Reachability for Two-Counter Machines with One Test and One Reset

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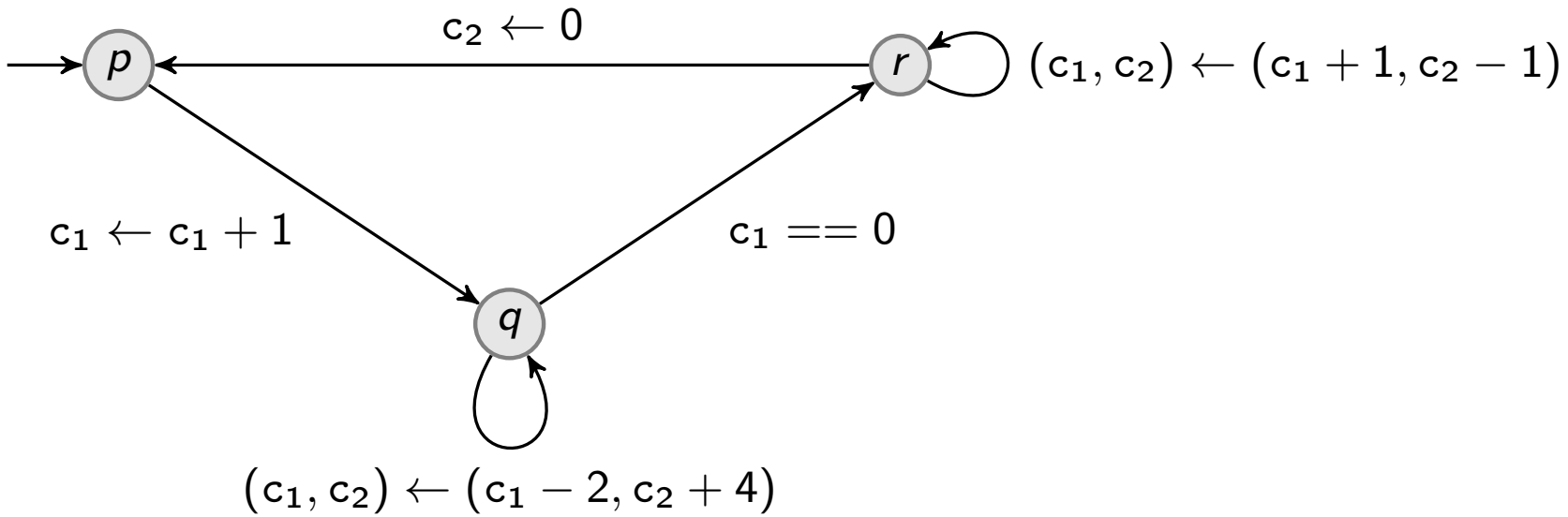
A Two-Counter Machine with One Test and One Reset



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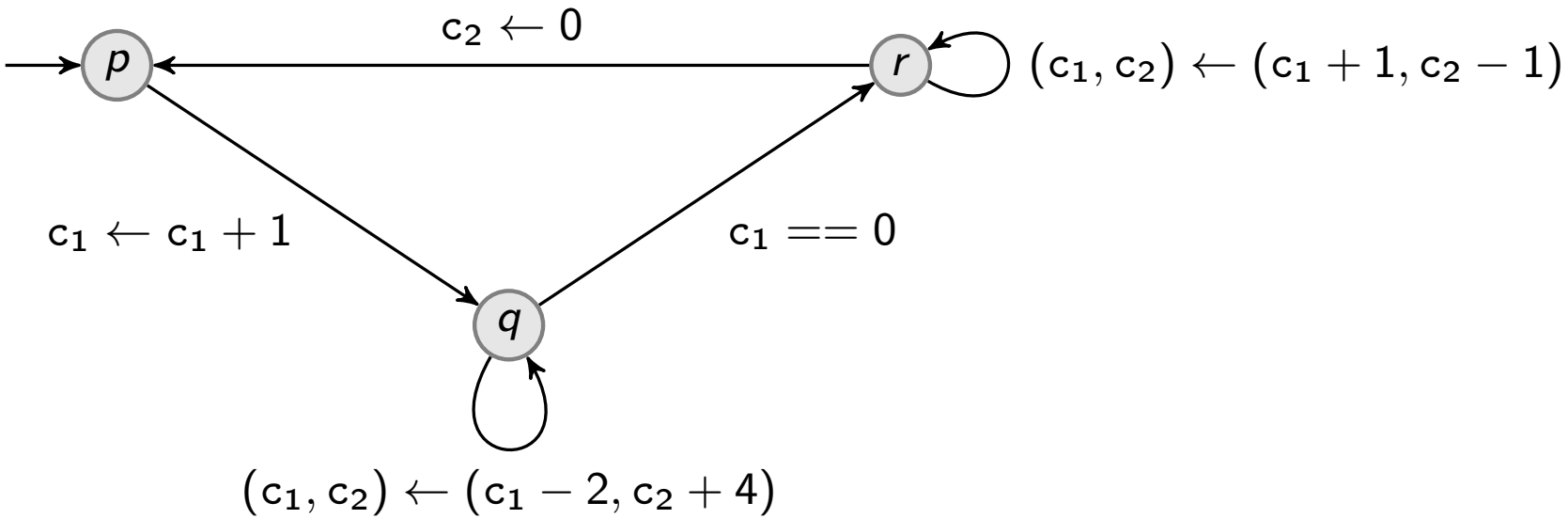
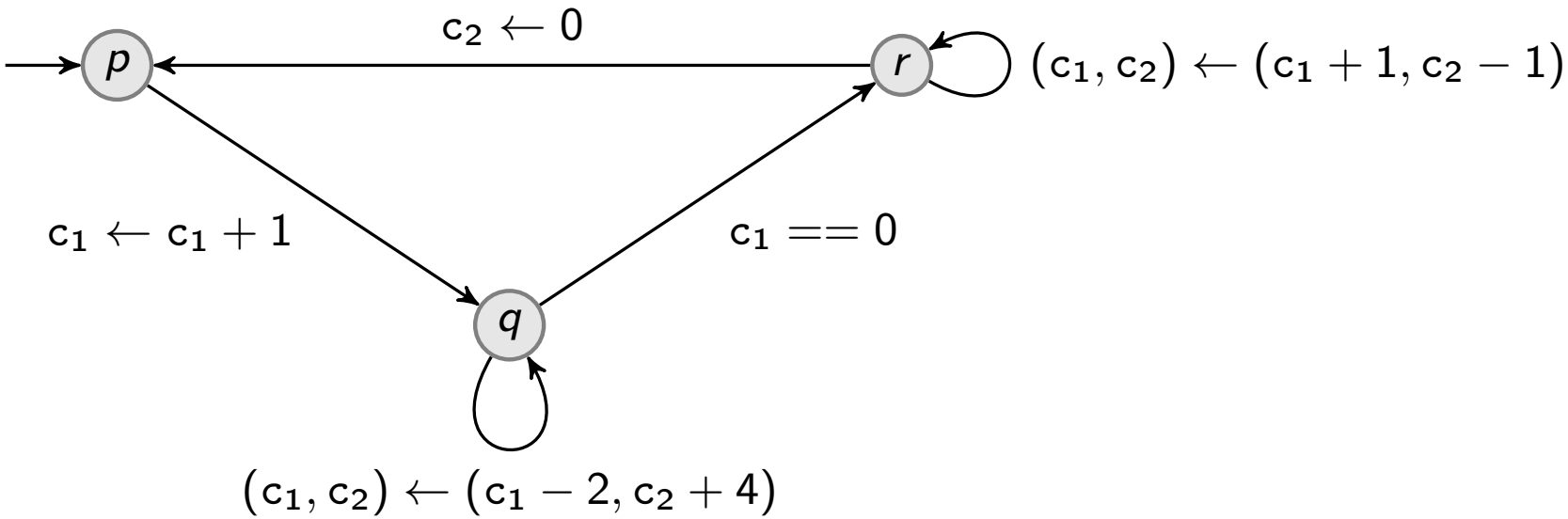


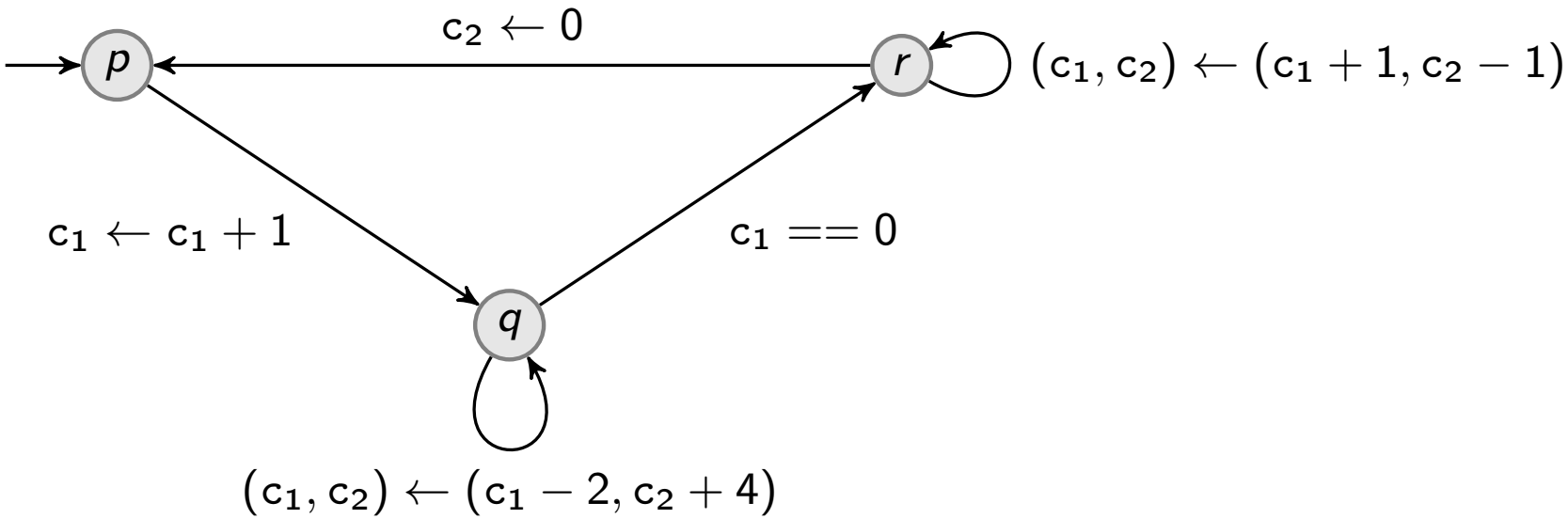
$$p(1,0) \rightarrow q(2,0) \rightarrow q(0,4) \not\rightarrow q$$

