# Separation by First-Order Logic

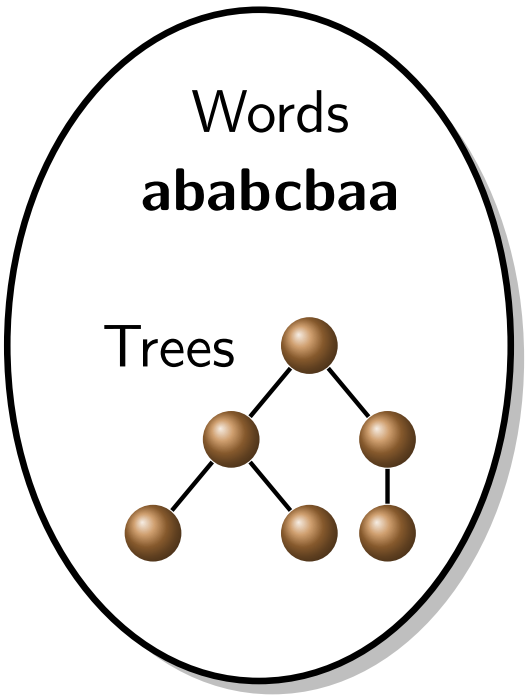
Thomas Place and Marc Zeitoun

LaBRI, Université Bordeaux, CNRS

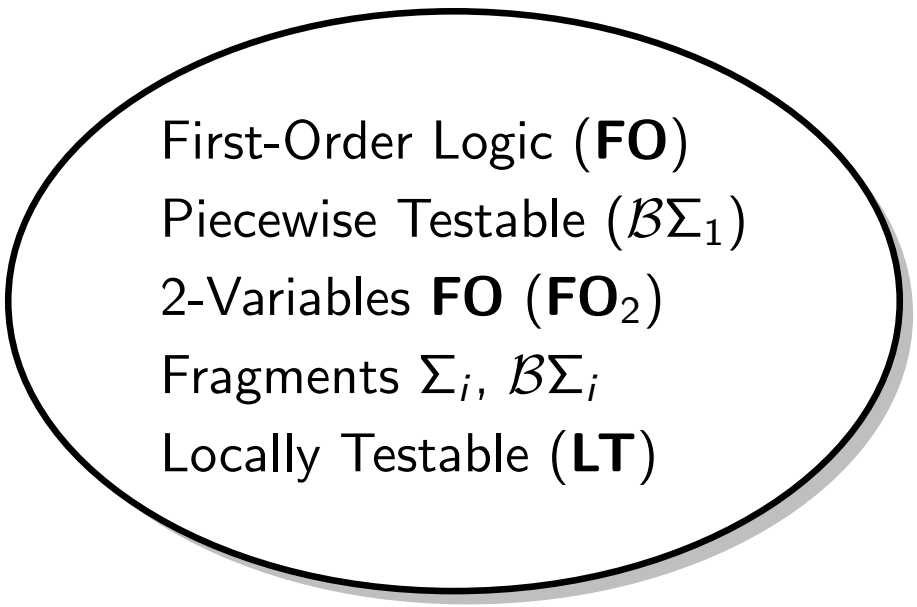
April 28, 2014

# Objects we consider

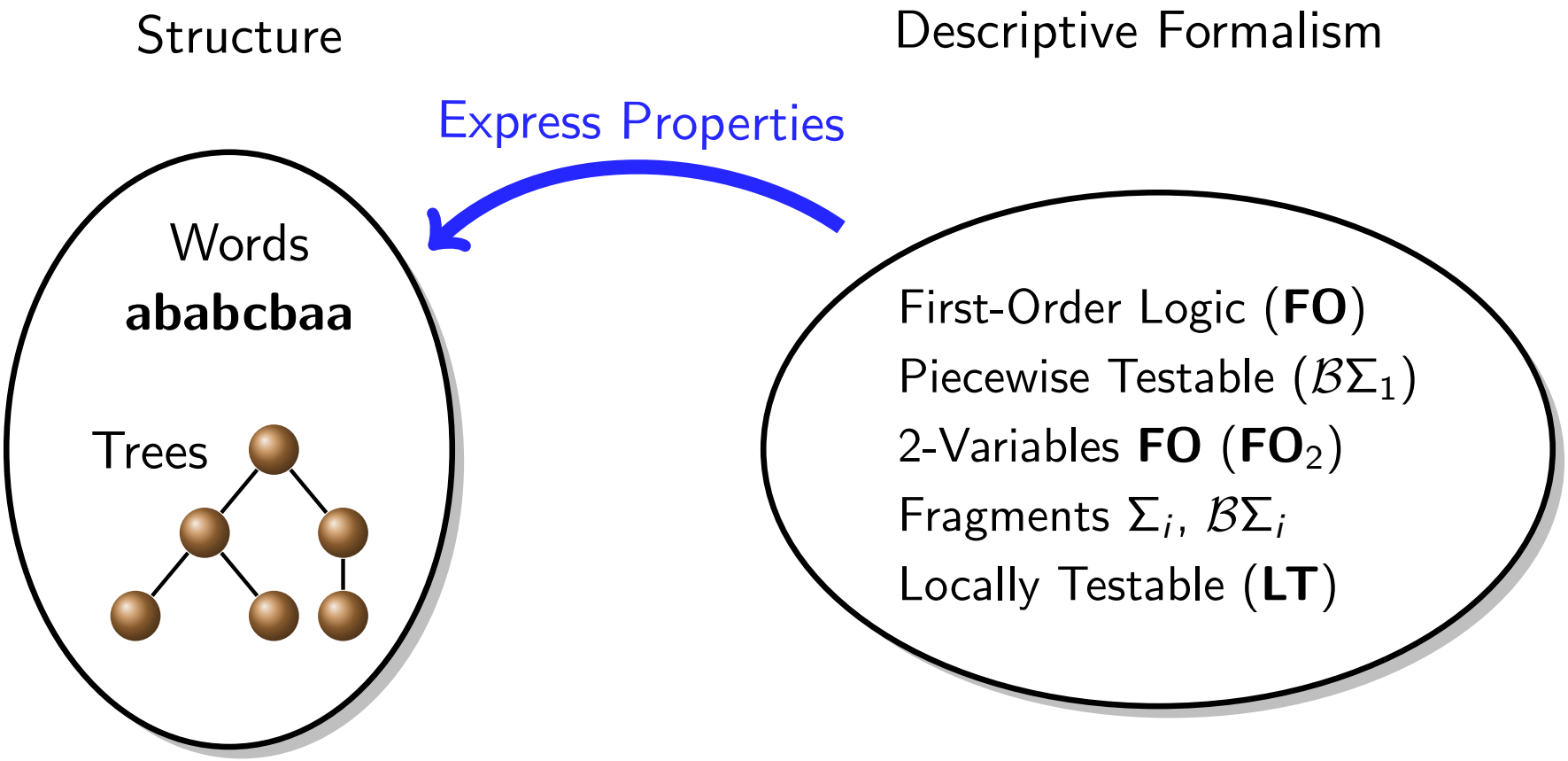
Structure




Descriptive Formalism



# Objects we consider

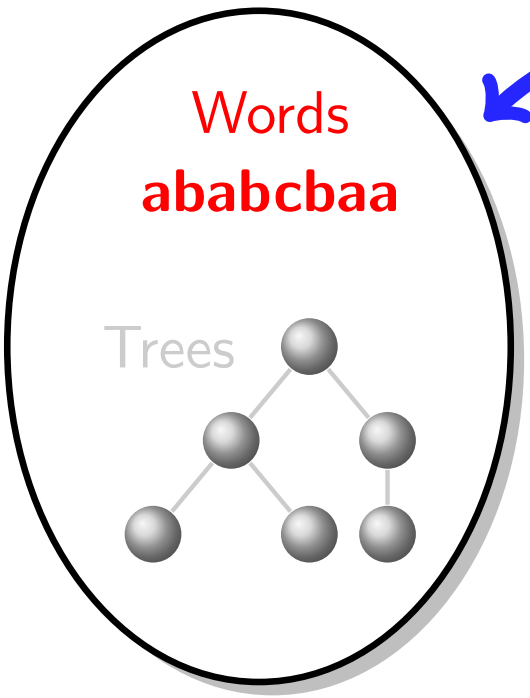


# Objects we consider

Structure

Descriptive Formalism

Express Properties




First-Order Logic (**FO**)

