\mathbf{toElect}(v) \equiv \neg \mathbf{isDefended}(v)$
- $\mathbf{headToUpdate}(v) \equiv \mathbf{firstHead}(v) \neq (0, \mathit{id}_v) \vee \mathbf{secondHead}(v) \neq \perp$
- $\mathbf{ordinaryToUpdate}(v) \equiv \neg \mathbf{correctFirstHead}(v) \vee \neg \mathbf{correctSecondHead}(v)$

Rules

- $\mathbf{RE}(v) : \neg \mathbf{Head}(v) \wedge \mathbf{toElect}(v) \longrightarrow \mathbf{firstHead}(v) := (0, \mathit{id}_v); \mathbf{secondHead}(v) := \perp;$
 - $\mathbf{RU}(v) : \neg \mathbf{Head}(v) \wedge \neg \mathbf{toElect}(v) \wedge \mathbf{ordinaryToUpdate}(v) \longrightarrow$
$$\mathbf{computingFirstHead}(v); \mathbf{computingSecondHead}(v);$$
 - $\mathbf{RR}(v) : \mathbf{Head}(v) \wedge \mathbf{toResign}(v) \longrightarrow$
$$\mathbf{computingFirstHead}(v); \mathbf{computingSecondHead}(v);$$
 - $\mathbf{RC}(v) : \mathbf{Head}(v) \wedge \neg \mathbf{toResign}(v) \wedge \mathbf{headToUpdate}(v) \longrightarrow$
$$\mathbf{firstHead}(v) := (0, \mathit{id}_v); \mathbf{secondHead}(v) := \perp;$$
-

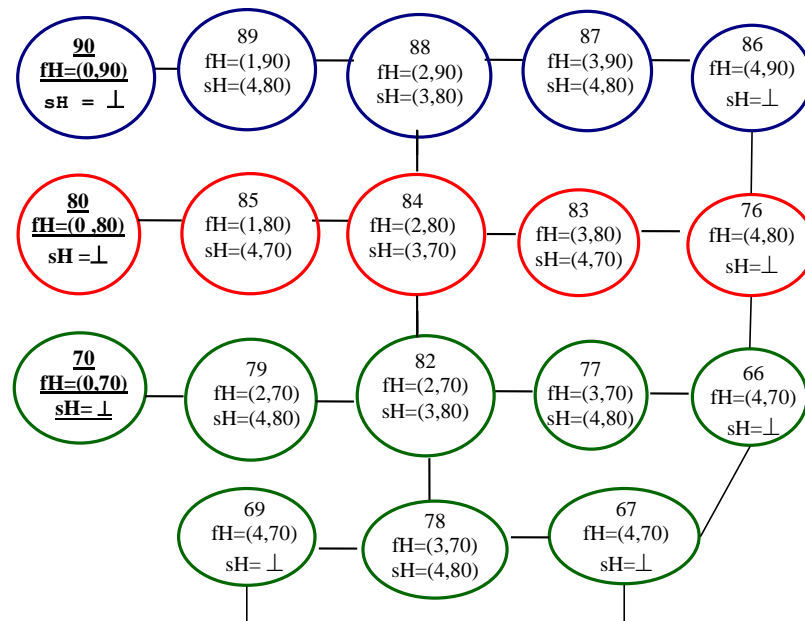
3.2 Code of the protocol SID

A node v is said to be a head if $\text{firstHead}(v) = id_v^+$; otherwise it is an ordinary node. The heads set built by the protocol SID (defined in protocol 1) is a k -independent dominating set. The code of the macros are defined in the protocols 2 to 7.

The variable $\text{firstHead}(v)$ contains the identifier of the closest head to v (with its distance to v).

The variable $\text{secondHead}(v)$ contains the identifier of the second closest head to v (with its distance to v) inside its k -neighborhood. If a node v does not have two heads in its k -neighborhood then $\text{secondHead}(v)$ is set to \perp . The execution of the rule **RU** or the rule **RC** updates the two variables $\text{firstHead}(v)$, and $\text{secondHead}(v)$ without changing the status of v (i.e. v stays ordinary or head).

In the figure 1 is presented a terminal configuration of SID in the case where $k = 4$. In each node, it is indicated the value of firstHead , denoted by fH , and the value of secondHead denoted by sH . The terminal configuration has three heads. On the same network with k being equal to 4, is presented another terminal configuration having a single head in the figure 2.