A Two-Counter Machine with One Test and One Reset



$$p(1,0) \to q(2,0) \to q(0,4) \not\to q$$
$$\to r(0,4) \to\!\to r(2,2) \to p(2,0)$$

A Two-Counter Machine with One Test and One Reset



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$$\to r(0,4) \to\!\to r(2,2) \to p(2,0)$$

$$p(2,0) \to q(3,0) \to q(1,4) \not\to q$$
$$\not\to r$$

VASS Extended with Tests, Resets and Transfers

Definition

A d-dim VASS is a triple $\langle Q, \Sigma, \Delta \rangle$ where

- Q : finite set of states
- Σ : finite set of actions, with

$$\Sigma \subseteq \mathbb{Z}^d$$

- $\Delta \subseteq Q \times \Sigma \times Q$: finite set of transitions

Configurations: $q(\boldsymbol{x})$ with $q \in Q$ and $\boldsymbol{x} \in \mathbb{N}^d$

$(a_1, a_2) \in \mathbb{Z}^2$ stands for $(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$

VASS Extended with Tests, Resets and Transfers

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A d-dim Extended VASS is a triple $\langle Q, \Sigma, \Delta \rangle$ where

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- Σ : finite set of actions, with

$$\Sigma \quad \subseteq \quad \mathbb{Z}^d \ \cup \ \{T(i), R(i), Tr(i,j) \mid i,j \in \{1,\ldots,d\}, i \neq j\}$$

- $\Delta \subseteq Q \times \Sigma \times Q$: finite set of transitions

Configurations: $q(\boldsymbol{x})$ with $q \in Q$ and $\boldsymbol{x} \in \mathbb{N}^d$

$(a_1, a_2) \in \mathbb{Z}^2$	stands for	$(c_1, c_2) \leftarrow (c_1 + a_1, c_2 + a_2)$
$T(1)$		$c_1 == 0$
$R(2)$		$c_2 \leftarrow 0$
$Tr(1,2)$		$c_2 \leftarrow c_2 + c_1 \ ; \ c_1 \leftarrow 0$

Verification of Extended VASS

Reachability

Instance: an EVASS and two configurations $p(\boldsymbol{x})$, $q(\boldsymbol{y})$.

Question: is $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$?

Coverability

Instance: an EVASS and two configurations $p(\boldsymbol{x})$, $q(\boldsymbol{y})$.

Question: is $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{z})$ for some $\boldsymbol{z} \geq \boldsymbol{y}$?

Boundedness

Instance: an EVASS and a configuration $p(\boldsymbol{x})$.

Question: is $\{q(\boldsymbol{y}) \mid p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})\}$ finite ?

Verification of Extended VASS — Some Known Results

	Boundedness	Coverability	Reachability
d-dim VASS	ExpSpace-c		Dec
$T(1)$	Dec		
$T(\star)$	Und ($d \geq 2$)		
$R(1)$	Dec^1		
$R(1)$, $R(2)$	Dec	Dec^2	Und ($d \geq 3$)
$R(\star)$	Und ($d \geq 3$)	Dec^2	Und ($d \geq 3$)
$Tr(1)$	Dec^1		
$Tr(\star)$	Dec^2		Und ($d \geq 3$)

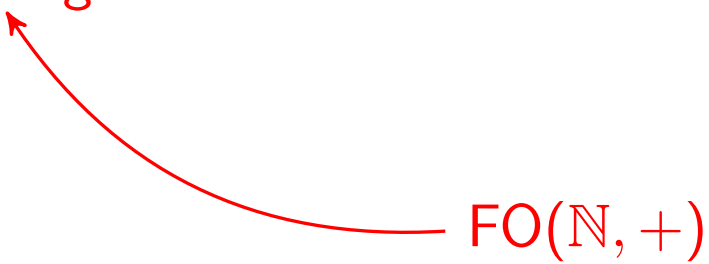
[1] Resets $R(i)$ and transfers $Tr(i,j)$ can be simulated by tests $T(i)$

[2] By general results on well-structured transition systems

Verification of 2-dim Extended VASS

Reachability, Coverability, Boundedness, ...

Is $post^*$ / pre^* / $\xrightarrow{*}$ Presburger-definable? If so can we compute it?



Verification of 2-dim Extended VASS

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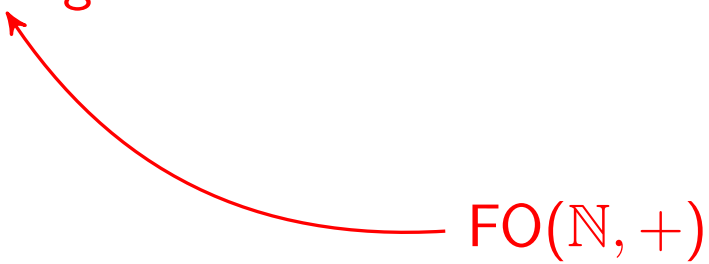


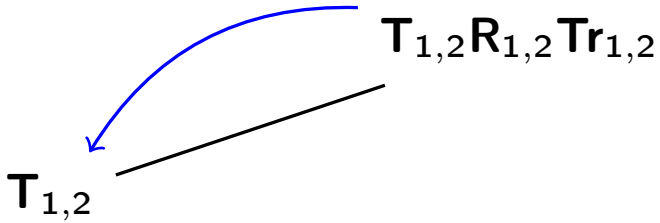
Classes $\mathbf{T}_I\mathbf{R}_J\mathbf{Tr}_K$ with $I, J, K \subseteq \{1, 2\}$

I : counters that can be tested for zero

J : counters that can be reset

K : counters that can be transfered from

Verification of 2-dim Extended VASS — State of Art



Reachability undecidable
[Minsky 1967]

2-dim VASS

$post^*/pre^*$ eff. Presb.
[Hopcroft, Pansiot 1979]

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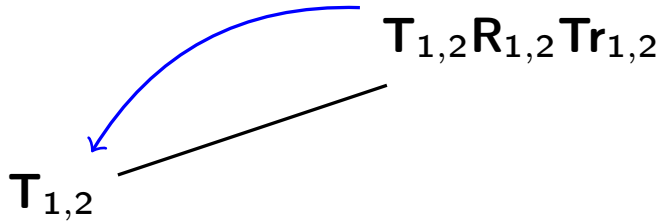

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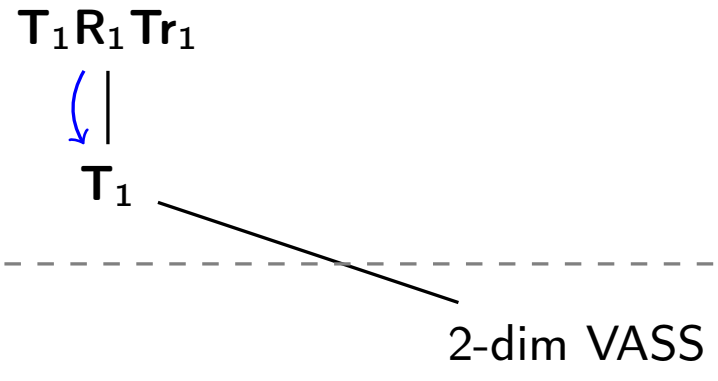

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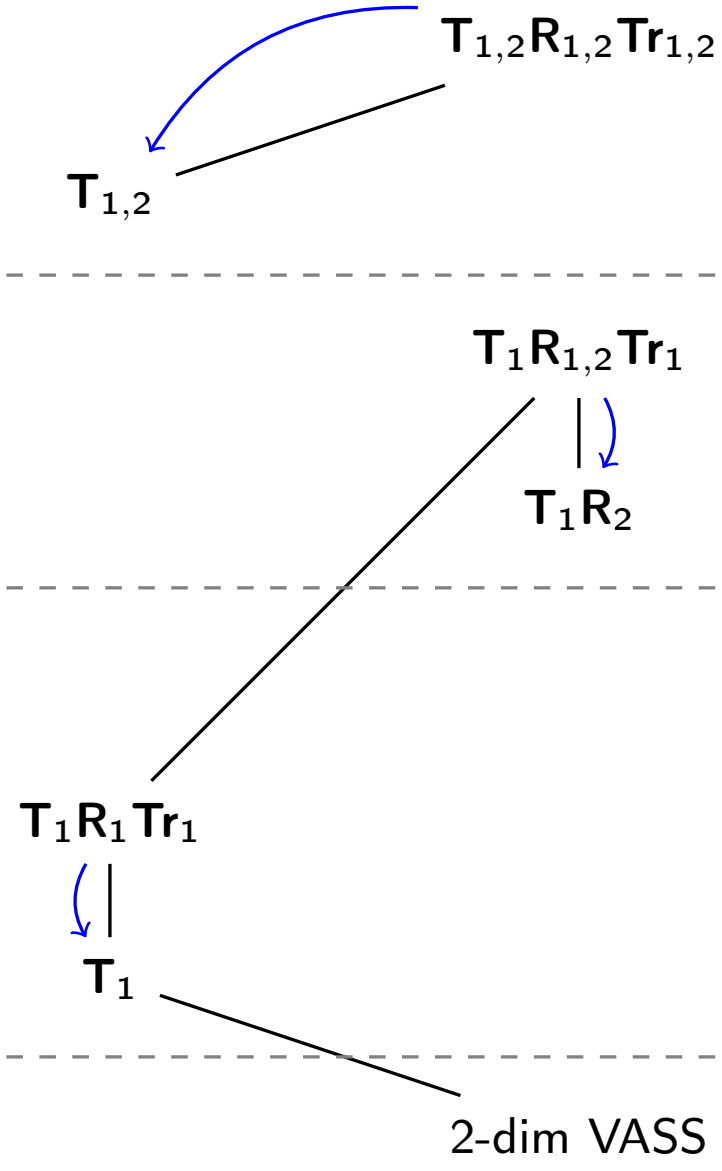