Piecewise Testable ($\mathcal{B}\Sigma_1$)

2-Variables **FO** ($\mathbf{FO}_2$)

Fragments $\Sigma_i$, $\mathcal{B}\Sigma_i$

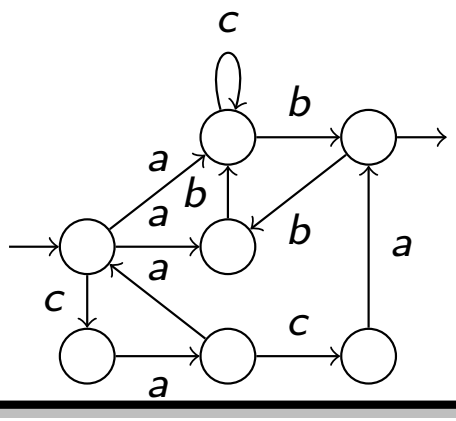
Locally Testable (**LT**)

For this talk

# A Reduced Problem: Decidable Characterizations

Decide the following problem:

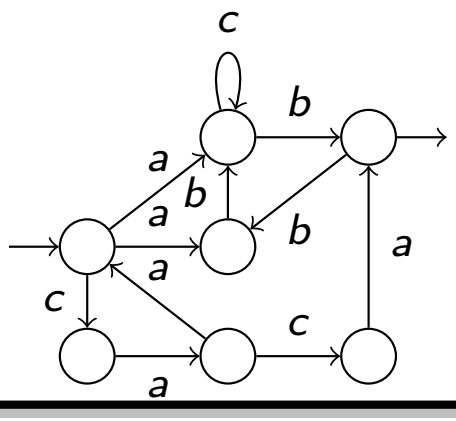
Take a regular language $L$



# A Reduced Problem: Decidable Characterizations

Decide the following problem:

Take a regular language $L$

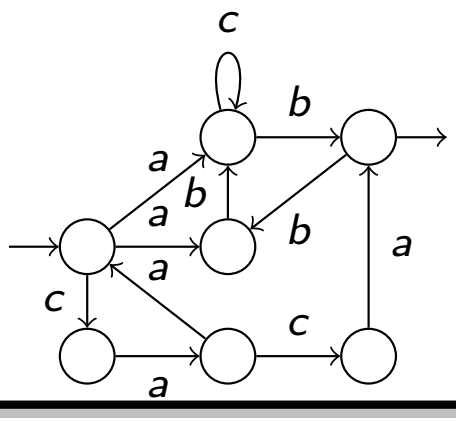



Can it be defined with a **FO** formula ?

# A Reduced Problem: Decidable Characterizations

Decide the following problem:

Take a regular language $L$




Can it be defined with a **FO** formula ?

## Schützenberger'65, McNaughton and Papert'71

For $L$ a regular language, the following are equivalent:

- $L$ is **FO**-definable.
- The syntactic monoid of $L$ satisfies $u^{\omega+1} = u^{\omega}$.

# Why we want more than decidable characterizations

If the characterization answer is yes for $L$:

- All subparts of the minimal automaton of $L$ are **FO**-definable.

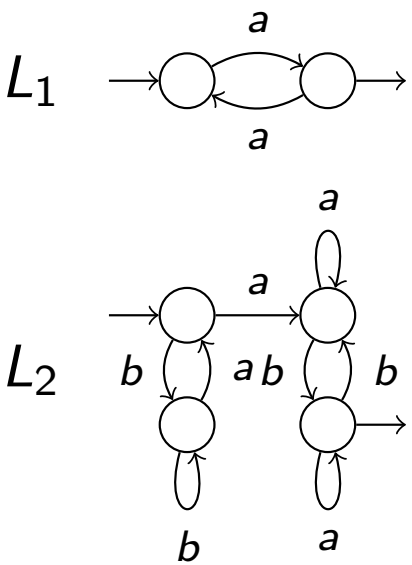
If the characterization's answer is no for $L$:

- We have little information.
- Defining $L$ would require differentiating some $u^\omega$ and $u^{\omega+1}$.
- Yet: the logic can still express facts on $L$.

# Separation

Decide the following problem:

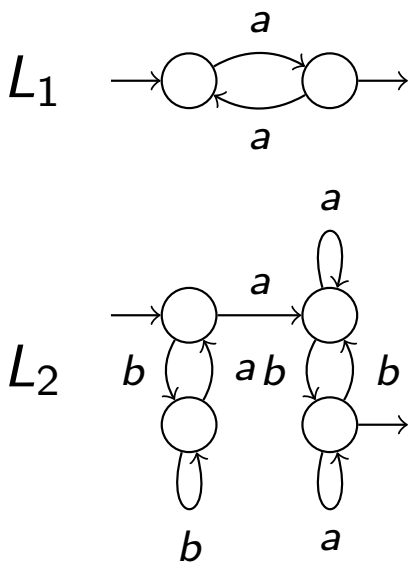
Take two regular languages $L_1, L_2$



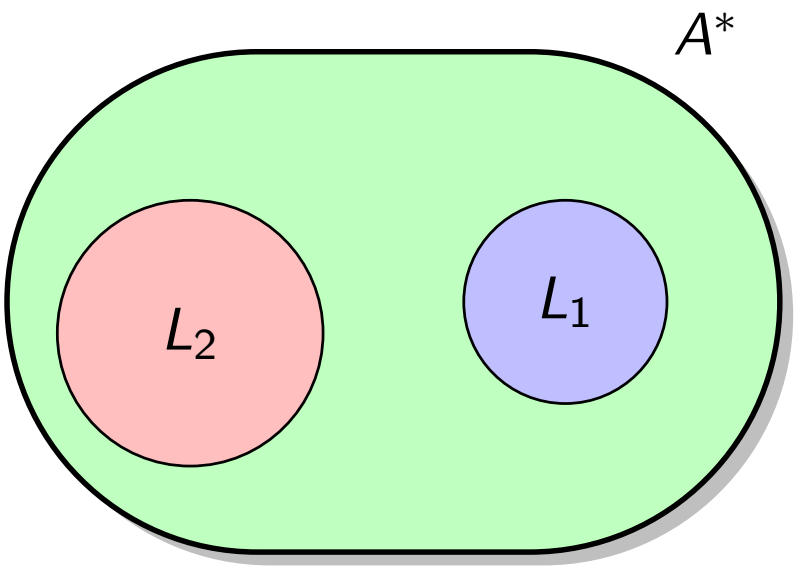
# Separation

Decide the following problem:

Take two regular languages $L_1, L_2$



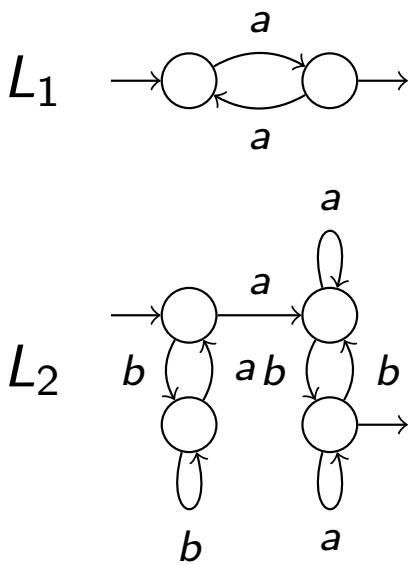

Can $L_1$ be separated from $L_2$ with a **FO** formula ?