The head identifiers are underlined.
The color of a node is the color of its closest head.

Figure 1: A terminal configuration of SID

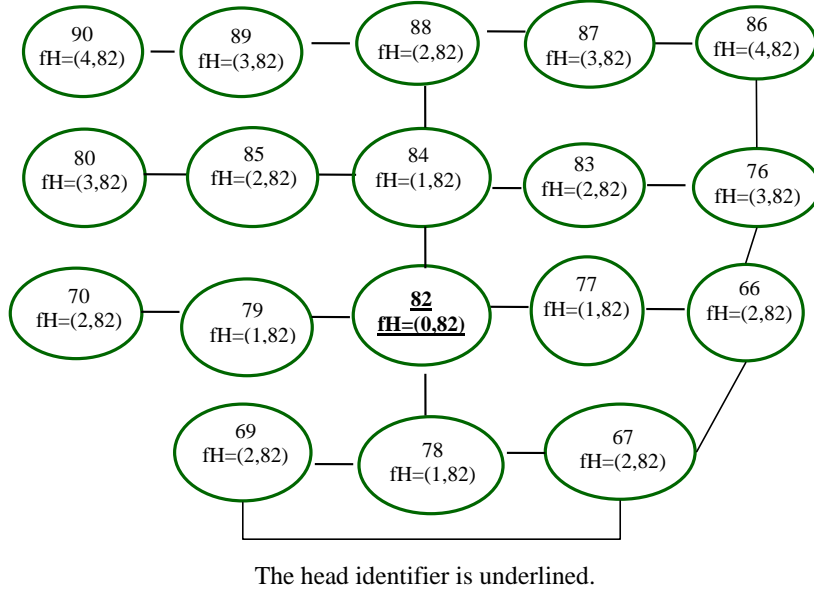


Figure 2: Another terminal configuration of SID

The macro $\text{isDefended}(v)$ (defined in protocol 2) returns true if the set $\text{firstAugmentedIdSet}(v)$ is not empty otherwise the macro returns false.

Protocol 2 : macro $\text{isDefended}(v)$

for $u \in N_v$ do
 if $\text{firstHead}(u).dist < k \wedge \text{firstHead}(u).id \neq id_v$ then return true;
 if $\text{secondHead}(u).dist < k \wedge \text{secondHead}(u).id \neq id_v$ then return true;
done; return false;

The macro $\text{isDominated}(v)$ (defined in protocol 3) returns true if a value x of $\text{firstAugmentedIdSet}(v)$ dominates the value id_v^+ ; otherwise the macro returns false.

The macro $\text{correctFirstHead}(v)$ (defined in protocol 4) returns true if the value of $\text{firstHead}(v)$ is $\min(\text{firstAugmentedIdSet}(v))$; otherwise the macro returns false. Notice that if the set $\text{firstAugmentedIdSet}(v)$ is empty the macro returns false;

The macro $\text{computingFirstHead}(v)$ (defined in protocol 5) returns $\min(\text{firstAugmentedIdSet}(v))$ if the set $\text{firstAugmentedIdSet}(v)$ is not empty; otherwise the macro returns \perp .

Protocol 3 : macro `isDominated(v)`

```
for  $u \in N_v$  do
  if firstHead(u).dist < k  $\wedge$  firstHead(u).id < id_v then return true;
  if secondHead(u).dist < k  $\wedge$  secondHead(u).id < id_v then return true;
done; return false;
```

Protocol 4 : Macro `correctFirstHead(v)`

```
beReal := false;
for  $u \in N_v$  do
  if firstHead(u).id  $\neq$  id_v  $\wedge$  firstHead(u).dist < k then
    if firstHead(v)  $\neq$  min(firstHead(v), firstHead(u) + 1) then return false;
    if firstHead(v) == (firstHead(u) + 1) then beReal := true;
  if secondHead(u)  $\neq$   $\perp$   $\wedge$  secondHead(u).id  $\neq$  id_v  $\wedge$  secondHead(u).dist < k then
    if firstHead(v)  $\neq$  min(firstHead(v), secondHead(u) + 1) then return false;
    if firstHead(v) == (secondHead(u) + 1) then beReal := true;
done; return beReal;
```

The macro `correctsecondHead(v)` (defined in protocol 6) returns true if the value of `secondHead(v)` is $\min(\text{firstAugmentedIdSet}(v) \cup \perp)$; otherwise the macro returns false.

