Weighted and counter systems

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Joint with Benedikt, Paul, Benjamin

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Barbizon 2011

Let's begin with the end

Thank you!

(Pot de départ fin juin)

Recipe: How to ask for a survey

1. Ask kindly for a talk (don't be too demanding)

P. Would you like to give a talk for Barbizon?

M. Sure. I'll be short in time, may I reuse an already prepared talk?

P. Perfect!

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P. Would you like to give a talk for Barbizon?

M. Sure. I'll be short in time, may I reuse an already prepared talk?

P. Perfect!

2. Wait 10 days

P. A survey would be *much* better, can you please give a title?

M. . . .

Counter and Weighted Systems

	Counter (mainly ∞)	Weighted (∞, Tempo, Mexico)
What for?	Represent runs of machines	Resources/functions
Motivation	MC for ∞ systems, complexity	Quantitative MC

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This talk: presentation of weighted automata on words.

Two popular types of Weighted Automata

1. Probabilistic automata.
2. Min-+ automata.

Can be presented uniformly.

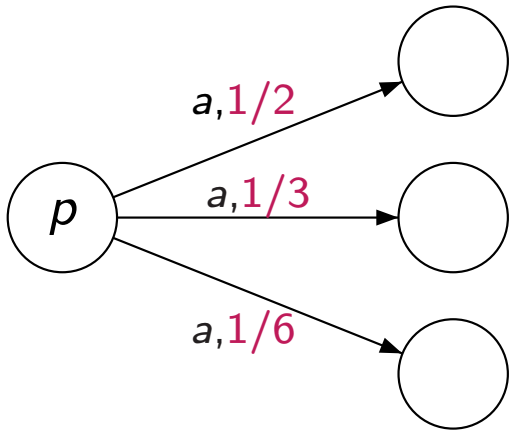
No uniform algorithms for typical questions.

Minsky machines vs. Probabilistic Automata

- Crux: Counters affect runs in Minsky machines, not in probabilistic automata.
- Runs are aggregated in probabilistic automata to compute a function.

	Minsky	Prob. Automaton
Guards	$= 0?$	None
Updates along transition	$+1, -1$	$\times \alpha$
Semantic of nondeterminism	Boolean	Sum

Probabilistic Automata



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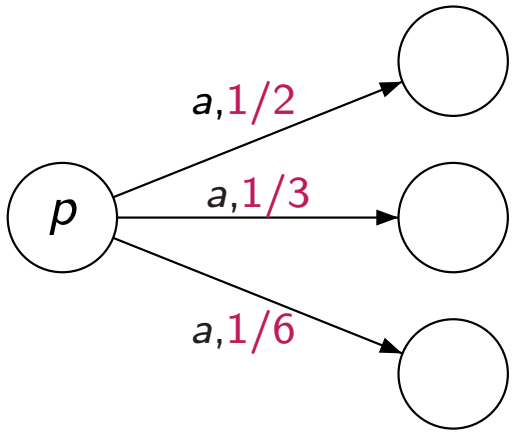
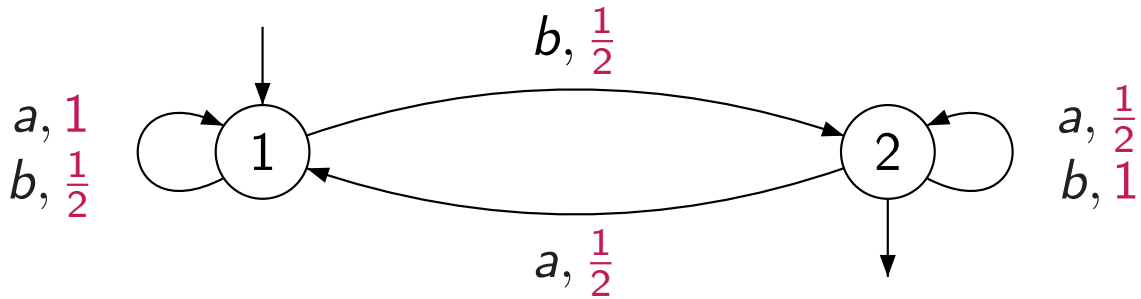


- Weight (probability) of a run: product of all transition weights.
  word: sum of weights of all successful runs.