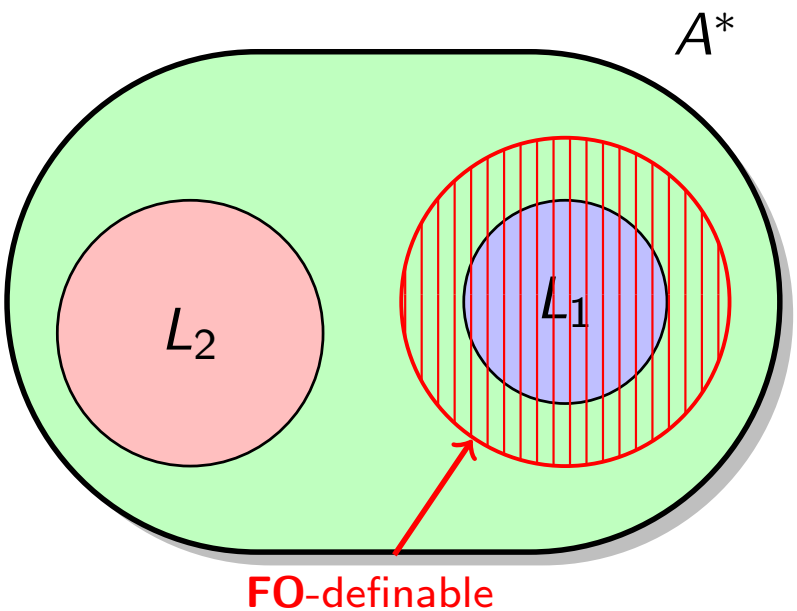
# Separation

Decide the following problem:

Take two regular languages $L_1, L_2$



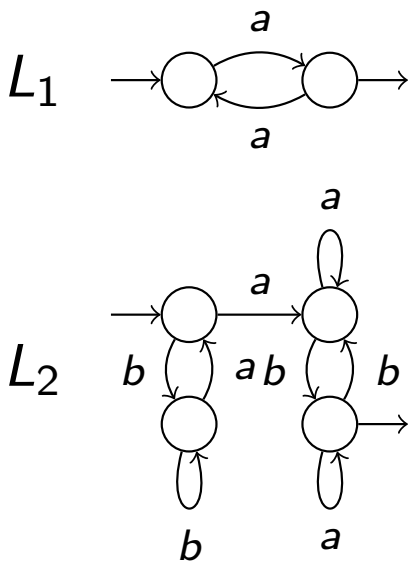

Can $L_1$ be separated from $L_2$ with a **FO** formula ?



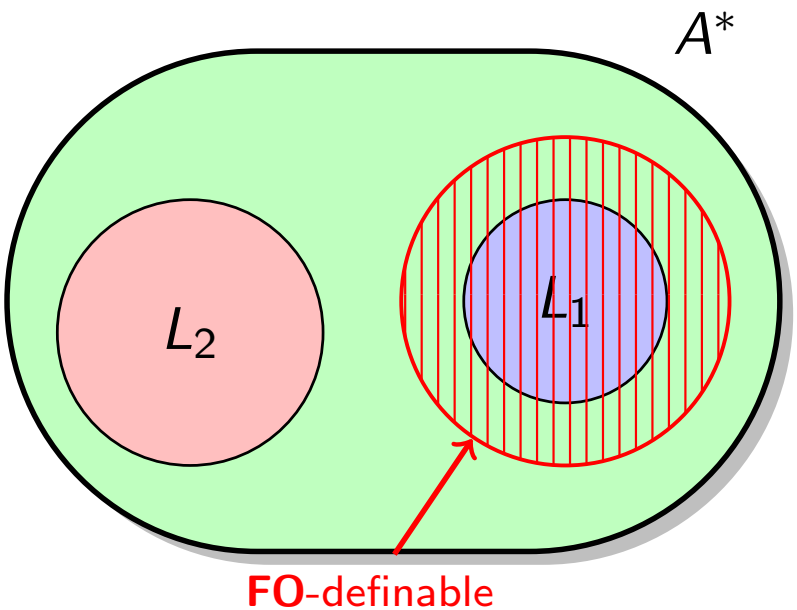
# Separation

Decide the following problem:

Take two regular languages $L_1, L_2$




Can $L_1$ be separated from $L_2$ with a **FO** formula ?




More general than getting a decidable characterization.

# Motivations for Separation

- More general: need **FO** techniques applying to all languages.

# Motivations for Separation

- More general: need **FO** techniques applying to all languages.
- Therefore, may give information to solve harder problems.

# Motivations for Separation

- More general: need **FO** techniques applying to all languages.
- Therefore, may give information to solve harder problems.
- For FO, already solved with such motivations by Henckell '88.

**Pointlike sets: the finest aperiodic cover of a finite semigroup**

*The research in this paper is motivated by the open question: "Is the complexity of a finite semigroup S decidable?"*

# Motivations for Separation

- More general: need **FO** techniques applying to all languages.
- Therefore, may give information to solve harder problems.
- For FO, already solved with such motivations by Henckell '88.

**Pointlike sets: the finest aperiodic cover of a finite semigroup**

*The research in this paper is motivated by the open question: "Is the complexity of a finite semigroup S decidable?"*

- Difficult algebraic techniques. Following the lead of the Presentation Lemma (Rhodes), we describe the finest cover on S that can be computed using an aperiodic semigroup and give an explicit relation. The central idea of the proof is that an aperiodic computation can be described by a new blow-up operator HW. The proof also relies on the Rhodes expansion of S and on Zeiger coding.

# An already known result: Henckell '88

**Guide to the paper**

*Chapter* 1. *Elementary definitions and notation* should be omitted on first reading and used as a reference as needed.

*Chapter* 2. *The* Pl-*functor* defines pointlike sets in a general setting and shows by an abstract compactness argument that $\mathrm{Pl}(S)$ can be computed by an aperiodic semigroup.

*Chapter* 3. *Definition of* $C^\omega(S)$ *and* $H^\omega$ defines $C^\omega(S)$, a collection of pointlike sets, in a constructive manner. $H^\omega$ is the 'blow-up-operator' that we will use in Chapter 5 to show $C^\omega(S) = \mathrm{Pl}(S)$. It has some examples in the end.

*Chapter* 4. *The Rhodes-expansion* defines the tools needed in Chapter 5.

*Chapter* 5. $C^\omega(S) = \mathrm{Pl}(S)$ shows the main result by actually constructing a relation $S \xrightarrow{R} \mathrm{CP}(S)$ computing $C^\omega(S)$ with $\mathrm{CP}(S)$ aperiodic. It uses $H^\omega$, generalized to $\hat{H}^\omega$ on $\hat{C}^\omega(S)$ 'to get rid of groups by blowing up'.