The macro `computingSecondHead(v)` (defined in protocol 7) returns $\min(\text{secondAugmentedIdSet}(v))$ if the set `secondAugmentedIdSet(v)` is not empty; otherwise the macro returns \perp .

Once the system is stabilized, the set `firstAugmentedIdSet(v)` contains some heads in k -neighborhood of v . More precisely, this set contains the closest and second closest head to v if there are in the k -neighborhood of v . If the k 's neighborhood of a node v does not contain any head then the set `firstAugmentedIdSet(v)` is empty. So the predicate `isDefended(v)` is not verified. If v is an ordinary node then v is enabled (the rule **RE** or the rule **RU** is enabled). Therefore, the heads set is a k -dominating set, in a terminal configuration.

If `toResign(v)` is verified then v has in its k -neighborhood a head u having a smaller identifier than v 's identifier (i.e. $id_v > id_u$). In this case, if v is a head, it is enabled. So, the set of heads is a k -independent set, in any terminal configuration.

The proof of the protocol *SID* has two parts.

- In the section 4, we prove that a terminal configuration of *SID* protocol is legitimate : the set of heads is a k -independent dominating set.

Protocol 5 : Macro computingFirstHead(v)

```
firstHead( $v$ ) :=  $\perp$ ;  
for  $u \in N_v$  do  
  if firstHead( $u$ ).id  $\neq id_v \wedge$  firstHead( $u$ ).dist  $< k$  then  
    firstHead( $v$ ) := min(firstHead( $v$ ), firstHead( $u$ ) + 1);  
  if secondHead( $u$ )  $\neq \perp \wedge$  secondHead( $u$ ).id  $\neq id_v \wedge$  secondHead( $u$ ).dist  $< k$  then  
    firstHead( $v$ ) := min(firstHead( $v$ ), secondHead( $u$ ) + 1);  
done;
```

Protocol 6 : Macro correctSecondHead(v)

```
beReal := false;  
for  $u \in N_v$  do  
  if firstHead( $u$ ).id  $\neq id_v \wedge$  firstHead( $u$ ).id  $\neq$  firstHead( $v$ ).id  $\wedge$   
    firstHead( $u$ ).dist  $< k$  then  
    if secondHead( $v$ )  $\neq$  (min(secondHead( $v$ ), firstHead( $u$ ) + 1)) then  
      return false;  
    if secondHead( $v$ ) == firstHead( $u$ ) + 1 then beReal := true;  
  if secondHead( $u$ )  $\neq \perp \wedge$  secondHead( $u$ ).id  $\neq id_v \wedge$   
    secondHead( $u$ ).id  $\neq$  firstHead( $v$ ).id  $\wedge$  secondHead( $u$ ).dist  $< k$  then  
    if secondHead( $v$ )  $\neq$  min(secondHead( $v$ ), secondHead( $u$ ) + 1) then  
      return false;  
    if secondHead( $v$ ) == secondHead( $u$ ) + 1 then beReal := true;  
done;  
if  $\neg beReal$  and (secondHead( $v$ )  $\neq \perp$ ) then return false;  
return true;
```

Protocol 7 : Macro computingSecondHead(v)

```
secondHead( $v$ ) :=  $\perp$ ;  
for  $u \in N_v$  do  
  if firstHead( $u$ ).id  $\neq id_v \wedge$  firstHead( $u$ ).id  $\neq$  firstHead( $v$ ).id  $\wedge$   
    firstHead( $u$ ).dist  $< k$  then  
    secondHead( $v$ ) := (min(secondHead( $v$ ), firstHead( $u$ ) + 1));  
  if secondHead( $u$ )  $\neq \perp \wedge$  secondHead( $u$ ).id  $\neq id_v \wedge$   
    secondHead( $u$ ).id  $\neq$  firstHead( $v$ ).id  $\wedge$  secondHead( $u$ ).dist  $< k$  then  
    secondHead( $v$ ) := min(secondHead( $v$ ), secondHead( $u$ ) + 1);  
done;
```

- In the section 5, we prove that all maximal computations under any unfair distributed scheduler are finite by *reductio ad absurdam* arguments.