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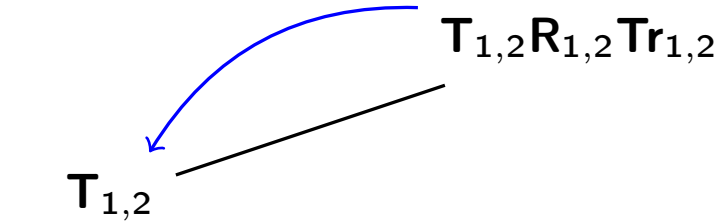

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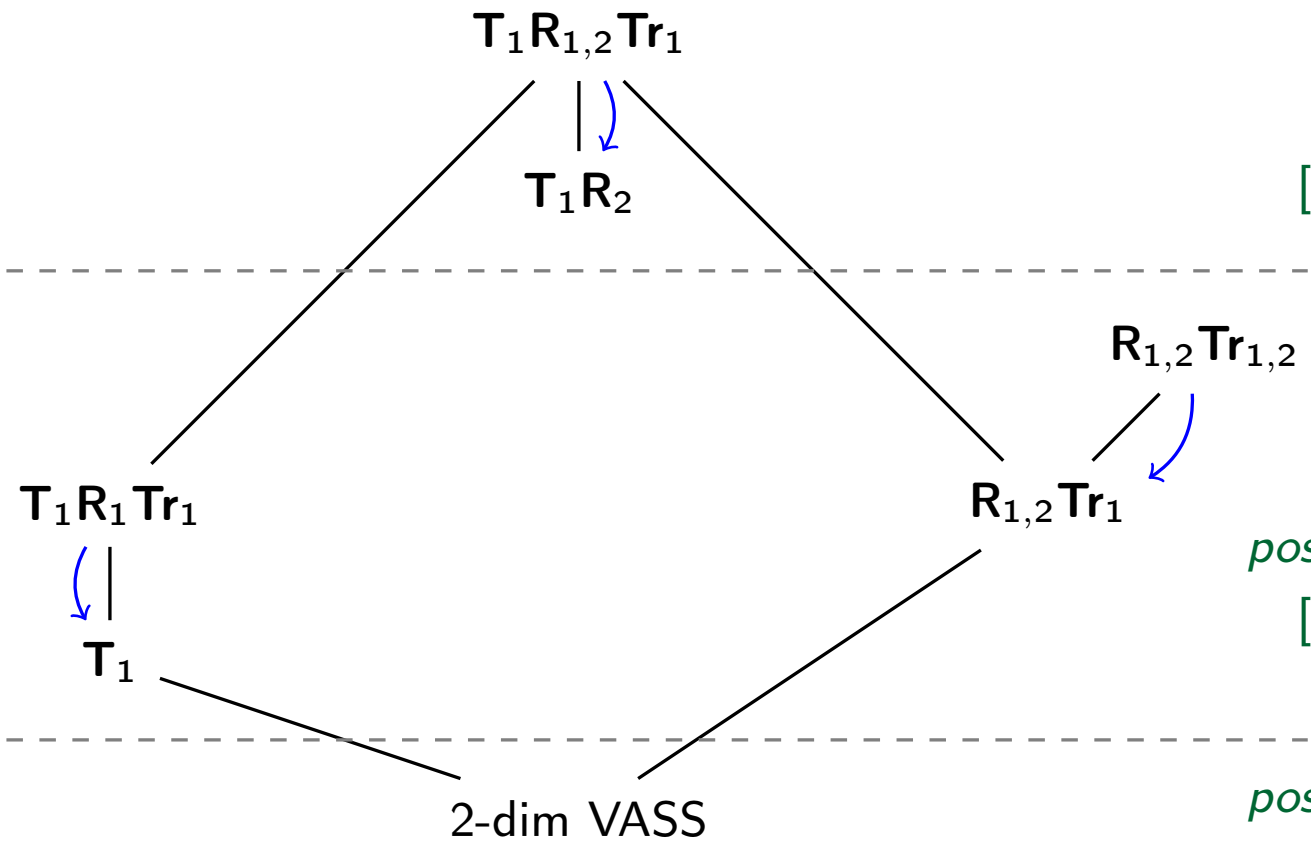

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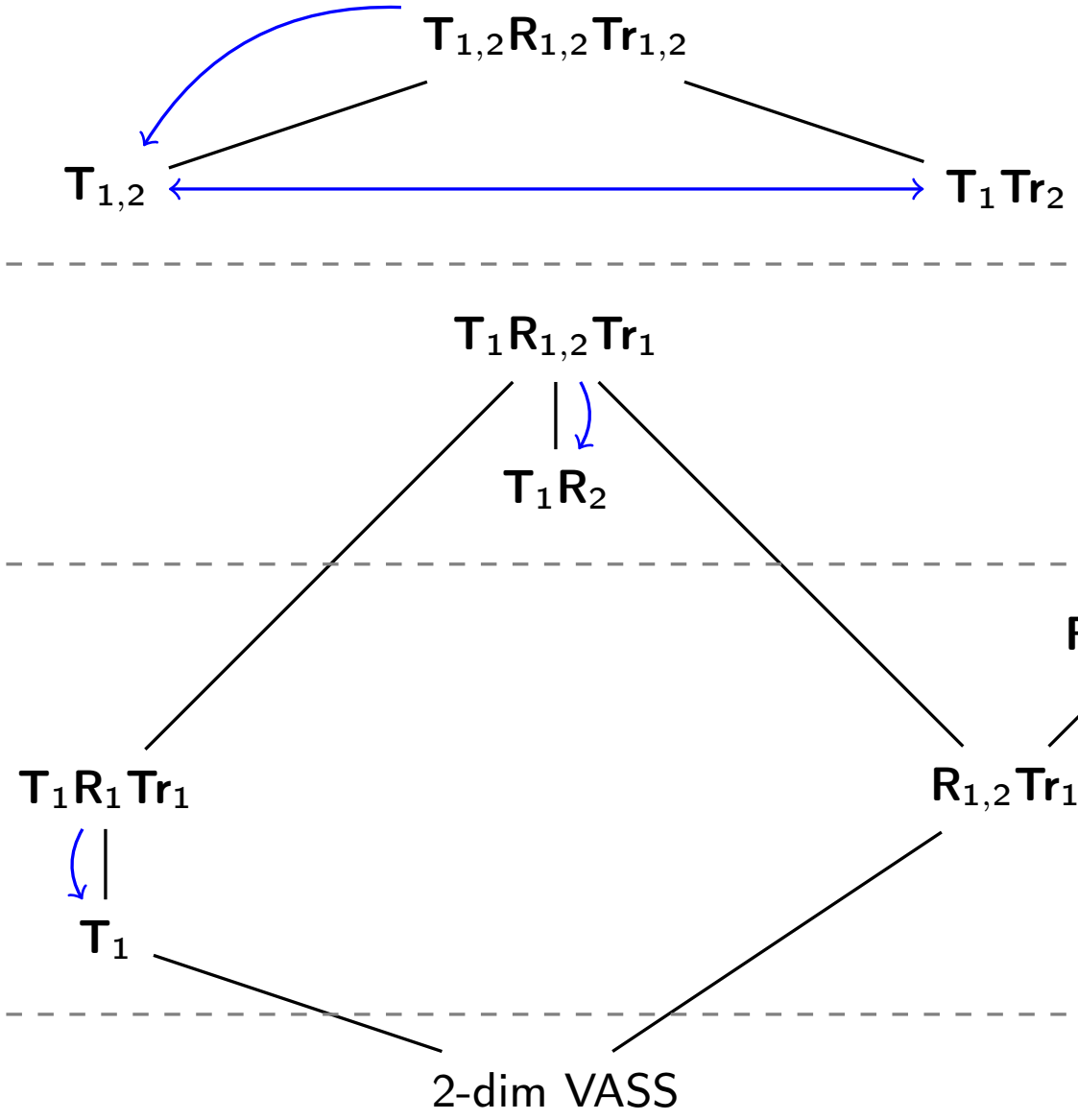

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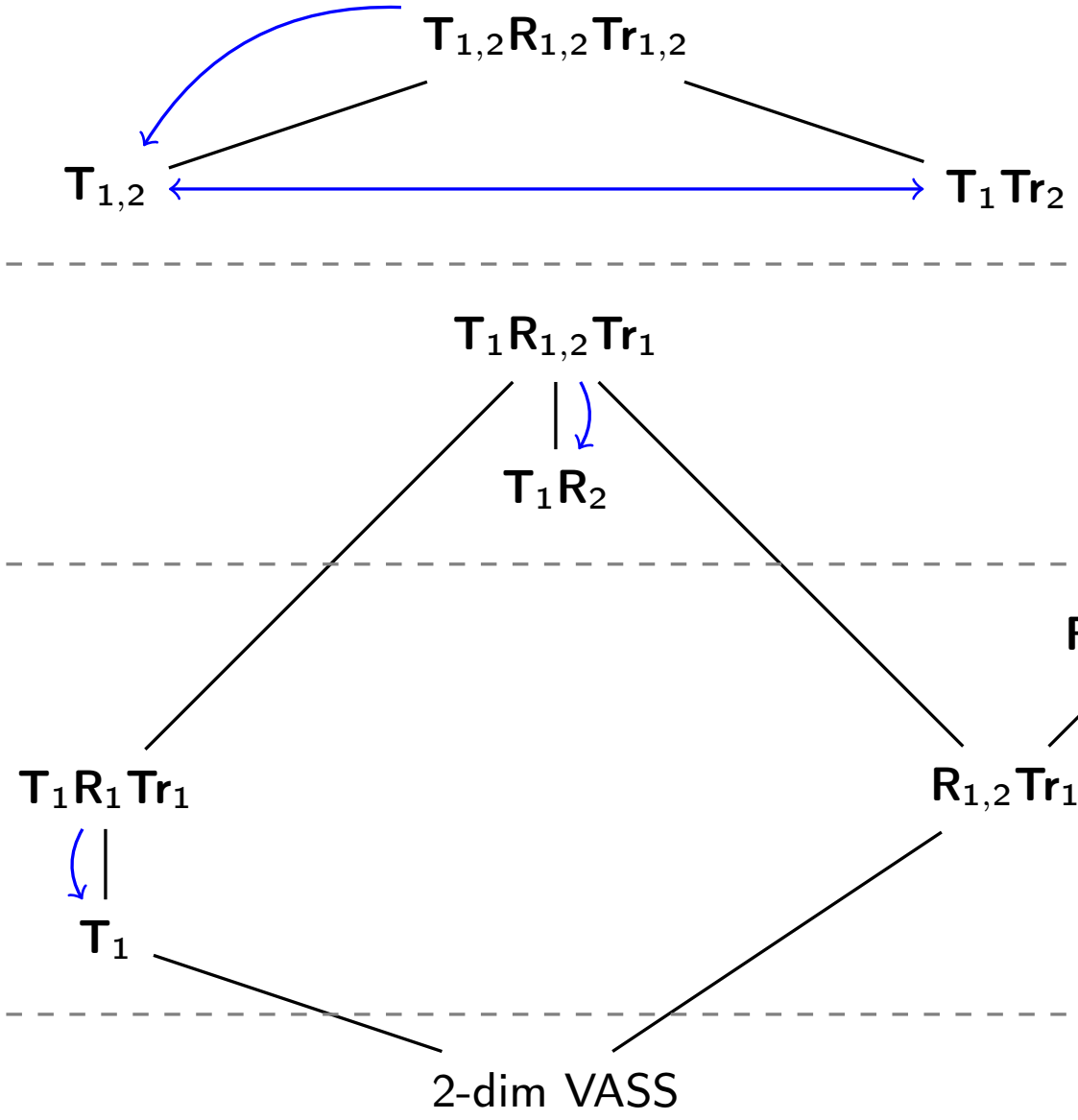