# Alternate formulations

## Several formulations of separation [Almeida'96]

The following are equivalent:

1. $L_1$ and $L_2$ are not FO-separable.
2. For all $k$, there exist $w_1 \in L_1$, $w_2 \in L_2$ with $w_1 \cong_k w_2$.
3. For all aperiodic $T$ and morphism $\beta : A^* \to T$,

$$\beta(L_1) \cap \beta(L_2) \neq \emptyset.$$

4. $\overline{L}_1 \cap \overline{L}_2 \neq \emptyset$ (closures taken in the pro-aperiodic semigroup).
5. $\overline{L}_1 \cap \overline{L}_2$ contains an $\omega$-term.

Actually, 5 may be exploited to prove decidability of separation.

# **FO** is hard, let's make it easy: Quantifier Rank

Quantifier rank of a formula: Nested depth of quantifiers.

$$\forall x \exists y \ (a(x) \implies \exists z \ (x < z < y \wedge b(y))) \quad \text{has quantifier rank 3}$$

If $k$ fixed: finitely many **FO** properties of rank $k \Rightarrow$ Separation is easy (test them all)

# **FO** is hard, let's make it easy: Quantifier Rank

Quantifier rank of a formula: Nested depth of quantifiers.

$$\forall x \exists y\ (a(x) \implies \exists z\ (x < z < y \wedge b(y))) \quad \text{has quantifier rank 3}$$

If $k$ fixed: finitely many **FO** properties of rank $k$ $\Rightarrow$ Separation is easy (test them all)

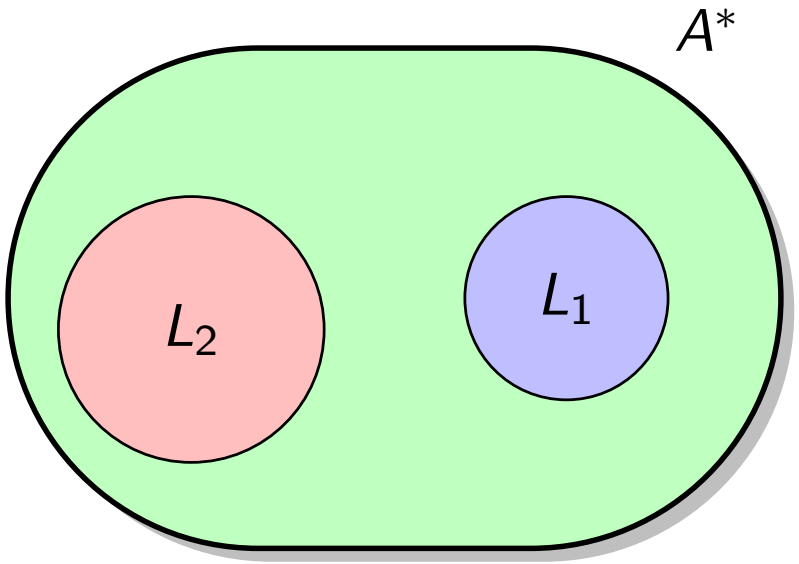
## $k$-equivalence for **FO**

Let $w_1, w_2$ be words:

$$w_1 \cong_k w_2 \text{ iff } w_1, w_2 \text{ satisfy the same formulas of rank } k$$

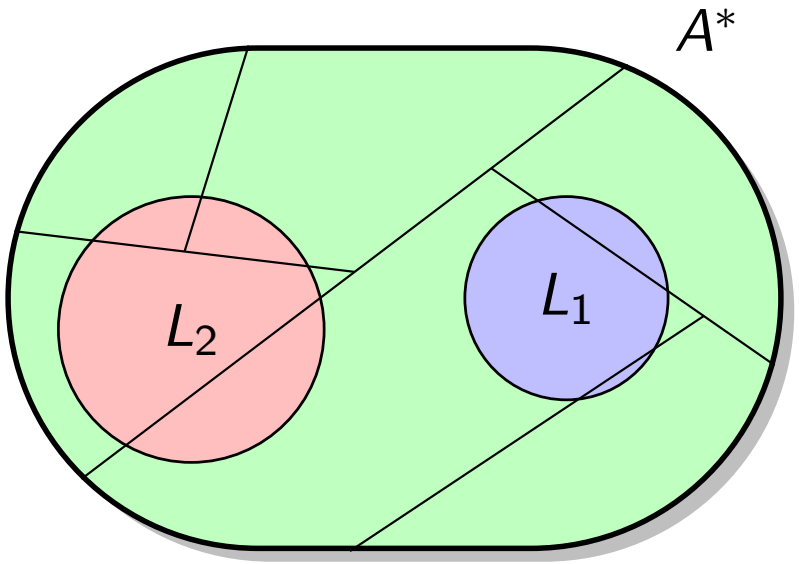
*All* **FO** *properties of rank $k$ are unions of classes of $\cong_k$.*

