

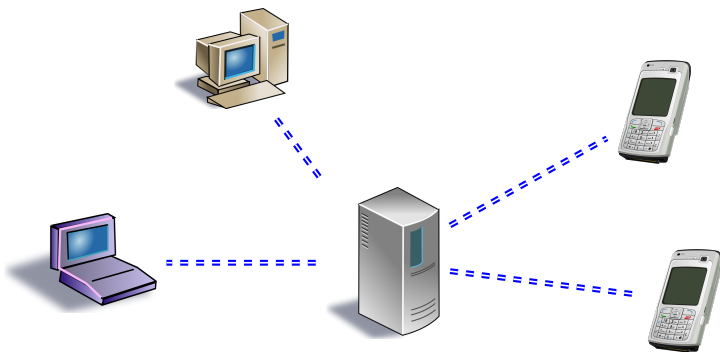
# On the Boundedness Problem for Pushdown Vector Addition Systems

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Kolloquiumsvortrag, University of Bamberg, January 29th 2014

# Verification of Concurrent Systems (1)



- ⇒ Concurrent systems everywhere!
- ⇒ Hard to design correctly
  - Complex and **unforeseen** interactions between components
  - Need for automated verification tools

# Verification of Concurrent Systems (2)

## Model-checking

$$\mathcal{M} \stackrel{?}{\models} \varphi$$

Need models  $\mathcal{M}$  with

- enough expressive power to represent the system to verify
- support for automatic verification (model-checking decidable)

Vector Addition Systems  $\simeq$  Petri nets

- Classical model for (parametrized) concurrent systems
  - ▶ Rendez-vous synchronization
  - ▶ Asynchronous communication via unbounded unordered buffers
  - ▶ Dynamic process creation
- Fundamental class that is often used as a toolbox

⇒ Can we have more expressive power while retaining decidability?

# Table of Contents

- 1 Vector Addition Systems
- 2 The Reduced Reachability Tree for VAS
- 3 Extension to Pushdown Vector Addition Systems
- 4 Conclusion

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# Vector Addition Systems with States in a Nutshell

A vector addition system with states is a finite-state automaton that is

- equipped with finitely many **counters**  $x, y, \dots$
- counters range over the set  $\mathbb{N}$  of **natural numbers**
- counter operations are:

- ▶ increment:

$$x := x + 1$$

- ▶ guarded decrement:

$$\text{assert}(x > 0) ; x := x - 1$$

Vector addition systems with states are similar to **Petri nets**

## Definition

A **vector addition system** is a pair  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  where

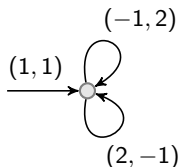
- $\mathbf{v}_{\text{init}} \in \mathbb{N}^d$  : **initial vector**
- $\mathbf{A} \subseteq \mathbb{Z}^d$  : finite set of **actions**

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$$\begin{aligned}\mathbf{A} &= \{(-1, 2), (2, -1)\} \\ \mathbf{v}_{\text{init}} &= (1, 1)\end{aligned}$$





## Vector Addition Systems — Semantics

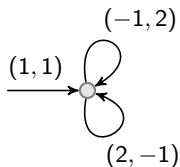
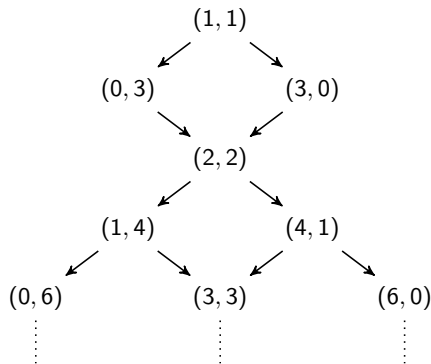
The **semantics** of a VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  is the transition system  $\langle \mathbb{N}^d, \mathbf{v}_{\text{init}}, \rightarrow \rangle$  whose transition relation  $\rightarrow$  is given by

$$\frac{\mathbf{a} \in \mathbf{A} \wedge \mathbf{v}' = \mathbf{v} + \mathbf{a} \geq \mathbf{0}}{\mathbf{v} \rightarrow \mathbf{v}'}$$