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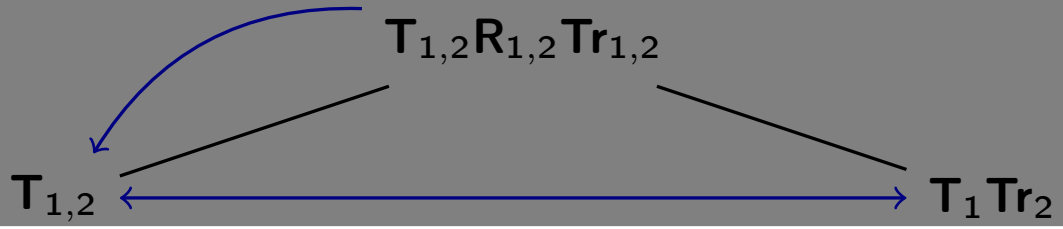

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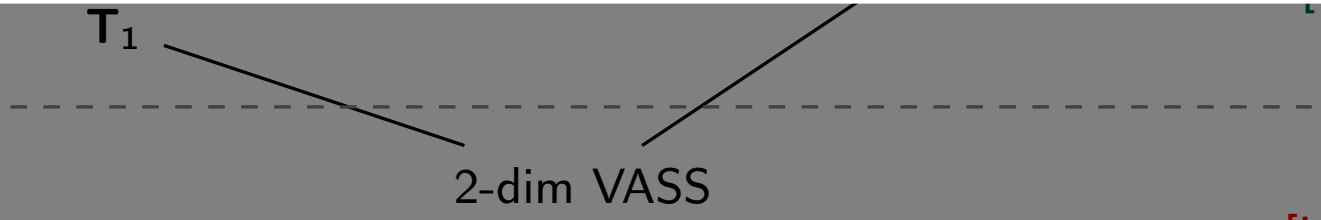
- 2-dim VASS reachability problem:
  - PSPACE-complete with integers in binary [Blondin, Finkel, Göller, Haase, McKenzie 2015]
  - NL-complete with integers in unary [Englert, Lazic, Totzke 2016]
- *d*-dim Hierarchically-Extended VASS:
  - decidability status of reachability, coverability, termination and deadlock-freedom [Akshay, Chakraborty, Das, Jagannath, Sandeep 2017]



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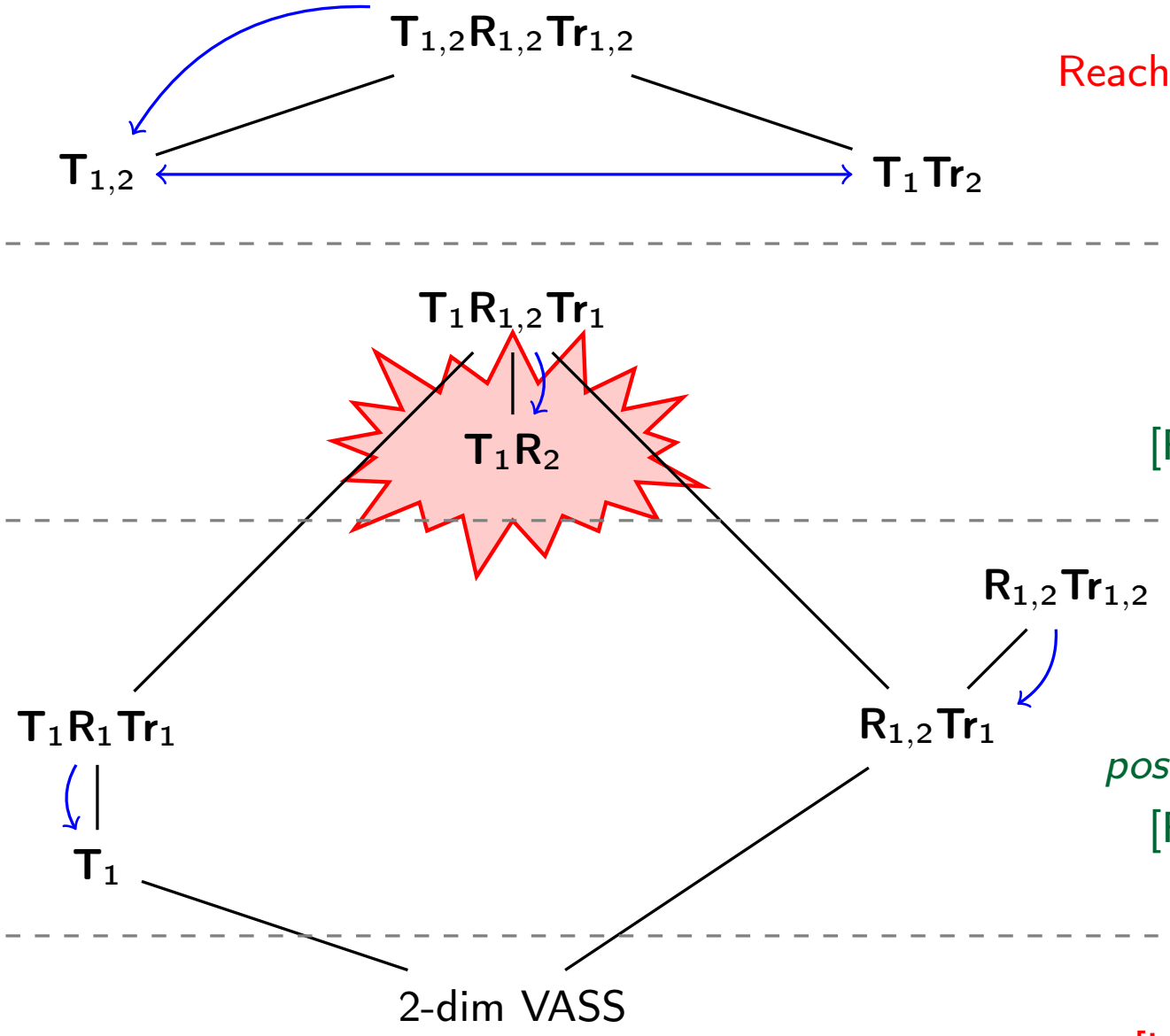

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Main Contribution of the Paper

$$\text{TRVASS} \;=\; \mathbf{T}_1\mathbf{R}_2 \;=\; \text{2-dim VASS extended with } T(1) \text{ and } R(2)$$

Theorem

The reachability relation of a TRVASS is effectively Presburger-definable.

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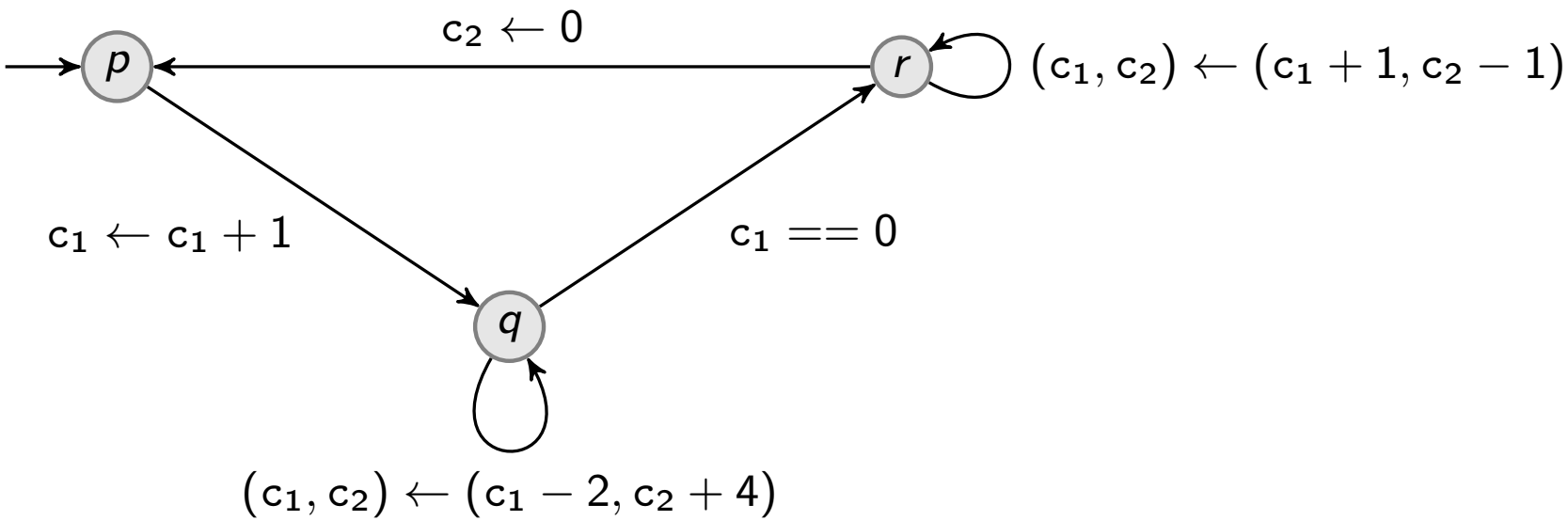