# Fixed Quantifier Rank $k$




Let's add the $\cong_k$-classes

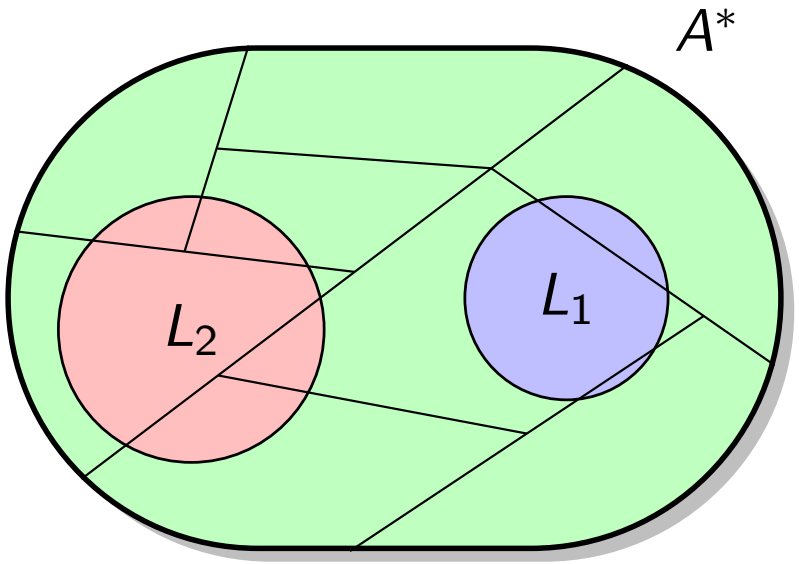
# Fixed Quantifier Rank $k$




Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

# Fixed Quantifier Rank $k$

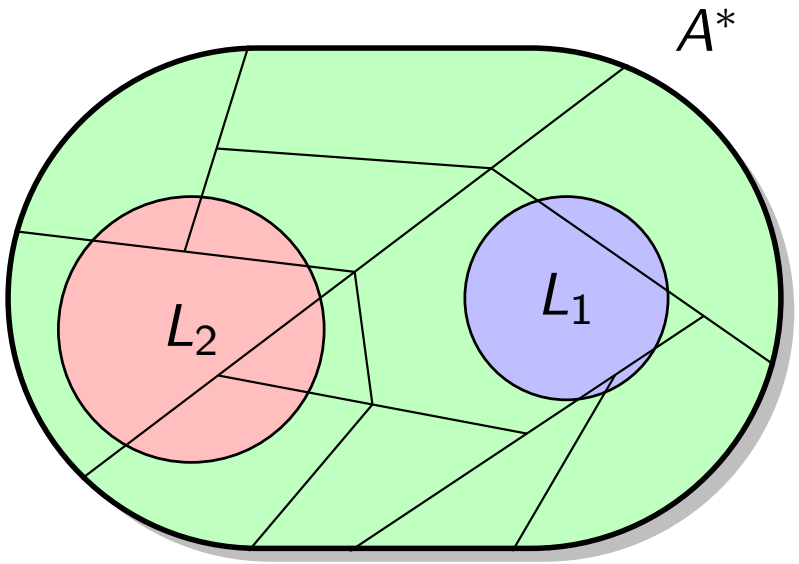



Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

When $k$ gets larger, $\cong_k$ is refined but it never ends

# Fixed Quantifier Rank $k$

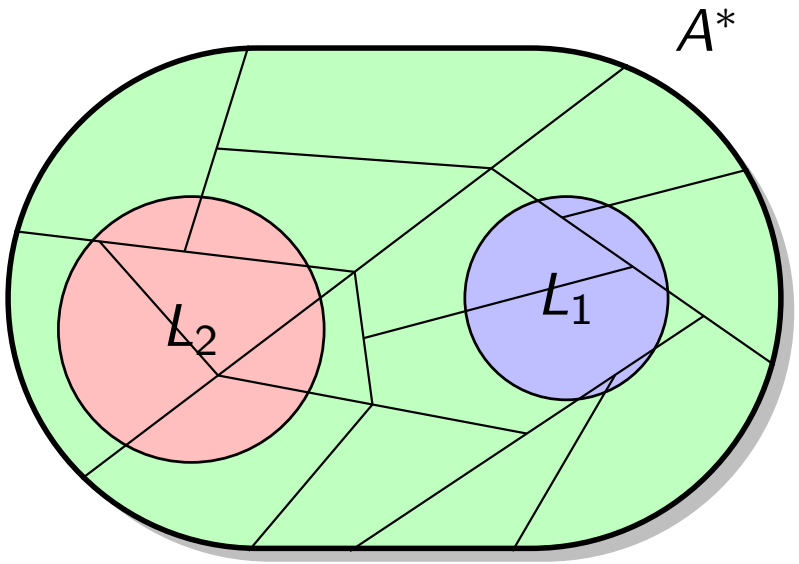



Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

When $k$ gets larger, $\cong_k$ is refined but it never ends

# Fixed Quantifier Rank $k$

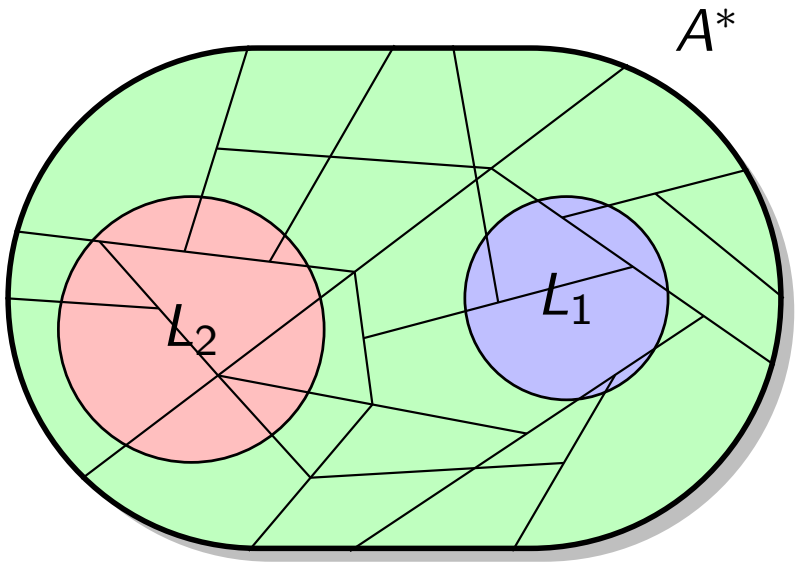



Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

When $k$ gets larger, $\cong_k$ is refined but it never ends

# Fixed Quantifier Rank $k$

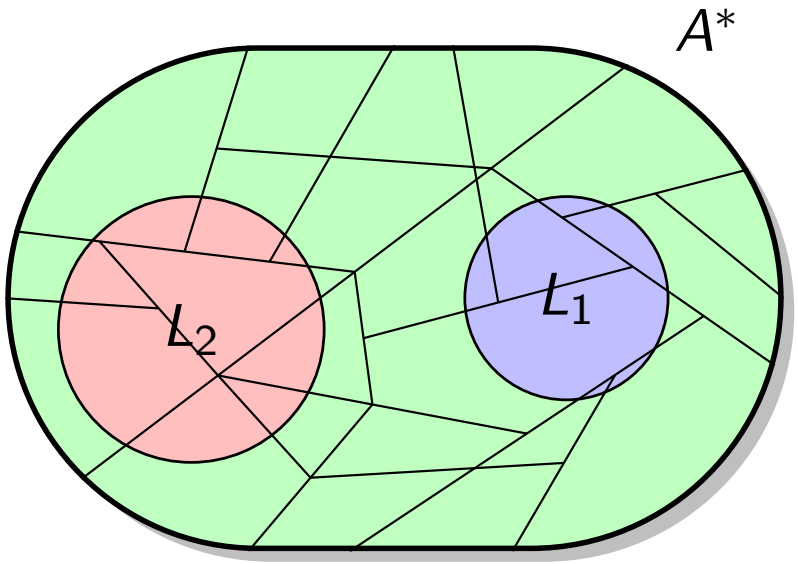



Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

When $k$ gets larger, $\cong_k$ is refined but it never ends

# Fixed Quantifier Rank $k$




Separable with rank $k$ iff no $\cong_k$-class intersects both languages

For full **FO** we want to know if there exists such a $k$
$\Rightarrow$ Compute a 'limit' for $\cong_k$.

When $k$ gets larger, $\cong_k$ is refined but it never ends

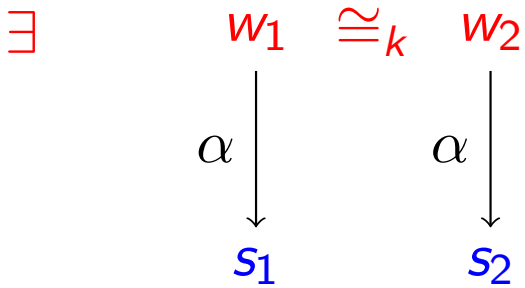
Idea. Abstract $\cong_k$ on a monoid recognizing both $L_1$ and $L_2$.

# "Pair" analysis

Recall from Thomas' talk:

**FO-indistinguishable pairs for $\alpha : A^* \to M$**

$(s_1, s_2) \in \mathrm{I}_k[\alpha]$ if




- Smaller and smaller sets: $\mathrm{I}_{k+1}[\alpha] \subseteq \mathrm{I}_k[\alpha]$
- Limit set: $\mathrm{I}[\alpha] = \bigcap_k \mathrm{I}_k[\alpha]$.
- Computing these pairs solves separation:

$$(s_1, s_2) \in \mathrm{I}[\alpha] \iff \alpha^{-1}(s_1) \text{ and } \alpha^{-1}(s_2) \text{ not separable}$$

# The Separation Criterion

## Separation Criterion

$L_1, L_2$ recognized by $\alpha : A^* \to M$ are not separable

iff

there are accepting elements $s_1, s_2 \in M$ for $L_1, L_2$ s.t. $(s_1, s_2) \in \mathrm{I}[\alpha]$.

# The Separation Criterion

## Separation Criterion

$L_1, L_2$ recognized by $\alpha : A^* \to M$ are not separable

iff

there are accepting elements $s_1, s_2 \in M$ for $L_1, L_2$ s.t. $(s_1, s_2) \in \mathrm{I}[\alpha]$.