- $[\![\mathcal{A}]\!](ba) = 1/4$
- Interpreting (a, b) as $(0, 1)$: $[\![\mathcal{A}]\!](a_1 \cdots a_n) = 0.a_n \cdots a_1$

3 Questions for Probabilistic Automata

1. Equality: $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$?

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   Undecidable (Paz'71) PCP

   From morphisms $f, g : \{0,1\}^* \to \{0,1\}^*$, compute $1/2[1 - (\overline{f(w)} - \overline{g(w)})]$

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3. 0 value: are there words accepted with arbitrarily small probability?

   Undecidable (Gimbert & Oualhadj'10)

Min-+ Automata

- Costs in $\mathbb{N} \cup \{\infty\}$.
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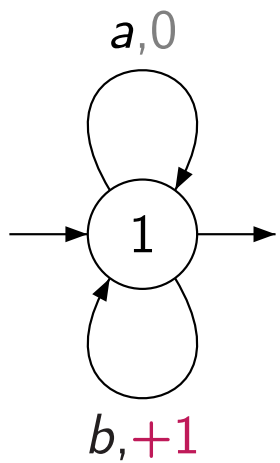
Weight of a word u = minimal cost for going from Init to Final reading u.



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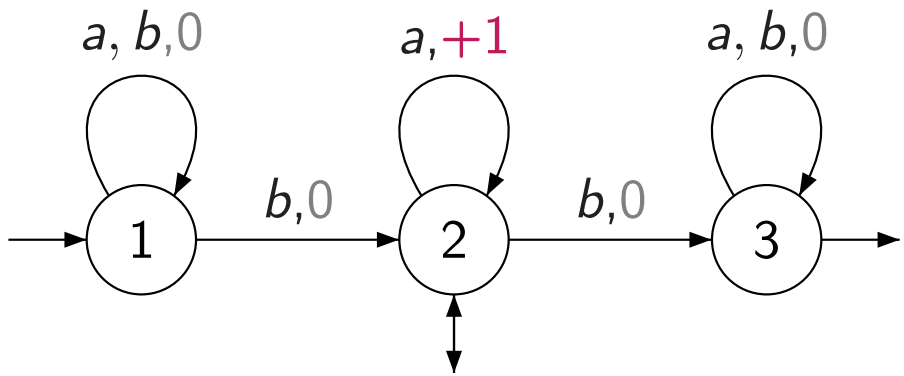

$[\![\mathcal{B}]\!](aaaabaabaaa) = 2$

$[\![\mathcal{B}]\!](u)$: min length of block of a's

From Probabilistic Automata to min-+ Automata

1. Transform product into sum: apply $\log \rightsquigarrow$ high probabilities yield small distances

$$([0,1],+,\times) \xrightarrow{-\log} (\mathbb{R}_{\geq 0} \cup \infty, \oplus, +)$$
$$xy \rightsquigarrow -\log(xy) = -\log(x) + -\log(y)$$
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The 3 Questions after log-transforming

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$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$	D	$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$
$[\![\mathcal{A}]\!] > \alpha$	U	$[\![\mathcal{A}]\!] \leq k$
0-value	U	Is $[\![\mathcal{A}]\!]$ bounded?

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Problems very sensitive to underlying semiring.