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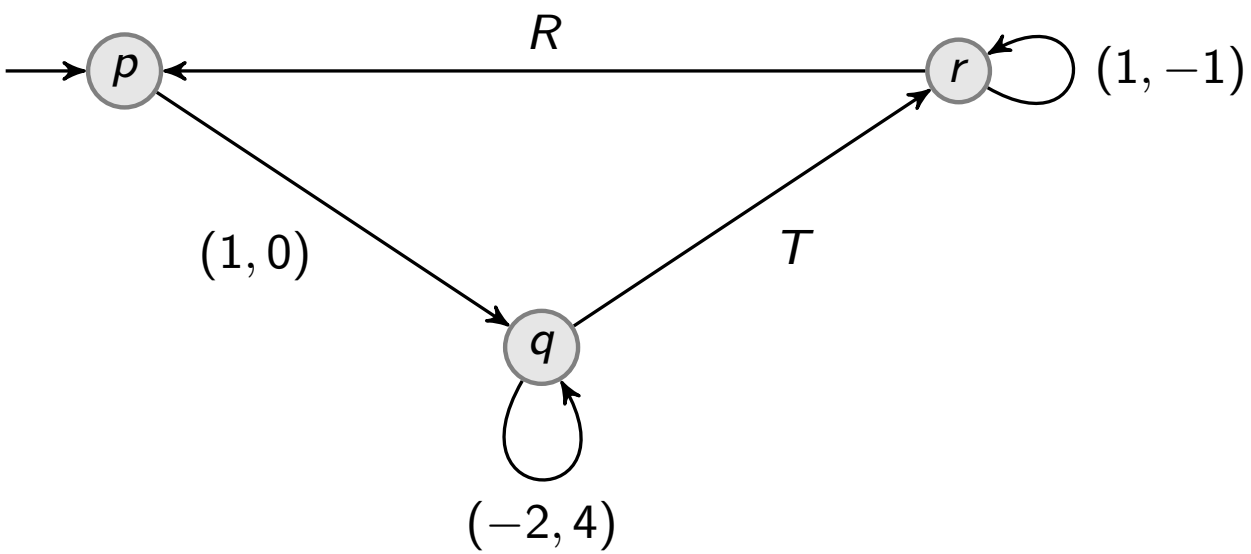
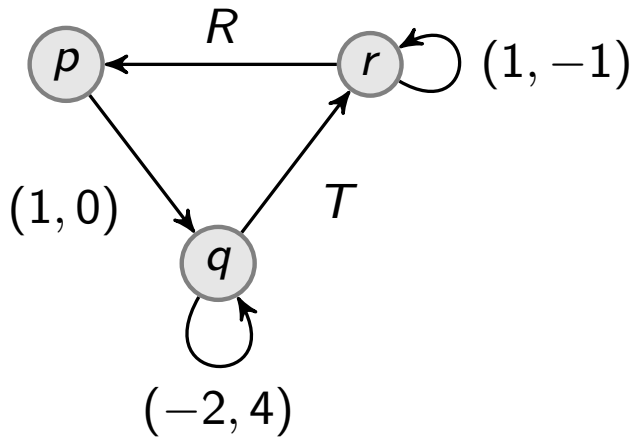


A Crucial Observation on Runs of TRVASS



$$\Sigma = \boldsymbol{A} \cup \{T, R\}$$
$$\boldsymbol{A} = \{(1,0), (-2,4), (1,-1)\}$$

Decomposition of runs from p in the example:

$$p(x_{\text{init}}, y_{\text{init}}) \xrightarrow{\boldsymbol{A}^*} \xrightarrow{T} r(0, y_1) \xrightarrow{\boldsymbol{A}^*} \xrightarrow{R} p(x_2, 0) \xrightarrow{\boldsymbol{A}^*} \xrightarrow{T} r(0, y_3) \cdots$$

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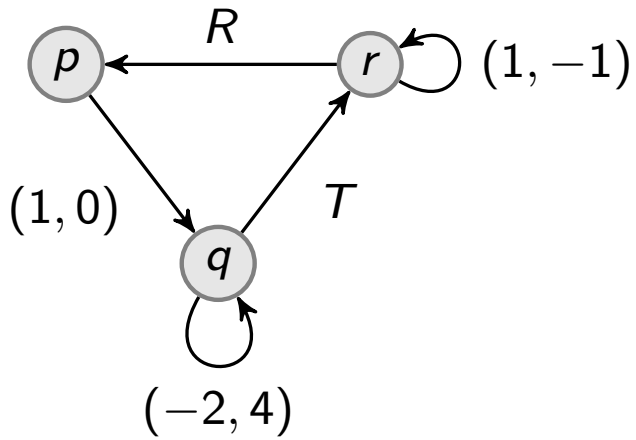

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Decomposition of runs in general:

$$c_{\text{init}} \xrightarrow{\boldsymbol{A}^*} \xrightarrow{T|R} c_1 \xrightarrow{\boldsymbol{A}^*} \xrightarrow{T|R} c_2 \cdots \xrightarrow{\boldsymbol{A}^*} \xrightarrow{T|R} c_k \xrightarrow{\boldsymbol{A}^*} c_{\text{final}}$$

where at least one counter is zero in c_i

Presburger (One-Counter) Automata

Definition

A Presburger automaton is a pair $\langle Q, \Delta \rangle$ where

- Q : finite set of states
- $\Delta \subseteq Q \times 2^{\mathbb{N}\times\mathbb{N}}_{\text{Presb}} \times Q$: finite set of transitions

Presburger-definable binary relations $S \subseteq \mathbb{N} \times \mathbb{N}$

Configurations: $q(x)$ with $q \in Q$ and $x \in \mathbb{N}$

Steps:

$$p(x) \to q(y) \quad \text{iff} \quad \exists\, (p \xrightarrow{S} q) \in \Delta : (x, y) \in S$$

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➪ Reachability is undecidable for Presburger automata in general

From TRVASS to Presburger Automata

Presburger automaton $\mathcal{P}$ associated to a TRVASS $\mathcal{V} = \langle Q, \Sigma, \Delta \rangle$

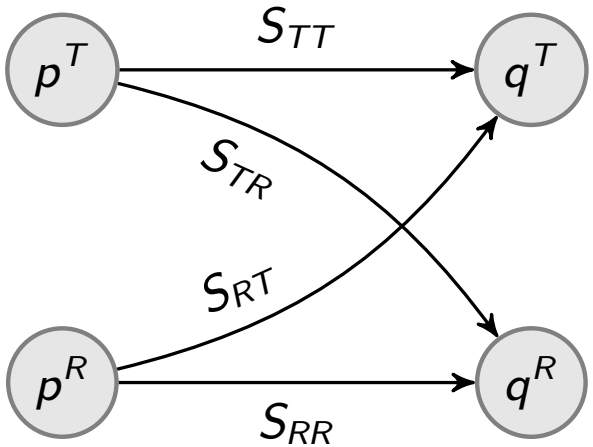
➪ Summarizes runs of the TRVASS between two extended actions

States of $\mathcal{P}$: $\{q^T, q^R \mid q \in Q\}$

$q^T(n) \approx q(0,n)$

$q^R(n) \approx q(n,0)$

Transitions of $\mathcal{P}$:



$$S_{TT}(m,n) \overset{\text{def}}{\Leftrightarrow} p(0,m) \xrightarrow{\boldsymbol{A}^* T} q(0,n)$$
$$S_{RT}(m,n) \overset{\text{def}}{\Leftrightarrow} p(m,0) \xrightarrow{\boldsymbol{A}^* T} q(0,n)$$
$$S_{TR}(m,n) \overset{\text{def}}{\Leftrightarrow} p(0,m) \xrightarrow{\boldsymbol{A}^* R} q(n,0)$$
$$S_{RR}(m,n) \overset{\text{def}}{\Leftrightarrow} p(m,0) \xrightarrow{\boldsymbol{A}^* R} q(n,0)$$

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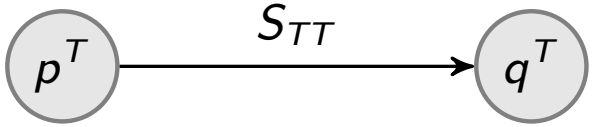

$$S_{TT}(m,n) \overset{\text{def}}{\Leftrightarrow} p(0,m) \xrightarrow{\boldsymbol{A}^* T} q(0,n)$$

Theorem ([Leroux, Sutre 2004])

The reachability relation $\xrightarrow{\boldsymbol{A}^*}$ is effectively Presburger-definable.