4 Correctness of the protocol SID

In this section, we prove that all terminal configuration of SID protocol are legitimate: the set of heads is a k -independent dominating set.

Definition 8 *The property $\text{OrdinaryPr}(i)$ defined for all $i \in [1, k]$ is verified if the two following statements are satisfied:*

- $\text{OrdinaryPrFirst}(i)$: for all ordinary node v , $\text{firstHead}(v) = (i, id_u)$ if and only if u is the closest head to v and i is the distance between u and v .
- $\text{OrdinaryPrSecond}(i)$: for all node v , $\text{secondHead}(v) = (i, id_w)$ if and only if w is the second closest head to v and i is the distance between w and v .

Observation 1 *In a terminal configuration,*

1. An ordinary node v does not verify $\text{OrdinaryToUpdate}(v)$;
so $\text{firstHead}(v) = \min(\text{firstAugmentedIdSet}(v))$ and
 $\text{secondHead}(v) = \min(\text{secondAugmentedIdSet}(v) \cup \perp)$.
2. A head u does not verify $\text{HeadToUpdate}(u)$;
3. Let w be a node (head or ordinary), $\text{firstHead}(w) \neq \perp$;
4. if v is an ordinary node then $\text{firstHead}(v).dist > 0$;
5. if $\text{secondHead}(v) \neq \perp$ then $\text{secondHead}(v).dist > 0$;
6. if $\text{secondHead}(v) \neq \perp$ then $\text{secondHead}(v).dist \geq \text{firstHead}(v).dist$ because
 $\text{secondAugmentedIdSet}(v) \subset \text{firstAugmentedIdSet}(v)$.

Lemma 2 *In a terminal configuration of protocol SID , the property $\text{OrdinaryPr}(1)$ is verified.*

Proof 2

Let v be an ordinary node, in a terminal configuration of protocol SID , named c . Assume that $(1, x) \in \text{firstAugmentedIdSet}(v)$. So v has a neighbor u such that $\text{firstHead}(u) = (0, x)$ or $\text{secondHead}(u) = (0, x)$. According to observation 1.4 $\text{secondHead}(u).dist > 0$ or $\text{secondHead}(u) = \perp$. So v has a neighbor u such that $\text{firstHead}(u) = (0, x)$. According to observation 1.3 u is a head; so $x = id_u$.

Notice that $\forall a \in \text{firstAugmentedIdSet}(v)$, we have $a.dist > 0$, in c .

Proof of OrdinaryPrFirst(1). If v has a head at distance 1 then v has a neighbor u such that $\text{firstHead}(u) = (0, id_u)$. So, we have $\text{firstHead}(v) = (1, id_u)$ with u being the head in v 's neighborhood having the smallest identifier.

If v has not a head at distance 1 then for any u neighbor, we have $\text{firstHead}(u).dist > 0$. and $\text{secondHead}(u).dist > 0$ or $\text{secondHead}(u) = \perp$ (according to observation 1.4). In this case, $\text{firstHead}(v).dist > 1$.

Proof of OrdinaryPrSecond(1). If v has several heads at distance 1 then v has a neighbor w such that $\text{firstHead}(w) = (0, id_w)$ with $id_w \neq \text{firstHead}(v).id$. So, $\text{secondHead}(v) = (1, id_w)$ with w being the head in v 's neighborhood having the second smallest identifier. If v has at most one head at distance 1 then v has not a neighbor w such that $\text{firstHead}(w) = (0, id_w)$ with $id_w \neq \text{firstHead}(v).id$. In this case, $\text{secondHead}(v).dist$ is larger than 1 or $\text{secondHead}(v) = \perp$. ■

Lemma 3 Let i be a positive integer smaller than k . In a terminal configuration of protocol *STD*, if the properties $\text{OrdinaryPr}(j)$ are verified for all $j \in [1, i]$ then the property $\text{OrdinaryPr}(i + 1)$ is verified.

Proof 3 Let us assume that the properties $\text{OrdinaryPr}(j)$ are verified for all $j \in [1, i]$ in any terminal configuration of protocol *STD*.