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- Equality $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$: Undecidable! [Krob'94].
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Finite power property

- Given L regular,

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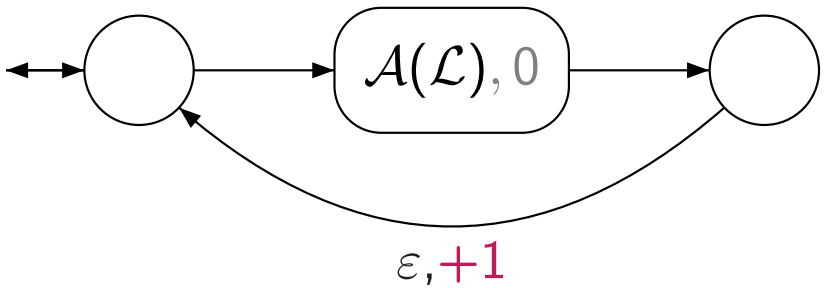
- Thomson's construction for star:



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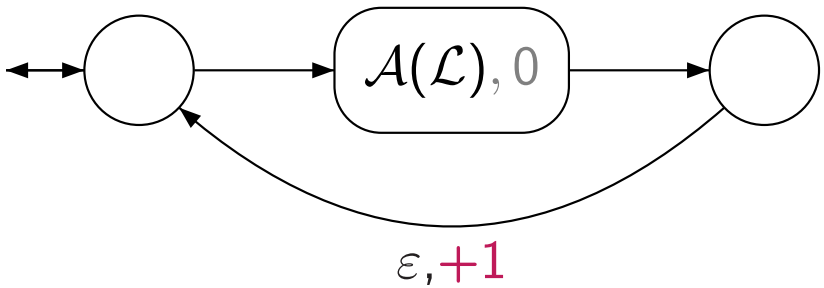
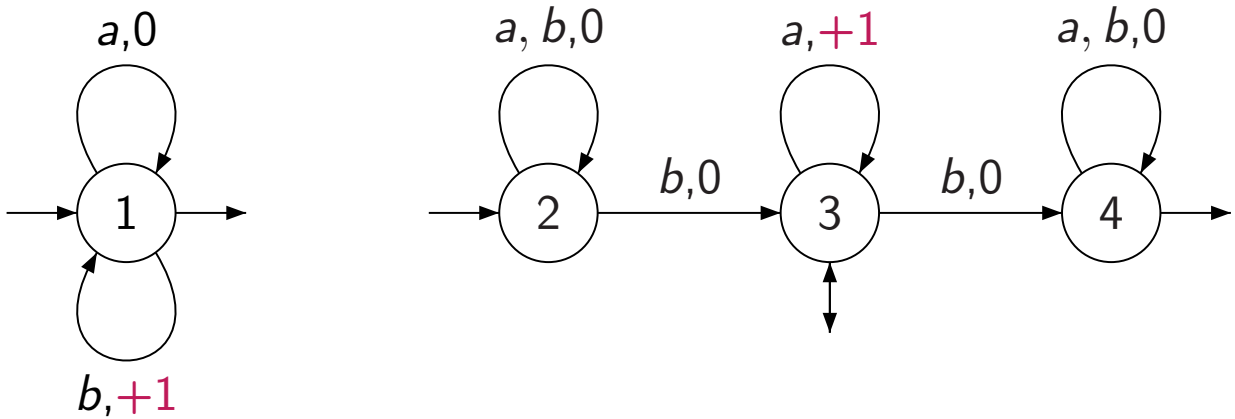


New automaton is limited iff. L has the finite power property.

- Generalization to multiple counters (Kirsten) $\rightsquigarrow$ star-height problem.
- Note: universality is a particular case of limitedness