From TRVASS to Presburger Automata — Continued

Presburger automaton $\mathcal{P}$ associated to a TRVASS $\mathcal{V} = \langle Q, \Sigma, \Delta \rangle$

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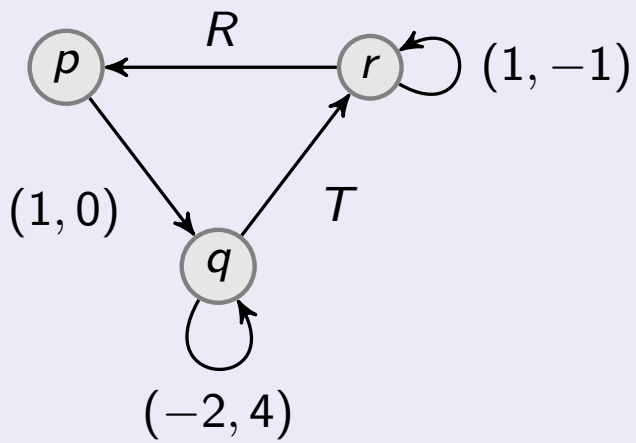
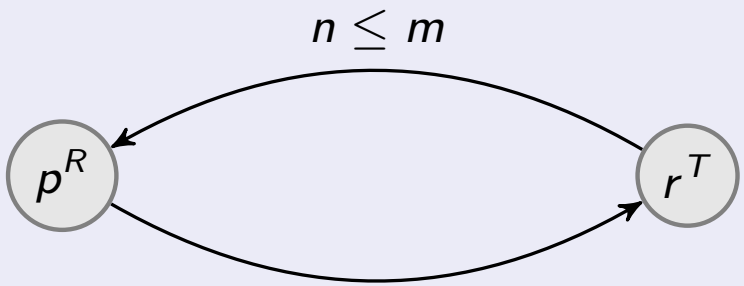
$\mathcal{V}$



$\mathcal{P}$ limited to p^R and r^T



$m \in (1 + 2\mathbb{N}) \wedge n = 2(m + 1)$

If $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable then so is $\xrightarrow{*}_{\mathcal{V}}$

From TRVASS to Presburger Automata — Continued

Presburger automaton $\mathcal{P}$ associated to a TRVASS $\mathcal{V} = \langle Q, \Sigma, \Delta \rangle$

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$$\xrightarrow{*}_{\mathcal{V}} \quad = \quad \xrightarrow{\boldsymbol{A}^*} \ \cup \ \xrightarrow{\boldsymbol{A}^* \{T,R\} \boldsymbol{A}^*} \ \cup \ \xrightarrow{\boldsymbol{A}^* \{T,R\} \Sigma^* \{T,R\} \boldsymbol{A}^*}$$

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$p(x_1, x_2) \xrightarrow{\boldsymbol{A}^* \, T \, \Sigma^* \, R \, \boldsymbol{A}^*} q(y_1, y_2)$ iff there exist $r, s \in Q$ and $m, n \in \mathbb{N}$ such that

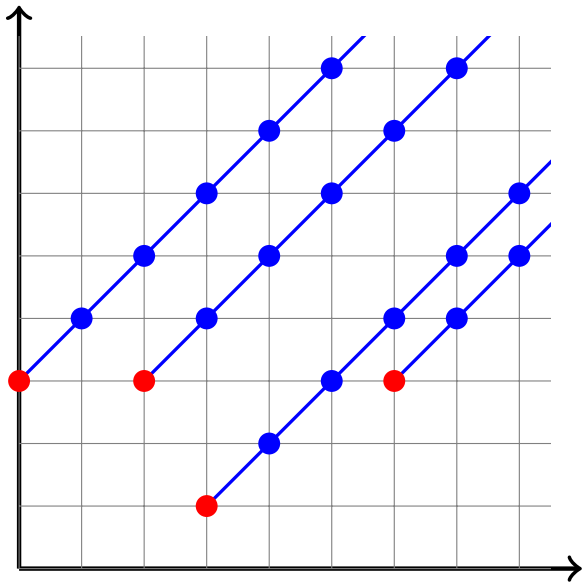
$$p(x_1, x_2) \xrightarrow{\boldsymbol{A}^* T} r(0, m) \ \wedge \ r^T(m) \xrightarrow{*}_{\mathcal{P}} s^R(n) \ \wedge \ s(n, 0) \xrightarrow{\boldsymbol{A}^*} q(y_1, y_2)$$

Diagonal Presburger-Definable Relations

Definition

A relation $S \subseteq \mathbb{N} \times \mathbb{N}$ is diagonal if

$$x\,S\,y \;\Rightarrow\; (x+c)\,S\,(y+c)$$



$\underbrace{p(0,m) \xrightarrow{\boldsymbol{A}^* T} q(0,n)}_{S_{TT}}$ and $\underbrace{p(m,0) \xrightarrow{\boldsymbol{A}^* R} q(n,0)}_{S_{RR}}$ are diagonal

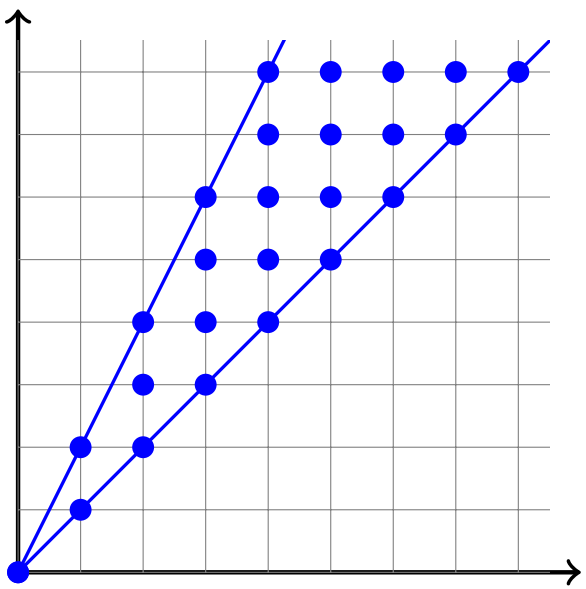
Theorem

Diagonal Presburger-definable relations are effectively closed under union, intersection, composition and transitive closure.

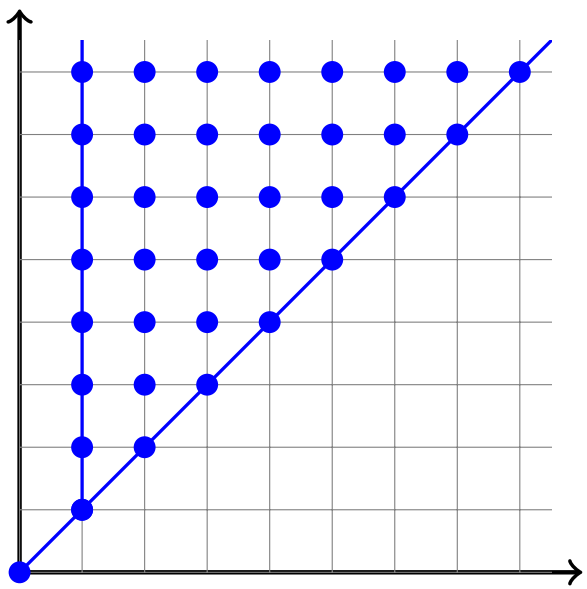
Diagonal Presburger-Definable Relations — Example

$$x\,S\,y \overset{\text{def}}{\Leftrightarrow} x \leq y \wedge y \leq 2x$$

$$x\,S^{+}\,y \Leftrightarrow x \leq y \wedge (x = 0 \Leftrightarrow y = 0)$$



S



S^{+}

Diagonal Presburger-Definable Relations — Proof

$$I_S = \{x \mid \exists y > x : x\, S\, y\} \qquad D_S = \{y \mid \exists x > y : x\, S\, y\}$$

$$\alpha = \min\{\delta > 0 \mid \forall x \in I_S : (x, x+\delta) \in S\} \qquad (0 \text{ if } I_S = \emptyset)$$
$$\beta = \min\{\delta > 0 \mid \forall y \in D_S : (y+\delta, y) \in S\} \qquad (0 \text{ if } D_S = \emptyset)$$

$$\mathtt{Inc}_S(x,y) \overset{\text{def}}{\Leftrightarrow} (x = y) \vee (x \in I_S \ \wedge\ y \in x + \mathbb{N}\alpha)$$
$$\mathtt{Dec}_S(x,y) \overset{\text{def}}{\Leftrightarrow} (x = y) \vee (y \in D_S \wedge x \in y + \mathbb{N}\beta)$$

Lemma

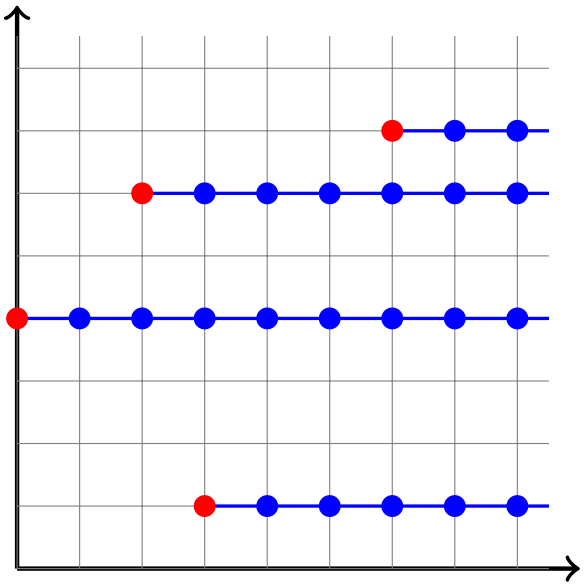
If $S \subseteq \mathbb{N} \times \mathbb{N}$ *is diagonal then* $S^+ = \mathtt{Inc}_S \cdot (S \cup \cdots \cup S^{\alpha+\beta+1}) \cdot \mathtt{Dec}_S$.

Horizontal Presburger-Definable Relations

Definition

A relation $S \subseteq \mathbb{N} \times \mathbb{N}$ is horizontal if

$$x\, S\, y \;\Rightarrow\; (x + c)\, S\, y$$



$\underbrace{p(0,m) \xrightarrow{\boldsymbol{A}^* R} q(n,0)}_{S_{TR}}$ is horizontal

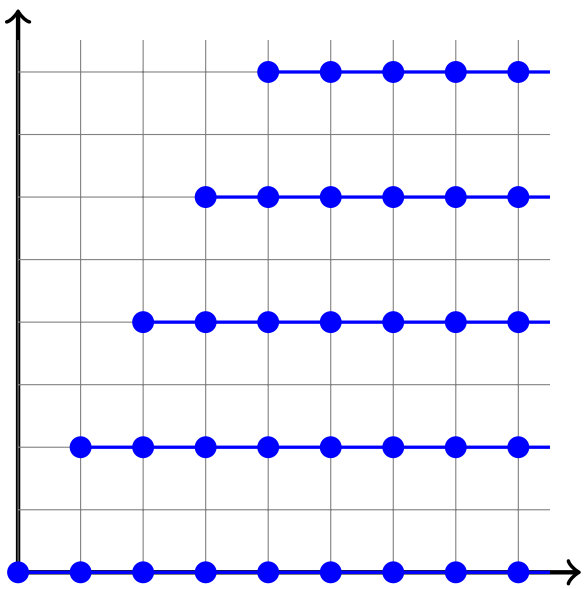
Theorem

Horizontal Presburger-definable relations are effectively closed under union, intersection, composition and transitive closure.