In a terminal configuration c , $(j, x) \in \text{firstAugmentedIdSet}(v)$ iff v has a neighbor u such that $\text{firstHead}(u) = (j - 1, x)$, or $\text{secondHead}(u) = (j - 1, x)$. If $j = 1$ then u is a head in c , according to Observation 1. If $1 < j \leq i + 1$ then x is the identifier of a head in c at distance $j - 1$ of u , according to the property $\text{OrdinaryPr}(j - 1)$. So x is the identifier of a head at distance at most j of v , in c .

Proof of OrdinaryPrFirst(i+1). Let v' be the closest head to v and d' the distance from v' to v in the terminal configuration c . Assume that $0 < d' \leq i + 1$. v has a neighbor u at distance $d' - 1$ to v' . In c , the node v' is the closest head of u ; so $\text{firstHead}(u) = (d' - 1, id_{v'})$, according to the properties $\text{OrdinaryPr}(d' - 1)$. According to the properties $\text{OrdinaryPr}(j) \forall j \in [1, i]$, in c , we have the following properties,

- if $(l, id) \in \text{firstAugmentedIdSet}(v)$ then $l \geq d'$; and
- if $(d', id) \in \text{firstAugmentedIdSet}(v)$ then $id \geq id_{v'}$. In c ,

We conclude that $\text{firstHead}(v) = (d', id_{v'})$, in c .

Proof of OrdinaryPrSecond(i+1). Assume that the network has several heads. Let v'' be the second closest head to v and d'' the distance from v'' to v , in a terminal configuration c . v has a neighbor u at distance $d'' - 1$ to v'' in c . (we have $d'' > 0$). v'' is the first or second closest head to u , in c . Assume that $d'' \leq i + 1$. According to the property $\text{OrdinaryPr}(d'' - 1)$, $\text{firstHead}(u) = (d'' - 1, id_{v''}) \vee \text{secondHead}(u) = (d'' - 1, id_{v''})$, in c . According to the properties $\text{OrdinaryPr}(j) \forall j \in [1, i]$, in c , we have the following properties,

- if $(l, id) \in \text{secondAugmentedIdSet}(v)$ then $l \geq d''$;

- if $(d^n, id) \in \text{secondAugmentedIdSet}(v)$ then $id \geq id_{v^n}$.

We conclude that $\text{secondHead}(v) = (d^n, id_{v^n})$. ■

The following corollary is a direct result of lemmas 2 and 3. It establishes that the set of heads is a k -dominating set.

Corollary 1 *Let v be a ordinary node, in a terminal configuration of protocol SID . $\text{firstHead}(v).id$ is the closest head to v ; their distance is $\text{firstHead}(v).dist \leq k$. If $\text{secondHead}(v) = \perp$ then v has a single head in its k -neighborhood; otherwise $\text{secondHead}(v).id$ is the second closest head to v ; their distance is $\text{secondHead}(v).dist$.*

The following theorem establishes that the set of heads is a k -independent set.

Theorem 1 *Let v be a head, in a terminal configuration of protocol SID , named c . v has not head in its k -neighborhood.*

Proof 4 *We will prove that if a head has another head in its k -neighborhood then the configuration c is not terminal.*

Let $wrongHeadSet$ the set of heads having one or several heads in their k -neighborhood. Assume that $wrongHeadSet$ is not empty. We denoted by $v1$ the node of $wrongHeadSet$ having the largest identifier. We denote by $v2$, the closest head to $v1$ and by d the distance between $v1$ and $v2$. We have $0 < d \leq k$ and $id_{v2} < id_{v1}$.

The node $v1$ has a neighbor u at distance $d-1$ of $v2$. The node $v2$ is the first or the second closest head to u . According to corollary 1, $(d-1, id_{v2}) = \text{firstHead}(u)$ or $(d-1, id_{v2}) = \text{secondHead}(u)$. $v1$ is enabled because $v1$ satisfied the predicate $\text{toResign}(v1)$. ■

5 Termination of the protocol SID

In this section, we prove that all maximal computations of protocol SID under any unfair distributed scheduler are finite by *reductio ad absurdam* arguments.