Computing $\mathrm{I}[\alpha]$ suffices to solve separation.

# Two approaches

2 approaches to compute the relation $\mathrm{I}[\alpha]$:

### Brute-force:

- Computing $\mathrm{I}_k[\alpha]$ easy for fixed $k$.
- $\mathrm{I}[\alpha] = \mathrm{I}_k[\alpha]$ for some $k$ depending on $\alpha$.
- $\Rightarrow$ Prove a bound $k = f(\alpha)$ and compute $\mathrm{I}_k[\alpha]$.

### Algorithm:

Find an algorithm that bypasses the bound $k$ and computes $\mathrm{I}[\alpha]$ directly.

# Two approaches

2 approaches to compute the relation $\mathrm{I}[\alpha]$:

### Brute-force:

- Computing $\mathrm{I}_k[\alpha]$ easy for fixed $k$.
- $\mathrm{I}[\alpha] = \mathrm{I}_k[\alpha]$ for some $k$ depending on $\alpha$.
- $\Rightarrow$ Prove a bound $k = f(\alpha)$ and compute $\mathrm{I}_k[\alpha]$.

### Algorithm:

Find an algorithm that bypasses the bound $k$ and computes $\mathrm{I}[\alpha]$ directly.

We use approach 2.

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

1st Property of **FO**

$w \cong_k w$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

1st Property of **FO**

$w \cong_k w$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

2nd Property of **FO**

$w_1 \cong_k w_2$ and $u_1 \cong_k u_2 \quad \Rightarrow \quad w_1 u_1 \cong_k w_2 u_2$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

2nd Property of **FO**

$w_1 \cong_k w_2$ and $u_1 \cong_k u_2 \quad \Rightarrow \quad w_1 u_1 \cong_k w_2 u_2$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $(s_1, s_2) \in \mathrm{I}[\alpha]$ and $(t_1, t_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1 t_1, s_2 t_2) \in \mathrm{I}[\alpha]$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

3rd Property of **FO**

$\forall k\ \exists n\ \forall w_1, w_2 \in A^* \qquad w_1 \cong_k w_2 \Rightarrow (w_1)^n \cong_k (w_2)^{n+1}$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $(s_1, s_2) \in \mathrm{I}[\alpha]$ and $(t_1, t_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1 t_1, s_2 t_2) \in \mathrm{I}[\alpha]$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

**3rd Property of FO**

$$\forall k\ \exists n\ \forall w_1, w_2 \in A^* \qquad w_1 \cong_k w_2 \Rightarrow (w_1)^n \cong_k (w_2)^{n+1}$$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $(s_1, s_2) \in \mathrm{I}[\alpha]$ and $(t_1, t_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1 t_1, s_2 t_2) \in \mathrm{I}[\alpha]$
3. Operation $\omega$: $(s_1, s_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1^\omega, s_2^{\omega+1}) \in \mathrm{I}[\alpha]$

# A first (non complete) Algorithm computing $\mathrm{I}[\alpha]$

Idea. Start with trivial pairs and add more pairs via a fixpoint algorithm.

3rd Property of **FO**

$\forall k\ \exists n\ \forall w_1, w_2 \in A^* \qquad w_1 \cong_k w_2 \Rightarrow (w_1)^n \cong_k (w_2)^{n+1}$

1. Trivial pairs: for all $w \in A^*$ $(\alpha(w), \alpha(w)) \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $(s_1, s_2) \in \mathrm{I}[\alpha]$ and $(t_1, t_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1 t_1, s_2 t_2) \in \mathrm{I}[\alpha]$
3. Operation $\omega$: $(s_1, s_2) \in \mathrm{I}[\alpha] \Rightarrow (s_1^\omega, s_2^{\omega+1}) \in \mathrm{I}[\alpha]$