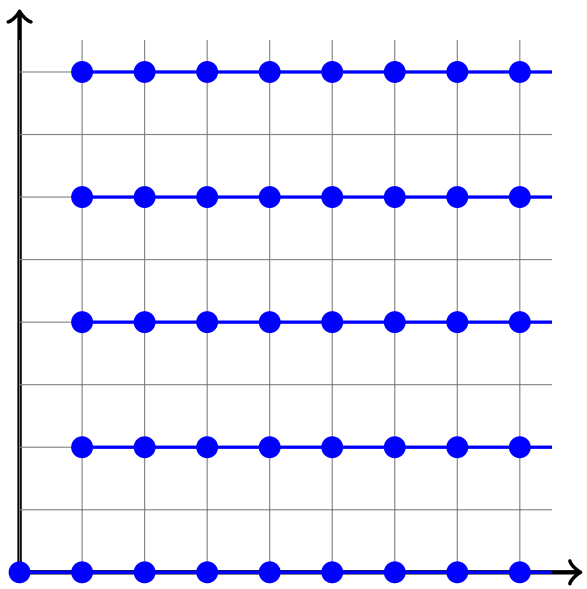
Horizontal Presburger-Definable Relations — Example

$$x\,S\,y \quad \overset{\text{def}}{\Leftrightarrow} \quad y \in 2\mathbb{N} \wedge y \leq 2x$$

$$x\,S^{+}\,y \quad \Leftrightarrow \quad y \in 2\mathbb{N} \wedge (x = 0 \Rightarrow y = 0)$$



S



S^{+}

Horizontal Presburger-Definable Relations — Proof

Lemma

If $S \subseteq \mathbb{N} \times \mathbb{N}$ is horizontal then $x\, S^+\, y$ iff there exists z such that

$$z\, S\, y \ \wedge\ \forall u \in \{x, \ldots, z-1\}, \exists v > u : u\, S\, v$$

Assume $x = x_0\, S\, x_1\, S \cdots S\, x_k = y$.

If $x_{i-1} \geq x_i$ then $x_{i-1}\, S\, x_{i+1}$. So we may suppose w.l.o.g. that

$$x_0 < x_1 < \cdots < x_{k-1}$$

Hence $u\, S\, x_i$ for all $u \in \{x_{i-1}, \ldots, x_i - 1\}$ and $0 < i < k$.

So for all $u \in \{x_0, \ldots, x_{k-1} - 1\}$, there is $v > u$ with $u\, S\, v$.

Take $z = x_{k-1}$.

Horizontal Presburger-Definable Relations — Proof

Lemma

If $S \subseteq \mathbb{N} \times \mathbb{N}$ is horizontal then $x\, S^+\, y$ iff there exists z such that

$$z\, S\, y \ \wedge\ \forall u \in \{x, \ldots, z-1\}, \exists v > u : u\, S\, v$$

Assume $z\, S\, y$ and for all $u \in \{x, \ldots, z-1\}$, there is $v > u$ with $u\, S\, v$.

If $x \geq z$ then $x\, S\, y$. Otherwise, there exists $x_1 > x$ with $x\, S\, x_1$.

Repeat this process to build $x < x_1 < \cdots < x_k$ with $x\, S\, x_1\, S\, \cdots\, S\, x_k$ and $x_k \geq z$.

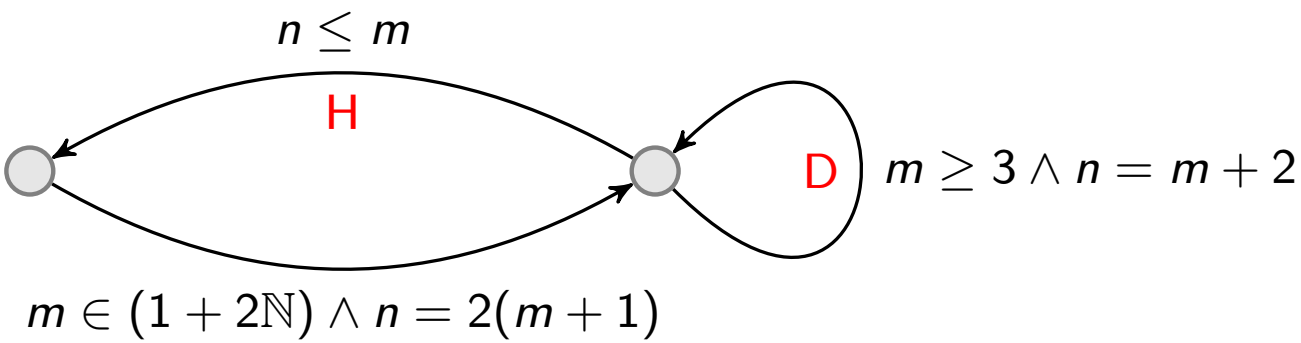
Since $z\, S\, y$, we get $x_k\, S\, y$.

Shallow Presburger Automata

Definition

A Presburger automaton is shallow if every cycle contains

only diagonal transitions or an horizontal transition



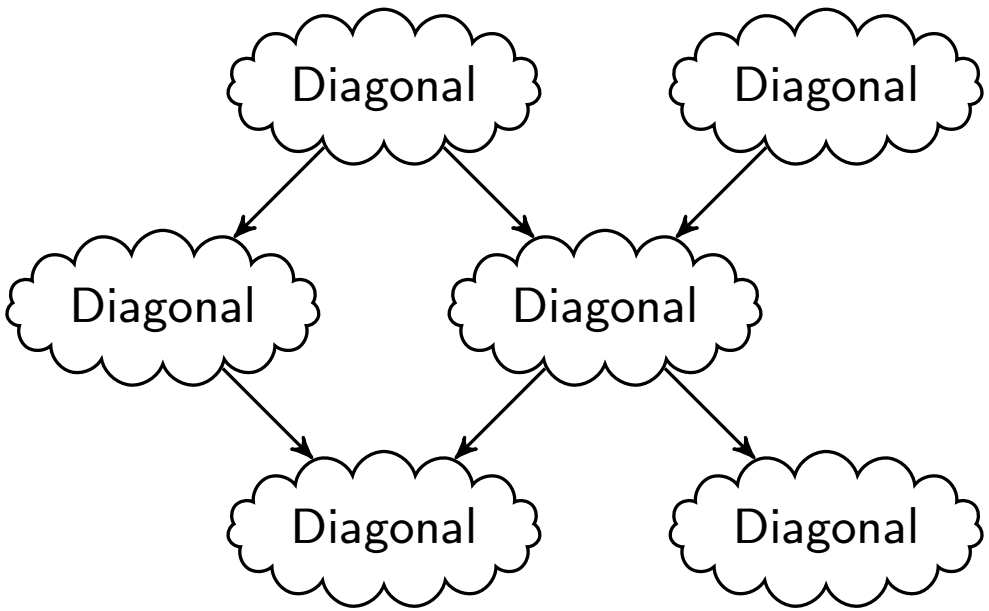
Theorem

The reachability relation of a shallow Presburger automaton is effectively Presburger-definable.

Shallow Presburger Automata — Proof

By induction on the number of horizontal transitions

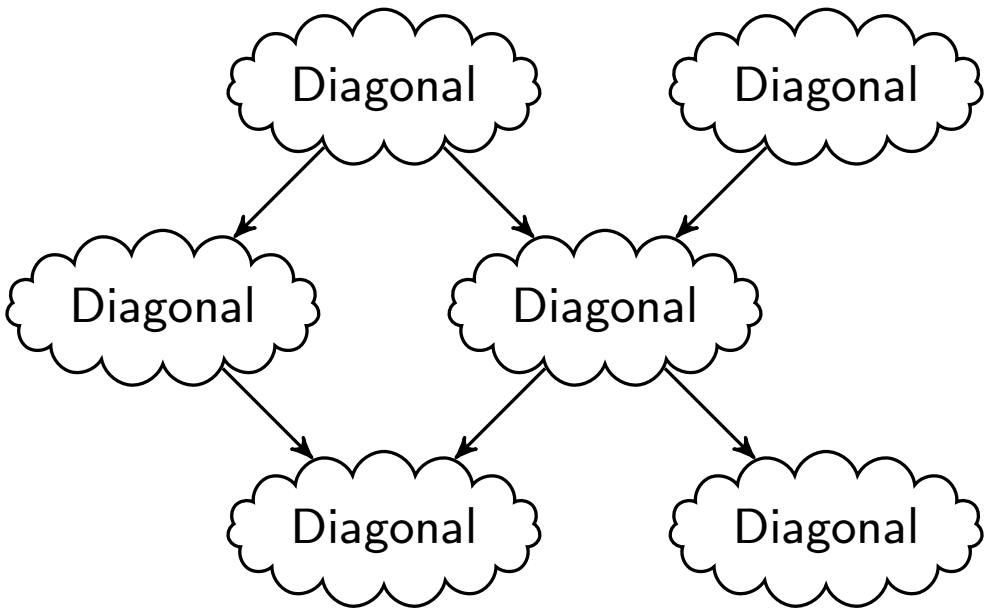
Base case: every SCC contains only diagonal transitions



Shallow Presburger Automata — Proof

By induction on the number of horizontal transitions

Base case: every SCC contains only diagonal transitions



Diagonal Presburger-definable relations effectively closed under union, composition and reflexive-transitive closure

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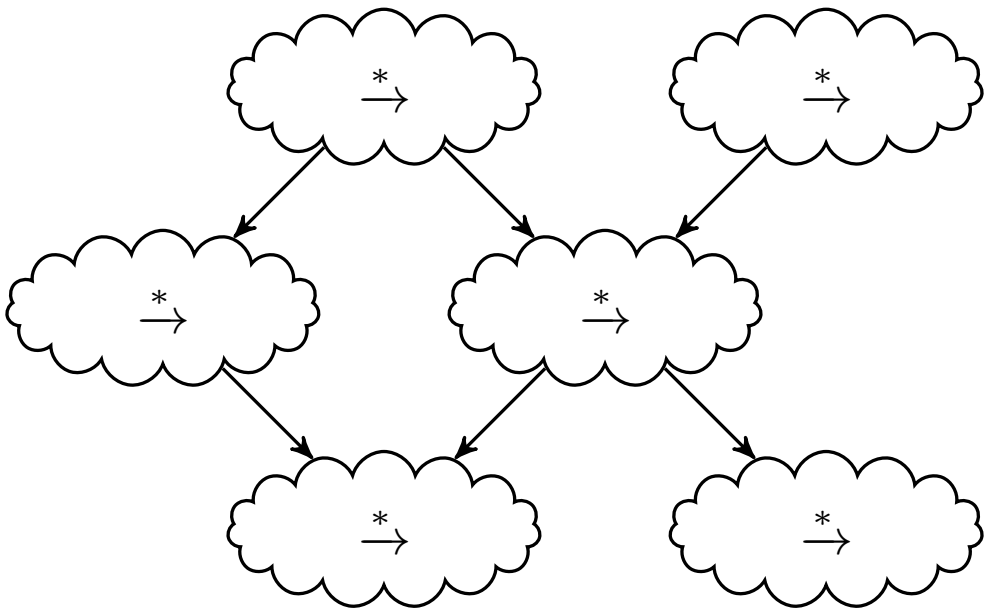