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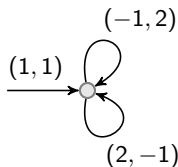
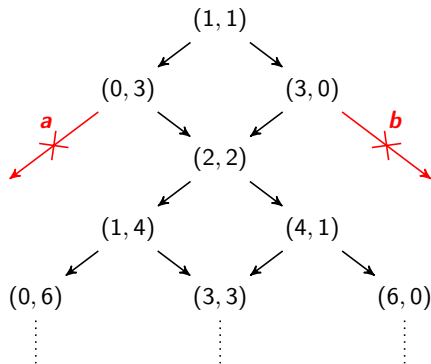
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## Definition

A **vector addition system with states** is a tuple  $\langle Q, q_{\text{init}}, \mathbf{v}_{\text{init}}, \Delta \rangle$  where

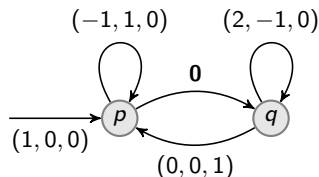
- $Q$  : finite set of **states**
- $q_{\text{init}} \in Q$  : **initial state**
- $\mathbf{v}_{\text{init}} \in \mathbb{N}^d$  : **initial vector**
- $\Delta \subseteq Q \times \mathbb{Z}^d \times Q$  : finite set of transition **rules**

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$$\begin{aligned}
 Q &= \{p, q\} \\
 q_{\text{init}} &= p \\
 \mathbf{v}_{\text{init}} &= (1, 0, 0) \\
 \Delta &= \{(p, (-1, 1, 0), p), \dots\}
 \end{aligned}$$



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$$\begin{array}{ccc}
 \text{VAS} & & \text{VASS} \\
 \langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle & \simeq & \langle \{q\}, q, \mathbf{v}_{\text{init}}, \{q\} \times \mathbf{A} \times \{q\} \rangle
 \end{array}$$

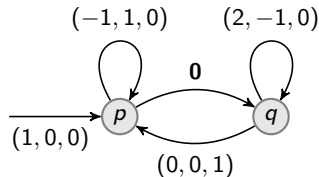
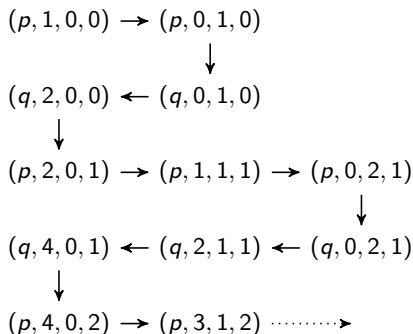
The **semantics** of a VASS  $\langle Q, q_{\text{init}}, \mathbf{v}_{\text{init}}, \Delta \rangle$  is the transition system  $\langle Q \times \mathbb{N}^d, (q_{\text{init}}, \mathbf{v}_{\text{init}}), \rightarrow \rangle$  whose transition relation  $\rightarrow$  is given by

$$\frac{(q, \mathbf{a}, q') \in \Delta \wedge \mathbf{v}' = \mathbf{v} + \mathbf{a} \geq \mathbf{0}}{(q, \mathbf{v}) \rightarrow (q', \mathbf{v}')}$$

# Vector Addition Systems with States — Semantics

The **semantics** of a VASS  $\langle Q, q_{\text{init}}, \mathbf{v}_{\text{init}}, \Delta \rangle$  is the transition system  $\langle Q \times \mathbb{N}^d, (q_{\text{init}}, \mathbf{v}_{\text{init}}), \rightarrow \rangle$  whose transition relation  $\rightarrow$  is given by

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$$\langle Q, q_{\text{init}}, \mathbf{v}_{\text{init}}, \Delta \rangle \stackrel{?}{\models} \varphi$$

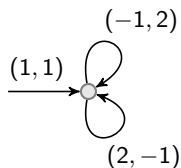
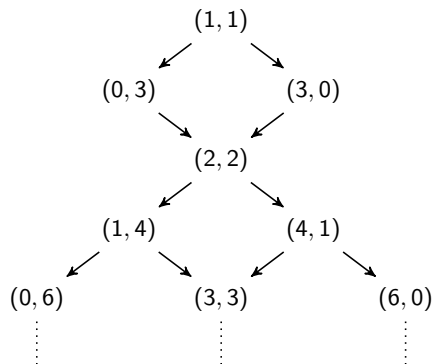
where  $\varphi$  specifies : reachability, coverability, boundedness, termination, liveness, regularity, ...