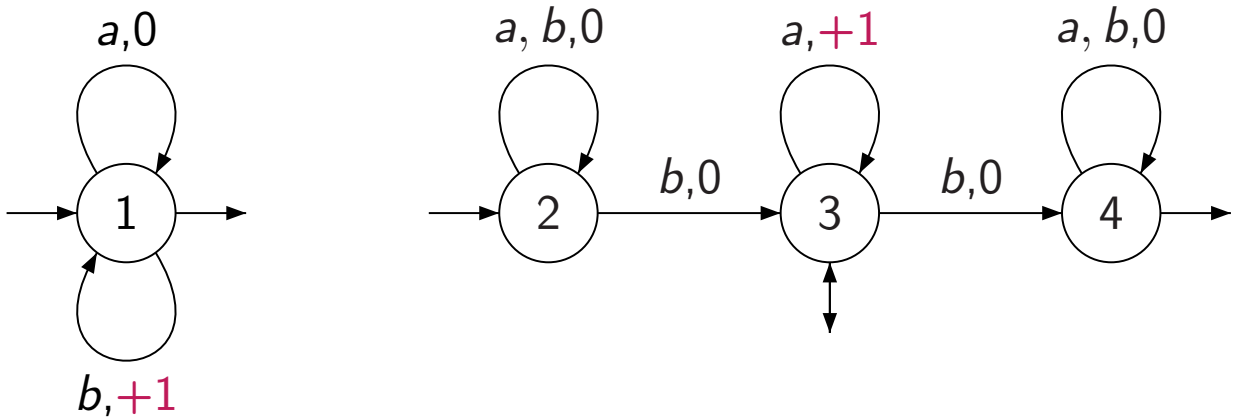
  ($\Longrightarrow$ limitedness is PSPACE-hard).

Limitedness is decidable



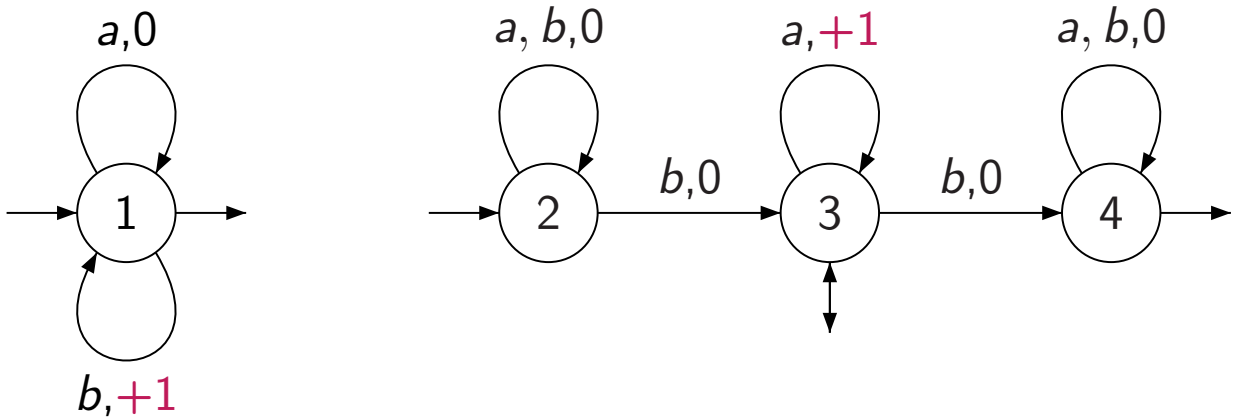
- Transformation monoid: $a = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix}$
- Problem: the monoid generated is infinite.
- Solution: Abstract by projecting onto $\{0, 1, \infty\}$: $0 \mapsto 0$, $n \mapsto 1$, $\infty \mapsto \infty$.

Limitedness is decidable



- Loss of precision in the abstraction: $aa = a = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix}$.
- Add a new value ω meaning "unbounded".
- New operation $\#$ "iterate a large number of times": $a^{\#} = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & \omega & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix}$
- Think of $(a^{\#}b)^{\#}$ as $(a^n b)^n$ for large n.

Limitedness is decidable



- Algorithm: compute the $(\#,\cdot)$ monoid generated by letter matrices.
- If for some matrix M, $\min(\text{Init}, M, \text{Final}) = \omega$: witness of non-limitedness.

$$\text{Example: } (a^{\#}b)^{\#}a^{\#} = \begin{pmatrix} \omega & \infty & \infty & \infty \\ \infty & 0 & \omega & \omega \\ \infty & \infty & \infty & \omega \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