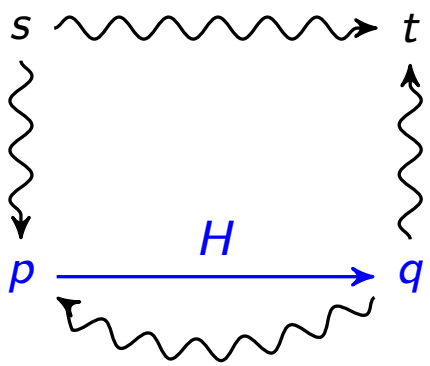
Diagonal Presburger-definable relations effectively closed under union, composition and reflexive-transitive closure

So $\xrightarrow{*}$ is effectively Presburger-definable within each SCC

Shallow Presburger Automata — Proof

By induction on the number of horizontal transitions

Induction step: pick an horizontal transition $p \xrightarrow{H} q$



A run from s to t

- either never takes $p \xrightarrow{H} q$
- or takes $p \xrightarrow{H} q$ at least once

Shallow Presburger Automata — Proof

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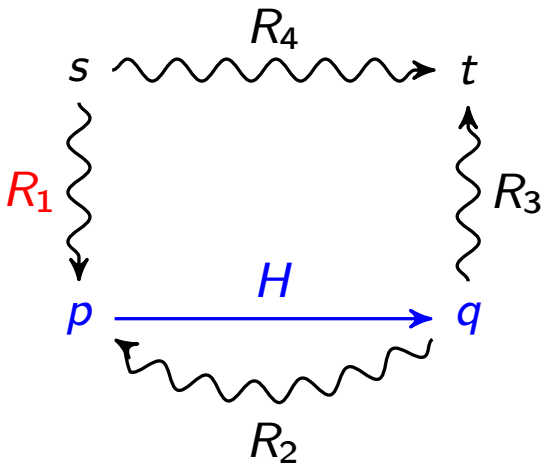

A run from s to t

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$m\, R_1\, n \overset{\text{def}}{\Leftrightarrow} s(m) \xrightarrow{*} p(n)$ in the Presburger automata without $p \xrightarrow{H} q$

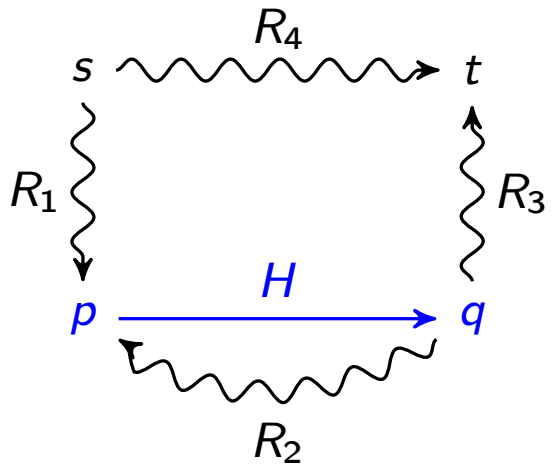
R_1, R_2, R_3, R_4 are effectively Presburger-definable by Ind. Hyp.

(Removing transitions preserves shallowness.)

Shallow Presburger Automata — Proof

By induction on the number of horizontal transitions

Induction step: pick an horizontal transition $p \xrightarrow{H} q$



A run from s to t

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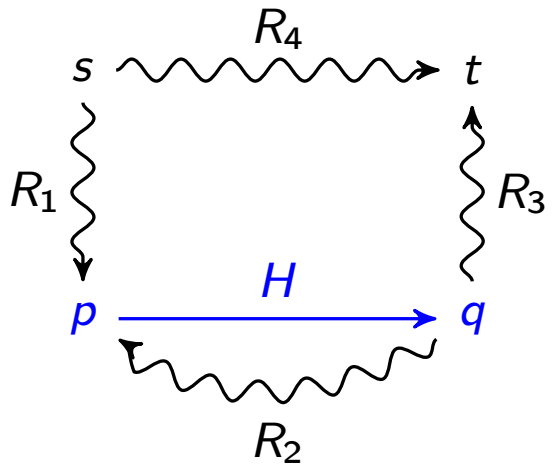
- or takes $p \xrightarrow{H} q$ at least once

$$
\begin{aligned}
& s(m) \xrightarrow{*} t(n) \\
\Leftrightarrow \quad & m\, R_4\, n \;\vee\; \exists m', n' : m\, R_1\, m' \;\wedge\; m' \left((H \cdot R_2)^* \cdot H\right) n' \;\wedge\; n'\, R_3\, n
\end{aligned}
$$

Shallow Presburger Automata — Proof

By induction on the number of horizontal transitions

Induction step: pick an horizontal transition $p \xrightarrow{H} q$



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\end{aligned}
$$

$H \cdot R_2$ is horizontal, so $(H \cdot R_2)^*$ is effectively Presburger-definable

Wrap-Up

Presburger automaton $\mathcal{P}$ associated to a TRVASS $\mathcal{V} = \langle Q, \Sigma, \Delta \rangle$

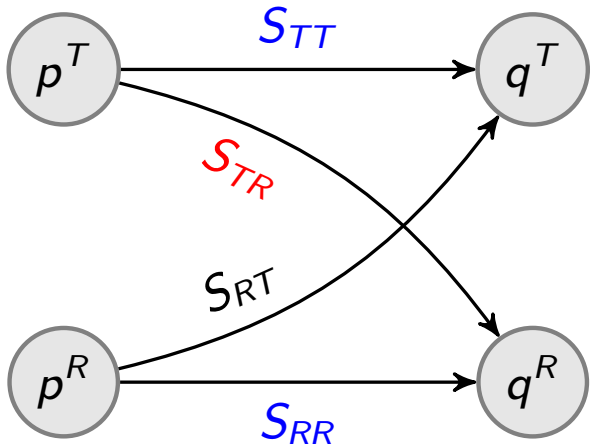
If $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable then so is $\xrightarrow{*}_{\mathcal{V}}$

Wrap-Up

Presburger automaton $\mathcal{P}$ associated to a TRVASS $\mathcal{V} = \langle Q, \Sigma, \Delta \rangle$

If $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable then so is $\xrightarrow{*}_{\mathcal{V}}$

Transitions of $\mathcal{P}$:



$S_{TT}(m, n)$: diagonal

$S_{RT}(m, n)$: ordinary

$S_{TR}(m, n)$: horizontal

$S_{RR}(m, n)$: diagonal

Wrap-Up

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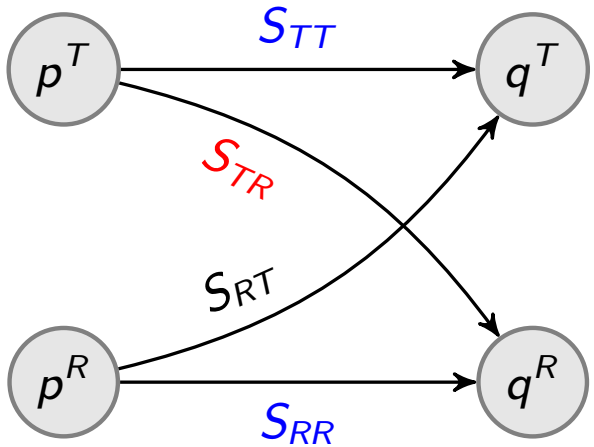

$S_{TT}(m, n)$: diagonal
$S_{RT}(m, n)$: ordinary
$S_{TR}(m, n)$: horizontal
$S_{RR}(m, n)$: diagonal

$\mathcal{P}$ is shallow so $\xrightarrow{*}_{\mathcal{P}}$ is effectively Presburger-definable

Summary

Class	*Post**	$\xrightarrow{*}$
$\mathbf{T}_1\mathbf{Tr}_2 \simeq \mathbf{T}_{1,2} \simeq \mathbf{T}_{1,2}\mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Reachability, . . . undecidable	
TRVASS $= \mathbf{T}_1\mathbf{R}_2 \simeq \mathbf{T}_1\mathbf{R}_{1,2}\mathbf{Tr}_1$	Eff. Presburger	**Eff. Presburger**
$\mathbf{R}_{1,2}\mathbf{Tr}_1 \simeq \mathbf{R}_{1,2}\mathbf{Tr}_{1,2}$	Eff. Presburger	Eff. Presburger
$\mathbf{T}_1 \simeq \mathbf{T}_1\mathbf{R}_1\mathbf{Tr}_1$	Eff. Presburger	Eff. Presburger
2-dim VASS	Eff. Presburger	Eff. Presburger

- Simple proof that unifies and generalizes all existing results
- *Post** is effectively Presburger-definable for TRVASS
  - Boundedness is decidable

Relies on the effective Presburger-definability of $\xrightarrow{*}$ for 2-dim VASS [Leroux, Sutre 2004]

Open Problems

- Boundedness for d-dim TRVASS ($d \geq 3$)
  - Decidable for d-dim RRVASS [Dufourd, Jančar, Schnoebelen 1999]
- Complexity of reachability for TRVASS
  - PSPACE-complete for 2-dim VASS in binary [Blondin, Finkel, Göller, Haase, McKenzie 2015]
  - NL-complete for 2-dim VASS in unary [Englert, Lazic, Totzke 2016]
- Reachability for 1-dim Pushdown VASS
  - Coverability is decidable [Leroux, Sutre, Totzke 2015]

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Thank You!