## Definition

The **reachability set** is the set of all configurations that are **reachable** from the initial configuration:

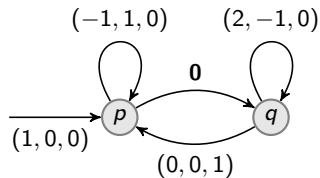
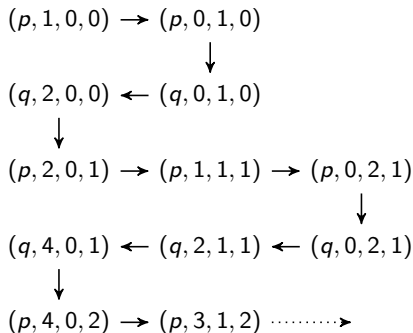
$$\left\{ (q, \mathbf{v}) \in Q \times \mathbb{N}^d \mid (q_{\text{init}}, \mathbf{v}_{\text{init}}) \xrightarrow{*} (q, \mathbf{v}) \right\}$$

# Reachability Set — VAS Example



The reachability set is

$$(1, 1) + \mathbb{N} \cdot (1, 1) + \mathbb{N} \cdot (3, 0) + \mathbb{N} \cdot (0, 3)$$



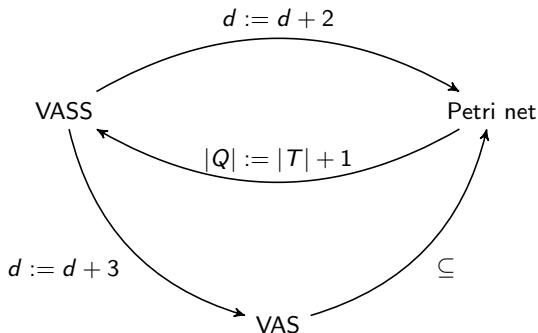
The reachability set is

$$\cup \left\{ \begin{array}{l} \{p\} \times \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + y \leq 2^z\} \\ \{q\} \times \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + 2y \leq 2^{z+1}\} \end{array} \right\}$$

VASs  $\simeq$  Petri nets  $\simeq$  VASSs

## Additional Feature of Petri nets

Test  $x \geq cst$  without modifying  $x$



**Input:** A VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  and a final vector  $\mathbf{v}_{\text{final}}$

## Reachability Problem

**Output:** Whether  $\mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v}_{\text{final}}$

- Decidable [Mayr 81, Kosaraju 82, Leroux 11]
- EXPSPACE-hard [Lipton 76]

## Coverability Problem

**Output:** Whether  $\exists \mathbf{v} : \mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v} \geq \mathbf{v}_{\text{final}}$

- Decidable [Karp&Miller 69]
- EXPSPACE-complete [Rackoff 78]

**Input:** A VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$

## Boundedness Problem

**Output:** Whether  $\{\mathbf{v} \in \mathbb{N}^d \mid \mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v}\}$  is finite

- Decidable [Karp&Miller 69]
- EXPSPACE-complete [Rackoff 78]

Other EXPSPACE-complete problems include:

- Termination: Are all runs finite?
- Regularity / Context-freeness: Is the language regular / context-free?

Some undecidable problems:

- CTL model-checking: Does  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  satisfy a given CTL formula?
- Equality: Are the reachability sets of two given VAS equal?

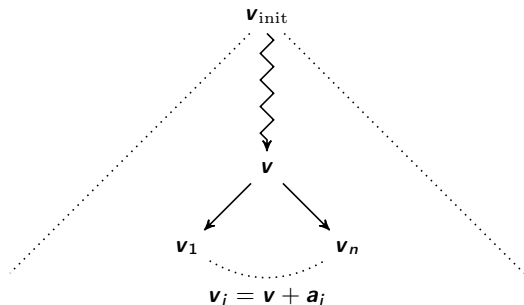
# Table of Contents

- 1 Vector Addition Systems
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# Reachability Tree

Consider a VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  with  $\mathbf{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$