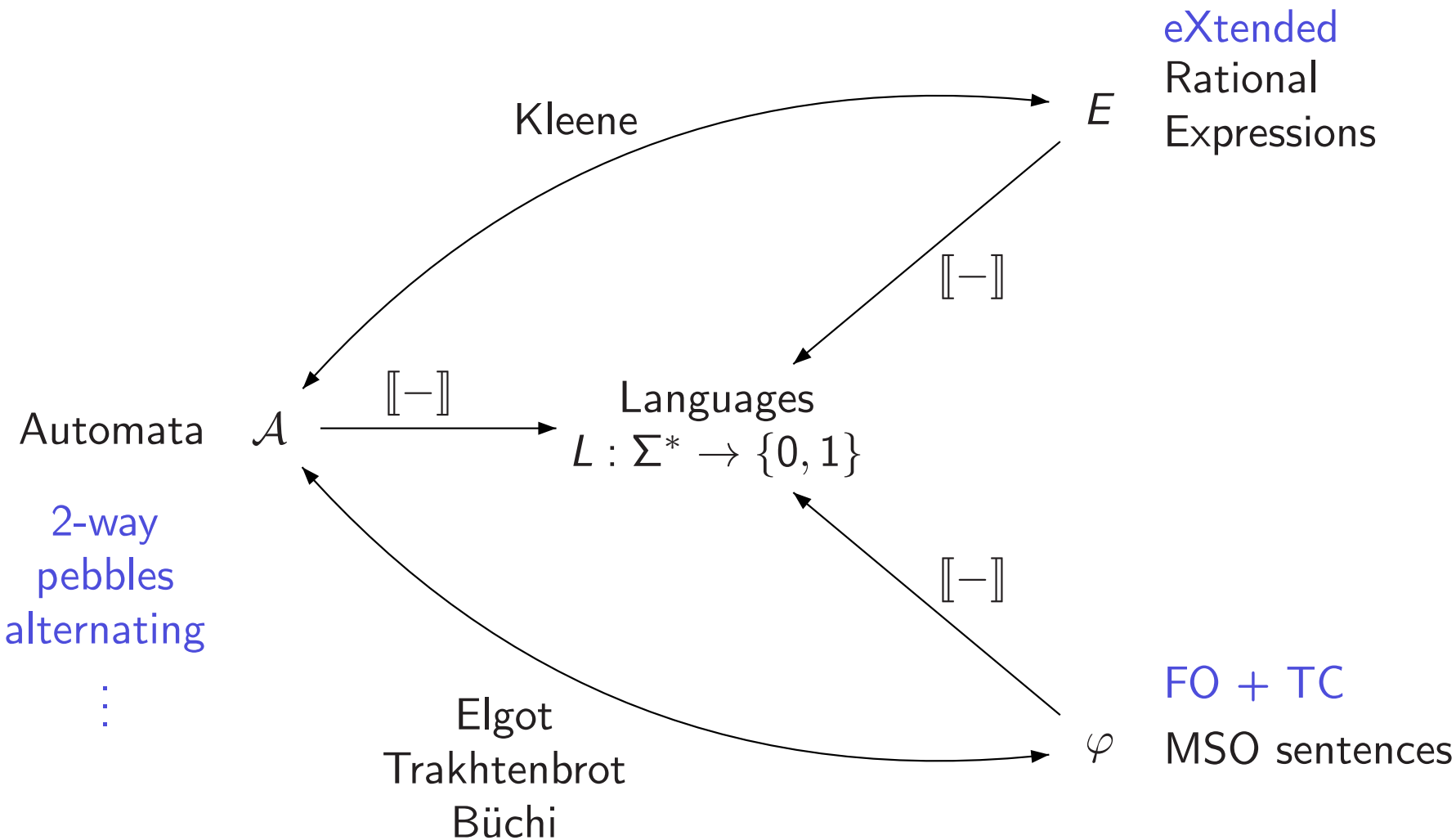
- Otherwise: this shows that "regular behaviors" à la $(a^n b)^n$ are limited.
  Difficult: show that all behaviors are also limited. Algebraic tools of I. Simon

Specifying quantitative properties

Develop high-level denotational formalism

- to express easily quantitative properties on words/trees,
- should allow us to compute arithmetic expressions (possibly guarded by logical conditions written in a standard language (eg., FO or XPath),
- should have an equivalent operational model.