Let e be a maximal computation starting from a configuration, named $c0$. In a configuration c reached by e , for all node v , $\text{firstHead}(v)_c.id$ is either the identifier of an node or this value appears in the initial configuration (i.e. there is a node u , such that $\text{firstHead}(v)_c.id = \text{firstHead}(u)_{c0}.id \vee \text{firstHead}(v)_c.id = \text{secondHead}(u)_{c0}.id$). So, the value taken by a variable firstHead in e belongs to a bounded set. Similary we prove also that the value taken by a variable secondHead in e belongs to a bounded set.

5.1 RR and RE rules

Along any computation, a node performs at most one time the rule **RC**.

Assume that a or several nodes perform infinitely often the action **RE** or the action **RR**. Between two consecutive actions **RE** by a node u , this node has performed on time the action **RR**. So a node u that infinitely often perform the action **RE** or the action **RR** changes its status infinitely often. We name u^+ the node the smallest identifier among nodes that changes their status infinitely often. e has a suffix $e1$ where only nodes having a identifier larger than id_{u^+} changes their status (i.e. to perform the action **RE** or the action **RR**).

As the set of value taken by $\mathbf{firstHead}(u^+)$ is bounded; along $e1$, infinitely often after the action **RR**(u^+), $\mathbf{firstHead}(u^+)$ has the same value, denoted $(l+1, id)$. Notice that $id < id_{u^+}$ and $0 < l < k$. So u^+ has a neighbor u_l such that, infinitely often before the action **RR**(u^+), u_l verifies $\mathbf{firstHead}(u_l) = (l, id)$ or $\mathbf{secondHead}(u_l) = (l, id)$.

At time, where u^+ becomes head, we have $\mathbf{firstAugmentedIdSet}(u^+) = \emptyset$. So, the values of u_l variables are infinitely often larger than (l, id) : So u_l gives infinitely often to one of its variables the value (l, id) , but also gives a larger value to the same variable.

Assume that $l > 0$. At time where u_l gives the value (l, id) to one of its variable : u_l has a neighbor u_{l-1} , having the value $(l-1, id)$. At time where u_l gives a larger value than (l, id) to the same variable : u_{l-1} has a larger value than $(l-1, id)$. We conclude that there is a series of $l+1$ nodes : $u_l, u_{l-1}, ..u_0$ such that u_i has infinitely often has the value (i, id) and infinitely often does not have this value along $e1$.

Along $e1$, u_0 performs infinitely often the action **RR** and the action **RE**. We have $id = id_{u_0} < id_{u^+}$: there is a contradiction.

So e has a suffix, named $e2$, in which the only rule performed is **RU**.

5.2 RU rule

Assume that a node or several nodes changing infinitely often their value $\mathbf{firstHead}$ or their value $\mathbf{secondHead}$ along $e2$. We named min^+ the smallest value infinitely often allocates to the variable $\mathbf{firstHead}$ or to the variable $\mathbf{secondHead}$ of one of these nodes. Let $e3$ be the suffix of $e2$ in which no variable $\mathbf{firstHead}$ and no variable $\mathbf{secondHead}$ gets a value smaller than min^+ . Along $e3$, infinitely often, a node, named u^+ , performs **RU** action to set the value min^+ to its variable $\mathbf{firstHead}$ or its variable $\mathbf{secondHead}$; and infinitely often, u^+ performs **RU** action to set to the same variable a value larger than min^+ .

Let $c \rightarrow c'$ be a computation step of $e3$ where u^+ performs **RU** action to set a value larger than min^+ to its variable $\mathbf{firstHead}$ or to its variable $\mathbf{secondHead}$. In c , min^+ is smaller than $min(\mathbf{firstAugmentedIdSet}(u^+))$ or min^+ is smaller than $min(\mathbf{secondAugmentedIdSet}(u^+))$. This property stays verified along $e3$: u^+ never sets the value min^+ to its variable $\mathbf{firstHead}$ (resp. to its variable $\mathbf{secondHead}$). There is a contradiction.

We have established that e_2 has a suffix e_4 where no rule is executed. A terminal configuration is reached.

6 Conclusion

A simple and silent self-stabilizing protocols building k -independent dominating sets is presented. The obtained k -independent dominating set contains at most $\lfloor (2n)/(k+2) \rfloor$ nodes. The protocol converges under the unfair distributed scheduler (the weakest scheduling assumption). The protocol is memory efficient : it requires only $2\log((k+1)n+1)+1$ bits per node.

The computation of the convergence time of the protocol STD is opened question.

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