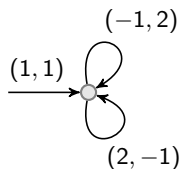
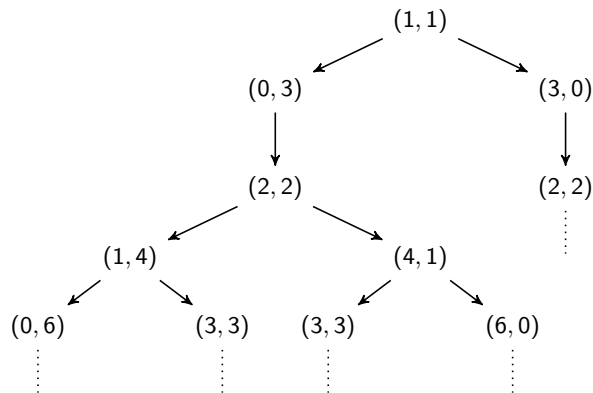
The **reachability tree** is the **potentially infinite** tree obtained by an exhaustive and enumerative forward exploration



Only keep the children  $\mathbf{v}_i$  such that  $\mathbf{v}_i = \mathbf{v} + \mathbf{a}_i \geq \mathbf{0}$



# Reachability Tree — VAS Example



# How to Truncate?

## Monotonicity Property of VAS

$$\forall \begin{array}{l} \mathbf{v} \rightarrow \mathbf{v}' \\ \mathbf{u} \rightarrow \mathbf{u}' \end{array} \exists \mathbf{v}' = \mathbf{u}' + (\mathbf{v} - \mathbf{u})$$

Consider a run

$$\mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v} \xrightarrow{*} \mathbf{v}'$$

such that

$$\mathbf{v} < \mathbf{v}'$$

Then

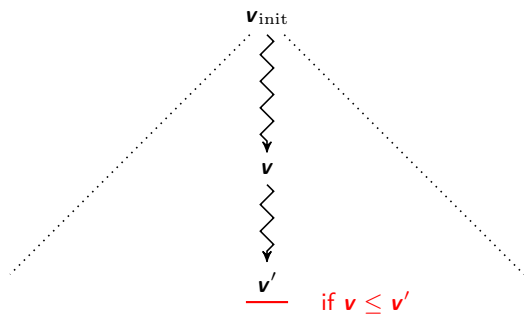
$$\mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v} \xrightarrow{*} \mathbf{v}' \xrightarrow{*} \mathbf{v}'' \xrightarrow{*} \mathbf{v}''' \dots$$

with

$$\mathbf{v} < \mathbf{v}' < \mathbf{v}'' < \mathbf{v}''' < \dots$$

# Reduced Reachability Tree [Karp&Miller 69, Finkel 90]

The **reduced reachability tree** is the **prefix** of the reachability tree obtained as follows



# Finiteness of the Reduced Reachability Tree

## Theorem

*The reduced reachability tree of a VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  is finite.*

**Proof.** By contradiction, assume that it is infinite.

The tree is finitely branching. So, by König's Lemma, there is an infinite branch

$$\mathbf{v}_{\text{init}} \rightarrow \mathbf{v}_1 \rightarrow \mathbf{v}_2 \cdots$$

## Dickson's Lemma

Every infinite sequence  $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$  of vectors in  $\mathbb{N}^d$  contains an increasing pair  $\mathbf{v}_i \leq \mathbf{v}_j$  with  $i < j$ .

By Dickson's Lemma, there exists  $i < j$  such that  $\mathbf{v}_i \leq \mathbf{v}_j$ .

Hence, by truncation,  $\mathbf{v}_j$  has no child. □

## Theorem

A VAS  $\langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  is unbounded if, and only if, its reduced reachability tree contains a leaf that is strictly larger than one of its ancestors.

## Proof.

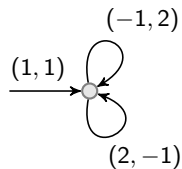
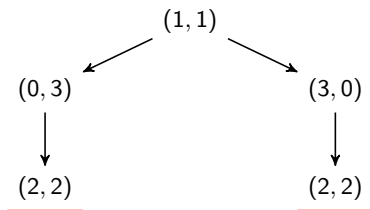
$\Leftarrow$  If  $\mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v} \xrightarrow{*} \mathbf{v}' > \mathbf{v}$  then, by monotonicity, there are infinitely many reachable configurations.

$\Rightarrow$  If the VAS is unbounded, then its reachability tree is infinite.

By König's Lemma and then Dickson's Lemma, there exists a node that is strictly larger than one of its ancestors.