On words: what remains true for weights?



Boolean: $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$

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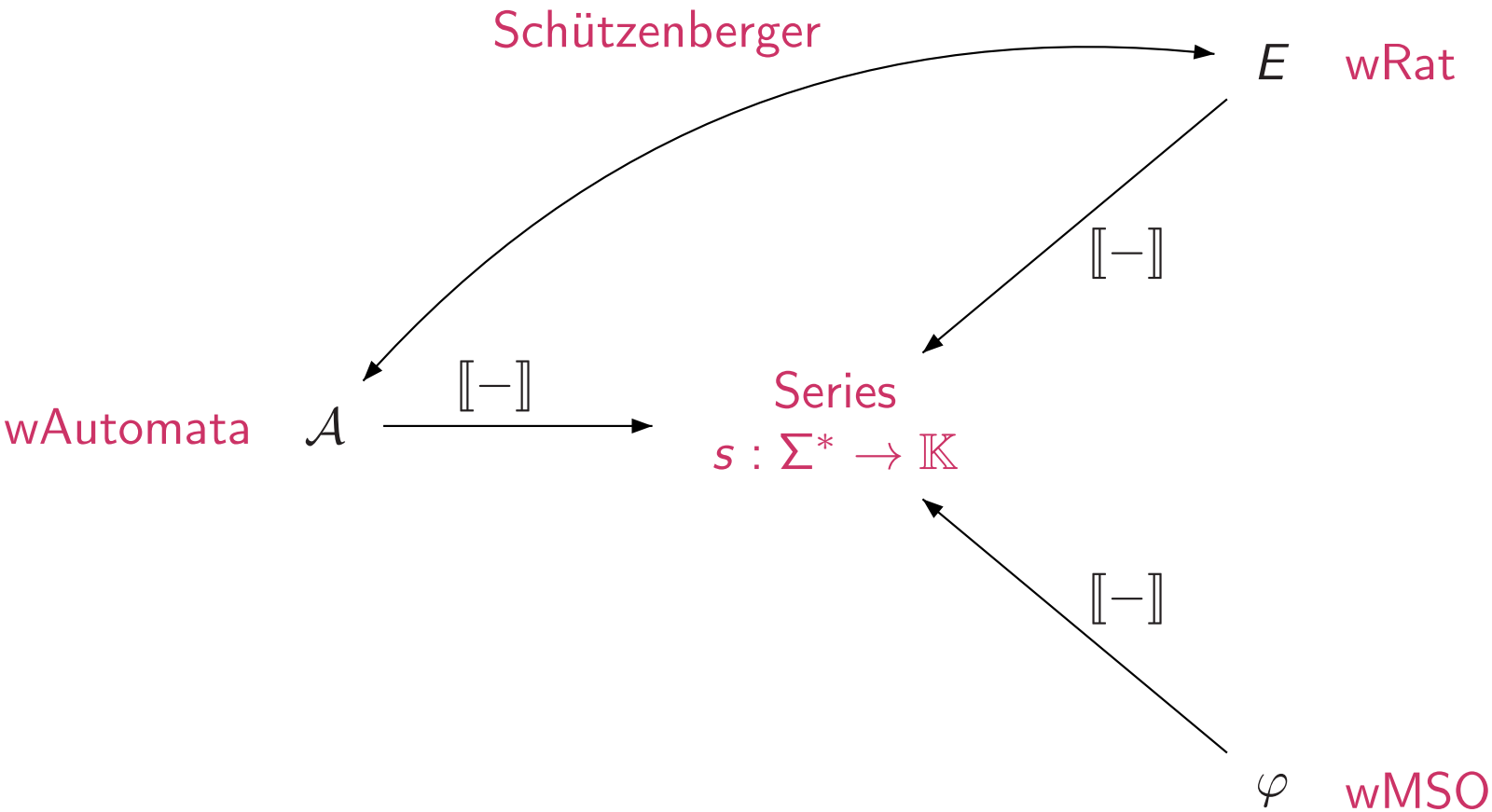