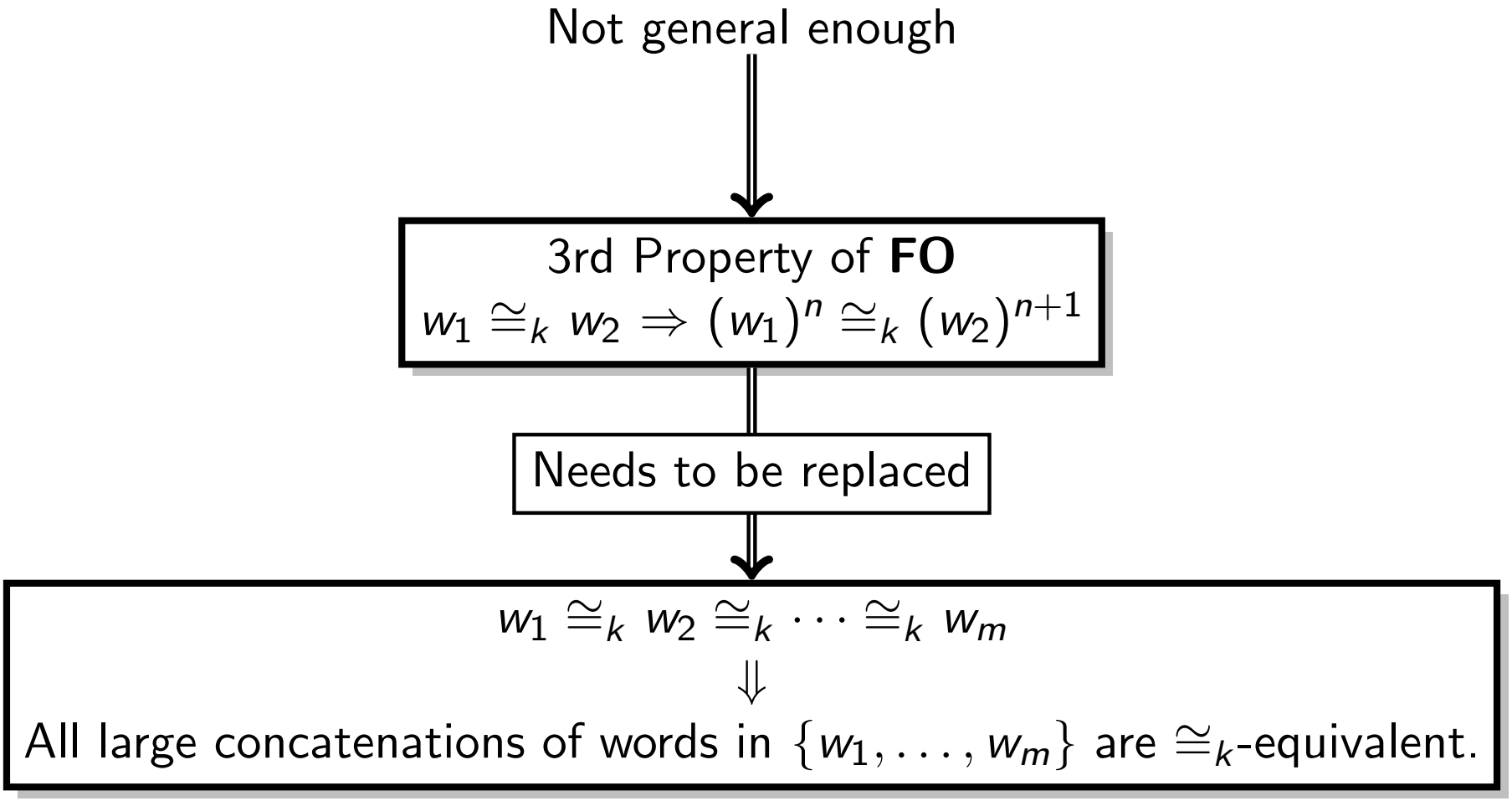
Correct by definition but not complete

# Why it does not work

**3rd Property of FO**

$$w_1 \cong_k w_2 \Rightarrow (w_1)^n \cong_k (w_2)^{n+1}$$

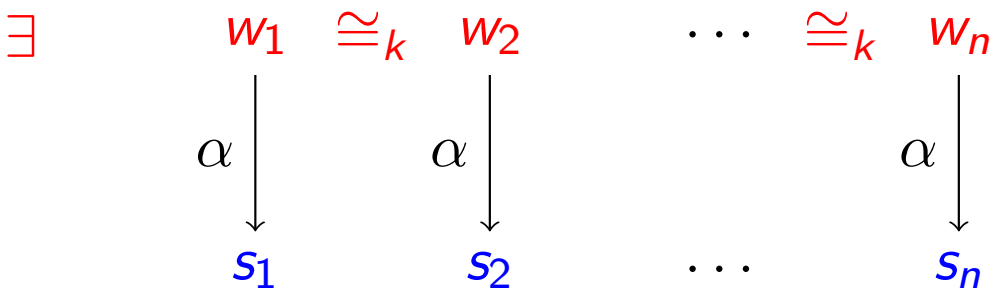
# Why it does not work



# Need for better analysis

A Generalization: FO-indistinguishable Sets for $\alpha : A^* \to S$:

- $\{s_1, s_2, \ldots, s_n\} \in \mathrm{I}_k[\alpha]$ if




- Limit set: $\mathrm{I}[\alpha] = \bigcap_k \mathrm{I}_k[\alpha]$.
- Computing these sets is more general than computing pairs.

  $\Rightarrow$ also solves separation (and gives much more).

# From Pairs to Sets

## New Objective

We want to compute the set $\mathrm{I}[\alpha] \subseteq 2^M$ such that:

$$S \in \mathrm{I}[\alpha] \text{ iff } S \in \mathrm{I}_k[\alpha],\ \forall k \in \mathbb{N}$$

# From Pairs to Sets

## New Objective

We want to compute the set $\mathrm{I}[\alpha] \subseteq 2^M$ such that:

$$S \in \mathrm{I}[\alpha] \text{ iff } S \in \mathrm{I}_k[\alpha],\ \forall k \in \mathbb{N}$$

## Remark

- With our new definition, we have $\mathrm{I}[\alpha] \subseteq 2^M$.
- $2^M$ is a monoid for the operation $S_1 \cdot S_2 = \{s_1 s_2 \mid s_1 \in S_1\ s_2 \in S_2\}$.

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

1st Property of **FO**

$$w \cong_k w$$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

1st Property of **FO**

$w \cong_k w$

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

**2nd Property of FO**

$w_1 \cong_k w_2$ and $u_1 \cong_k u_2 \quad \Rightarrow \quad w_1 u_1 \cong_k w_2 u_2$

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

**2nd Property of FO**

$w_1 \cong_k w_2$ and $u_1 \cong_k u_2 \quad \Rightarrow \quad w_1 u_1 \cong_k w_2 u_2$

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I}[\alpha] \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

3rd Property of **FO**

$$w_1 \cong_k w_2 \cdots \cong_k w_m$$

$$\Downarrow$$

All large concatenations of words in $\{w_1, \cdots, w_m\}$ are $\cong_k$-equivalent.

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I}[\alpha] \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

3rd Property of **FO**

$$w_1 \cong_k w_2 \cdots \cong_k w_m$$

$$\Downarrow$$

All large concatenations of words in $\{w_1, \cdots, w_m\}$ are $\cong_k$-equivalent.

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I}[\alpha] \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$
3. Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$

# A new (working) Algorithm

Idea : Start with trivial pairs and add more pairs via a fixpoint algorithm.

3rd Property of **FO**

$$w_1 \cong_k w_2 \cdots \cong_k w_m$$

$$\Downarrow$$

All large concatenations of words in $\{w_1, \cdots, w_m\}$ are $\cong_k$-equivalent.

1. Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
2. Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I}[\alpha] \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$
3. Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$

Correct by definition (e.g., use EF games)
Can be proved to be complete

# Alternate algorithms

- The algorithm reflects the equation $x^\omega = x^{\omega+1}$
- 2 other equivalent ways to characterize **FO**-definability:
  - All groups are trivial.
  - All $\mathcal{H}$-classes are trivial.
- The algorithm can be modified to reflect them too:

# Alternate algorithms

- The algorithm reflects the equation $x^\omega = x^{\omega+1}$
- 2 other equivalent ways to characterize **FO**-definability:
  - All groups are trivial.
  - All $\mathcal{H}$-classes are trivial.
- The algorithm can be modified to reflect them too:

New algorithm:

- Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
- Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I} \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$
- Operation: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$

# Alternate algorithms

- The algorithm reflects the equation $x^\omega = x^{\omega+1}$
- 2 other equivalent ways to characterize **FO**-definability:
  - All groups are trivial.
  - All $\mathcal{H}$-classes are trivial.
- The algorithm can be modified to reflect them too:

New algorithm:

  - Trivial sets: for all $w \in A^*$ $\{\alpha(w)\} \in \mathrm{I}[\alpha]$
  - Operation $\bullet$: $S_1 \in \mathrm{I}[\alpha]$ and $S_2 \in \mathrm{I} \Rightarrow S_1 S_2 \in \mathrm{I}[\alpha]$
  - Operation: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$

- Works also for $\mathcal{H}$-classes (similar to Henckell's algorithm).

# Alternate algorithms

3 variations of the 3rd operation.

- Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$.
- Operation $H$: $\mathcal{H}$ an $\mathcal{H}$-class of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{H}} S) \in \mathrm{I}[\alpha]$.
- Operation $G$: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$.

**Theorem**

All variations compute FO-indistinguishable sets, and all maximal ones.

# Alternate algorithms

3 variations of the 3rd operation.

- Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$.
- Operation $H$: $\mathcal{H}$ an $\mathcal{H}$-class of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{H}} S) \in \mathrm{I}[\alpha]$.
- Operation $G$: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$.

**Theorem**

All variations compute FO-indistinguishable sets, and all maximal ones.

- From $G$ to $\omega$: if $\mathcal{G} = \{T_1, \ldots, T_n\}$, then

$$T_1 \cup \cdots \cup T_n \subseteq (T_1^\omega \cup T_1^{\omega+1}) \cdots (T_n^\omega \cup T_n^{\omega+1})$$

# Alternate algorithms

3 variations of the 3rd operation.

- Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$.
- Operation $H$: $\mathcal{H}$ an $\mathcal{H}$-class of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{H}} S) \in \mathrm{I}[\alpha]$.
- Operation $G$: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$.

**Theorem**

All variations compute FO-indistinguishable sets, and all maximal ones.

- From $G$ to $\omega$: if $\mathcal{G} = \{T_1, \ldots, T_n\}$, then

$$T_1 \cup \cdots \cup T_n \subseteq (T_1^\omega \cup T_1^{\omega+1}) \cdots (T_n^\omega \cup T_n^{\omega+1})$$

- From $\omega$ to $H$: $S^\omega$ and $S^{\omega+1}$ are $\mathcal{H}$-equivalent.

# Alternate algorithms

3 variations of the 3rd operation.