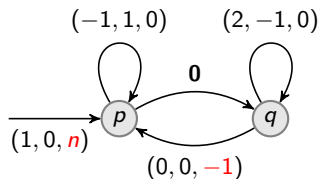
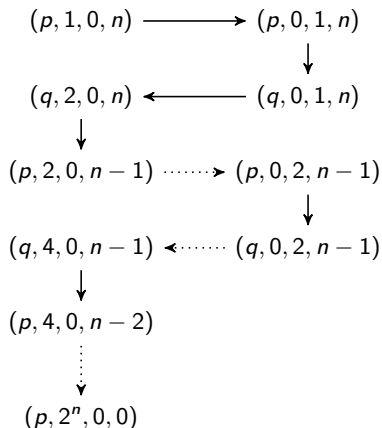
Pick the first one. It is a leaf in the reduced reachability tree.  $\square$

# Reduced Reachability Tree — VAS Example



This VAS is unbounded

# Reduced Reachability Tree — VASS Example (Variant)



This VAS is bounded

⇒ But the depth of the reduced reachability tree is at least  $2^n$

# How Big Can the Reduced Reachability Tree be?

## Theorem ([McAloon 84])

*The size of the reduced reachability tree of a VAS  $\mathcal{A} = \langle \mathbf{v}_{\text{init}}, \mathbf{A} \rangle$  is at most*

- *primitive-recursive in  $|\mathcal{A}|$  when the dimension  $d$  is fixed,*
- *Ackermannian in  $|\mathcal{A}|$  when the dimension  $d$  is part of the input.*



# How Big Can the Reduced Reachability Tree be?

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But the boundedness problem is  $\text{EXPSPACE}$ -complete [Rackoff 78]

If there is a witness

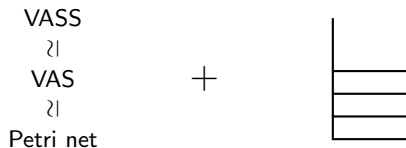
$$\mathbf{v}_{\text{init}} \xrightarrow{*} \mathbf{v} \xrightarrow{*} \mathbf{v}' > \mathbf{v}$$

then there is one of length at most doubly-exponential in  $|\mathcal{A}|$ .

# Table of Contents

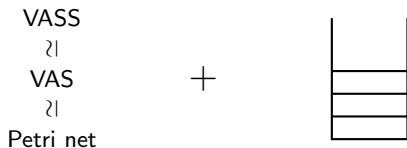
- 1 Vector Addition Systems
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# Pushdown Vector Addition Systems in a Nutshell

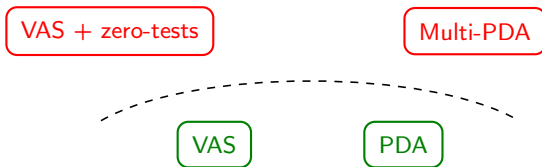


- ⇒ Richer model for the verification of concurrent systems
- Multi-threaded recursive programs
  - One recursive server + unboundedly many finite-state clients

# Pushdown Vector Addition Systems in a Nutshell



- ⇒ Richer model for the verification of concurrent systems
  - Multi-threaded recursive programs
  - One recursive server + unboundedly many finite-state clients
- ⇒ Study the decidability frontier



## Definition

A **pushdown VAS** is a triple  $\langle \mathbf{v}_{\text{init}}, \Gamma, \Delta \rangle$  where

- $\mathbf{v}_{\text{init}} \in \mathbb{N}^d$  : **initial vector**
- $\Gamma$  : finite **stack alphabet**
- $\Delta \subseteq (\mathbb{Z}^d \times \text{Op})$  : finite set of **actions**, with

$$\text{Op} = \{\varepsilon\} \cup \{\text{push}(\gamma), \text{pop}(\gamma) \mid \gamma \in \Gamma\}$$

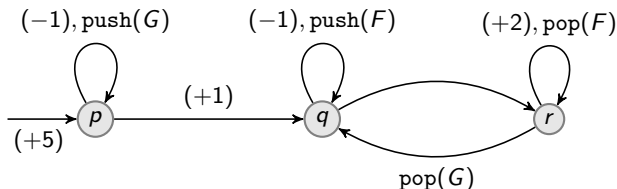
# Pushdown VAS(S) — Syntax

## Definition

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# Pushdown VAS(S) — Semantics (1)

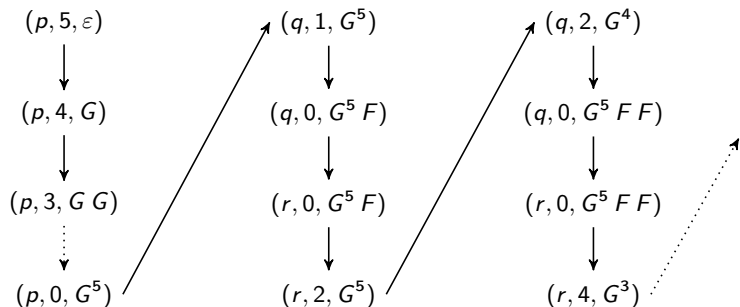
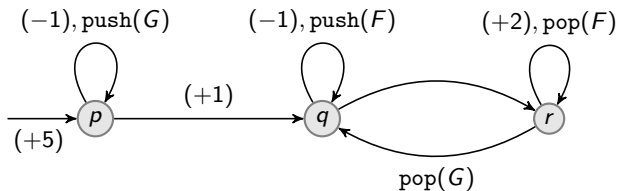
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$$\frac{(\mathbf{a}, \varepsilon) \in \Delta \wedge \mathbf{v}' = \mathbf{v} + \mathbf{a} \geq \mathbf{0}}{(\mathbf{v}, \sigma) \rightarrow (\mathbf{v}', \sigma)}$$

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$$\frac{(\mathbf{a}, \text{pop}(\gamma)) \in \Delta \wedge \mathbf{v}' = \mathbf{v} + \mathbf{a} \geq \mathbf{0}}{(\mathbf{v}, \sigma \cdot \gamma) \rightarrow (\mathbf{v}', \sigma)}$$

# Pushdown VAS(S) — Semantics (2)





# State of the Art

Reachability for pushdown VAS is TOWER-hard [Lazic 12]

- Decidability is open

Reachability is decidable for sub-classes of pushdown VAS

- VAS with one zero-tested counter [Reinhardt 08]
- VAS  $\cap$  CFL of finite index [Atig&Ganty 11]

Boundedness and place-boundedness are decidable for VAS with one zero-tested counter [Bonnet&Finkel&Leroux&Zeitoun 12]

⇒ We show that boundedness is decidable for pushdown VAS

- Reachability tree: similar to the VAS case
- Truncation technique adapted to pushdown VAS

# Truncation of the Reachability Tree — Failed Attempt

Consider a run

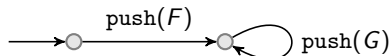
$$(\mathbf{v}_{\text{init}}, \varepsilon) \xrightarrow{*} (\mathbf{v}, \sigma) \xrightarrow{*} (\mathbf{v}', \sigma')$$

such that

$$\mathbf{v} \leq \mathbf{v}' \quad \text{and} \quad \sigma \leq_{\text{suffix}} \sigma'$$

Then  $(\mathbf{v}_{\text{init}}, \varepsilon) \xrightarrow{*} (\mathbf{v}, \sigma) \xrightarrow{*} (\mathbf{v}', \sigma') \xrightarrow{*} (\mathbf{v}'', \sigma'') \xrightarrow{*} (\mathbf{v}''', \sigma''') \dots$

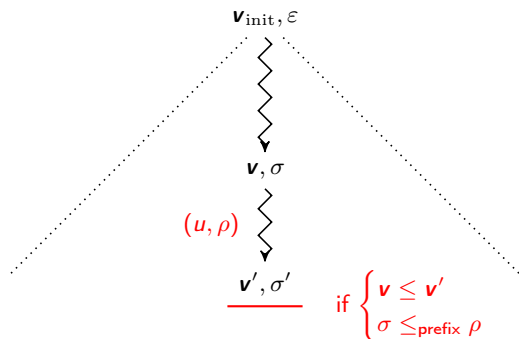
But:



No truncation, infinite branch!

# Reduced Reachability Tree for Pushdown VAS

The **reduced reachability tree** is the **prefix** of the reachability tree obtained as follows



# Finiteness of the Reduced Reachability Tree

## Theorem

*The reduced reachability tree of a pushdown VAS  $\langle \mathbf{v}_{\text{init}}, \Gamma, \Delta \rangle$  is finite.*

**Proof.** By contradiction, assume that it is infinite.

The tree is finitely branching. So, by König's Lemma, there is an infinite branch

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$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \cdots & & \\ & \mathbf{v} & & \mathbf{v}' \geq \mathbf{v} & & & \end{array}$$



# Finiteness of the Reduced Reachability Tree

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**Proof.** By contradiction, assume that it is infinite.