- Operation $\omega$: $S \in \mathrm{I}[\alpha] \Rightarrow (S^\omega \cup S^{\omega+1}) \in \mathrm{I}[\alpha]$.
- Operation $H$: $\mathcal{H}$ an $\mathcal{H}$-class of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{H}} S) \in \mathrm{I}[\alpha]$.
- Operation $G$: $\mathcal{G}$ a subgroup of $\mathrm{I}[\alpha] \Rightarrow (\bigcup_{S \in \mathcal{G}} S) \in \mathrm{I}[\alpha]$.

**Theorem**

All variations compute FO-indistinguishable sets, and all maximal ones.

- From $G$ to $\omega$: if $\mathcal{G} = \{T_1, \ldots, T_n\}$, then

$$T_1 \cup \cdots \cup T_n \subseteq (T_1^\omega \cup T_1^{\omega+1}) \cdots (T_n^\omega \cup T_n^{\omega+1})$$

- From $\omega$ to $H$: $S^\omega$ and $S^{\omega+1}$ are $\mathcal{H}$-equivalent.
- From $H$ to $G$: every $\mathcal{H}$-class is of the form $R.\mathcal{G}$ with $\mathcal{G}$ a group.

# Completeness

1. How to prove completeness?
2. Does this work for all logics?

# Completeness

1. How to prove completeness?
   $\Rightarrow$ Generalizing well-known characterization proof by Wilke

2. Does this work for all logics?

# Completeness

1. How to prove completeness?
   $\Rightarrow$ Generalizing well-known characterization proof by Wilke

2. Does this work for all logics?
   $\Rightarrow$ No, this works only for **FO** (deeply linked to the proof)

# Completeness: What we need to prove

Reminder: $\mathrm{I}[\alpha] = \bigcap_{k \in \mathbb{N}} \mathrm{I}_k[\alpha]$. In particular, for all $k$, $\mathrm{I}[\alpha] \subseteq \mathrm{I}_k[\alpha]$.

# Completeness: What we need to prove

Reminder: $\mathrm{I}[\alpha] = \bigcap_{k \in \mathbb{N}} \mathrm{I}_k[\alpha]$. In particular, for all $k$, $\mathrm{I}[\alpha] \subseteq \mathrm{I}_k[\alpha]$.

## What we prove

For $\ell = |M|(2^{|M|})!$, the algorithm computes all maximal subsets of $\mathrm{I}_\ell[\alpha]$.
In particular, we get the bound of the "brute-force" approach for free.

## Proof technique

To every $w \in A^*$, one can associate $\mathit{Gen}_k(w) \in \mathrm{I}_k[\alpha]$:

$$\mathit{Gen}_k(w) = \{s \in M \mid \exists w' \cong_k w \text{ s.t. } \alpha(w') = s\}$$

We prove that for all $w \in A^*$, $\mathit{Gen}_\ell(w)$ is computed by the algorithm.

$\Rightarrow$ We start with a $w \in A^*$, we need a way to decompose it in a way that respects the operations of our algorithm.

# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

$w = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$
- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$

$w =$ .....

# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$
- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$

$w =$ .....

In that Case:
$x^\omega = x^{\omega+1} \Rightarrow M = \{1_M\}$

# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$
- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$

$w = w_0\ aw_1\, aw_2\, aw_3\, aw_4 \cdots\cdots\cdots\cdots aw_m$

# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

$\alpha(w_0)$ detectable
(Induction on $|A|$)

$w = \boxed{w_0}\, aw_1\, aw_2\, aw_3\, aw_4 \cdots\cdots\cdots\cdots aw_m$

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$
- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$

# Wilke Proof of the **FO** characterization

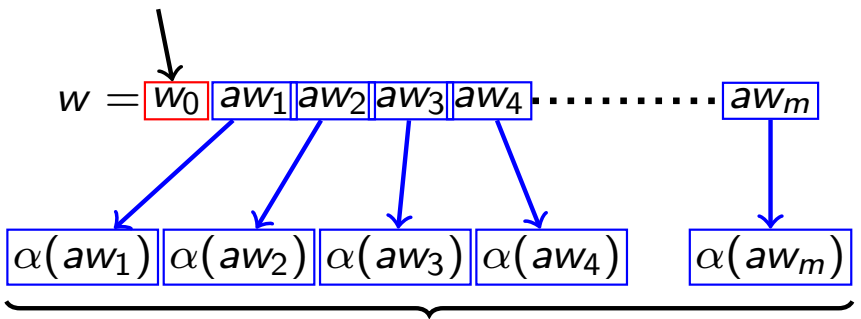
We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$

- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$

$\alpha(w_0)$ detectable
(Induction on $|A|$)




New meta-word on alphabet $\alpha(a)M$
Use a morphism $\beta : (\alpha(a)M)^* \to \alpha(a)M \subsetneq M$
Detectable by induction on $|M|$

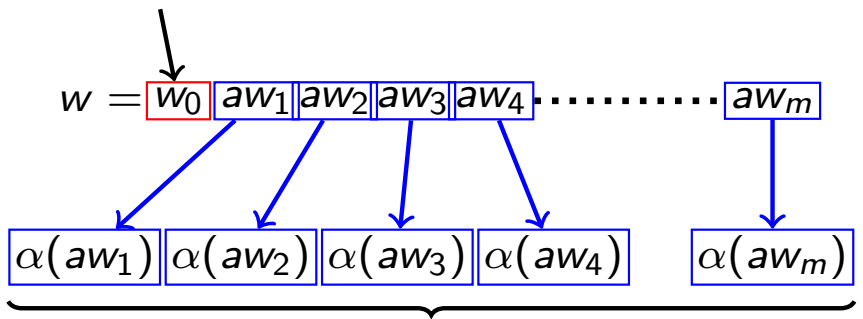
# Wilke Proof of the **FO** characterization

We have $\alpha : A^* \to M$ with $M$ satisfying $x^\omega = x^{\omega+1}$.
Let $w \in A^*$, how does **FO** proceeds to detect $\alpha(w)$?

$\alpha(w_0)$ detectable
(Induction on $|A|$)

Two Cases:

- For all $a \in A$, $\alpha(a)M = M$ and $M\alpha(a) = M$
- There exists $a \in A$ such that $\alpha(a)M \subsetneq M$ or $M\alpha(a) \subsetneq M$




New meta-word on alphabet $\alpha(a)M$
Use a morphism $\beta : (\alpha(a)M)^* \to \alpha(a)M \subsetneq M$
Detectable by induction on $|M|$

- Adapt induction: the algorithm works for smaller alphabets and smaller semigroups.
- Aperiodicity used only in the base case.

# Summary

We have

1. An algorithm for computing $\mathrm{I}[\alpha]$. Therefore, we can answer yes/no to the separation problem for **FO**.
2. A bound on the size of the separator: it is possible to compute a separator (in a very non-efficient way).

# Conclusion

We have the following results:

- Separation by **FO** is decidable (in $\mathrm{EXPTIME}$).
- Computing an actual separator formula can be done in an elementary way (but still with high complexity).
- Results can be (easily) generalized to infinite words.

Thank you!

# Computing a Separator

Assume $\mathrm{I}[\alpha] = \{S_1, \ldots, S_n\}$. By reversing the completeness proof, it is possible to compute $n$ **FO** formulas $\varphi_1, \ldots, \varphi_n$ of rank $k = |M|(2^{|M|})!$ such that:

- The associated languages are covering: $\{w \mid w \models \varphi_1 \vee \cdots \vee \models \varphi_n\} = A^*$.
- For all $i$, $w \models \varphi_i \Rightarrow \alpha(w_i) \in S_i$.
- The computation is inductive and elementary.

$\Rightarrow$ All information that can be expressed with **FO** as stated in $\mathrm{I}[\alpha]$ is a union of these formulas.