The tree is finitely branching. So, by König's Lemma, there is an infinite branch

$$\mathbf{v}_{\text{init}} \rightarrow \mathbf{v}_1 \rightarrow \mathbf{v}_2 \cdots$$



# Finiteness of the Reduced Reachability Tree

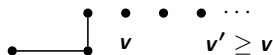
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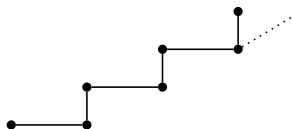
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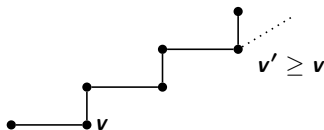
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## Theorem

A pushdown VAS  $\langle \mathbf{v}_{\text{init}}, \Gamma, \Delta \rangle$  is unbounded if, and only if, its reduced reachability tree contains a path

$$\underbrace{(\mathbf{v}, \sigma) \rightsquigarrow (\mathbf{v}', \sigma')}_{\sigma \text{ remains a prefix}}$$

such that  $\mathbf{v} \leq \mathbf{v}'$  and  $\sigma \leq_{\text{prefix}} \sigma'$ , and at least one of these inequalities is strict.

# How Big Can the Reduced Reachability Tree be?

Very very big. . .

# Fast Growing Functions

Functions  $F_\alpha : \mathbb{N} \rightarrow \mathbb{N}$ , indexed by ordinals  $\alpha \leq \omega^\omega$

$$\begin{aligned}F_0(n) &= n + 1, \\F_{\alpha+1}(n) &= F_\alpha^{n+1}(n) \\F_\lambda(n) &= F_{\lambda_n}(n) \quad \text{if } \lambda < \omega^\omega \text{ is a limit ordinal}\end{aligned}$$

$F_1$  : linear,  $F_2$  exponential,  $F_3$  tower of exponentials

$F_\omega$  is an Ackermannian function

$F_{\omega^\omega}$  is an **hyper-Ackermannian** function

## Example

$$\begin{aligned}F_{\omega^2+\omega+1}(2) &= F_{\omega^2+\omega}(F_{\omega^2+\omega}(F_{\omega^2+\omega}(2))) \\&= F_{\omega^2+\omega}(F_{\omega^2+\omega}(F_{\omega^2+3}(2))) \\&= F_{\omega^2+\omega}(F_{\omega^2+\omega}(F_{\omega^2+2}(F_{\omega^2+2}(F_{\omega^2+2}(2))))))\end{aligned}$$

## Theorem

*The reduced reachability tree of a pushdown VASS  $\mathcal{A}$  has at most  $F_{\omega\omega}(|\mathcal{A}|)$  nodes*

## Theorem

*For all  $n \in \mathbb{N}$ , there exists a pushdown VASS  $\mathcal{A}_n$ , of size quadratic in  $n$ , such that the reduced reachability tree of  $\mathcal{A}_n$  has at least  $F_{\omega\omega}(n)$  nodes*

# Table of Contents

- 1 Vector Addition Systems
- 2 The Reduced Reachability Tree for VAS
- 3 Extension to Pushdown Vector Addition Systems
- 4 Conclusion

- ⇒ Extension of the reduced reachability tree from VAS to Pushdown VAS
- ⇒ Boundedness and termination are decidable for pushdown VAS
- ⇒ Hyper-Ackermannian ( $F_{\omega^{\omega}}$ ) worst-case running time
  - The reduced reachability tree of a pushdown VASS  $\mathcal{A}$  has at most  $F_{\omega^{\omega}}(|\mathcal{A}|)$  nodes
  - This bound is tight



- ⇒ Complexity of the boundedness problem for pushdown VAS
  - Upper bound: Hyper-Ackermann
  - Lower bound: Tower of exponentials
- ⇒ Reachability for Pushdown VAS
  - Open even for 1-dim VASS + stack
- ⇒ Coverability for Pushdown VAS
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  - Coverability for Pushdown VAS is harder than reachability for VAS