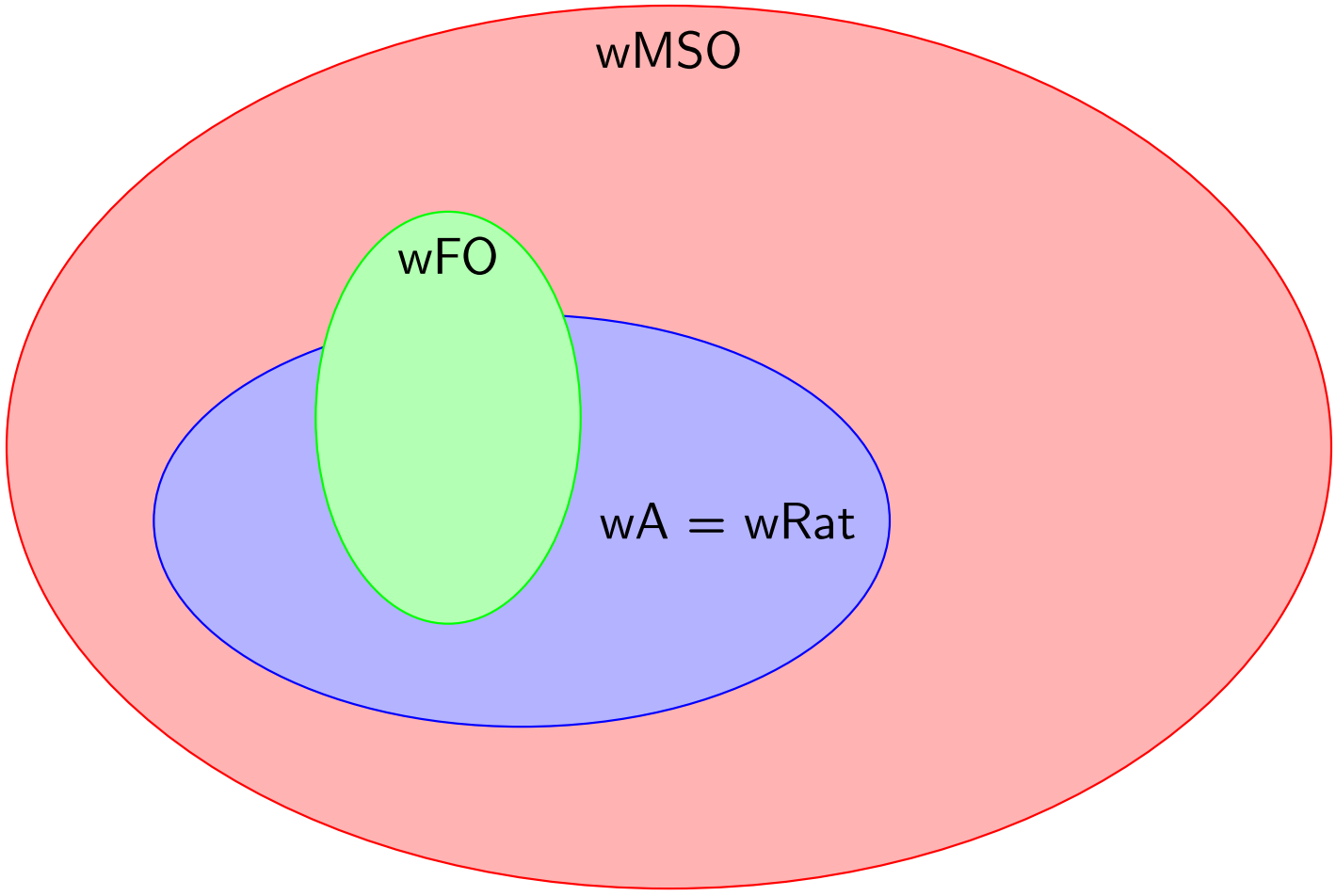


Quantitative: $\mathbb{K} = (K, +, \times, 0, 1)$

Expressiveness in the weighted setting



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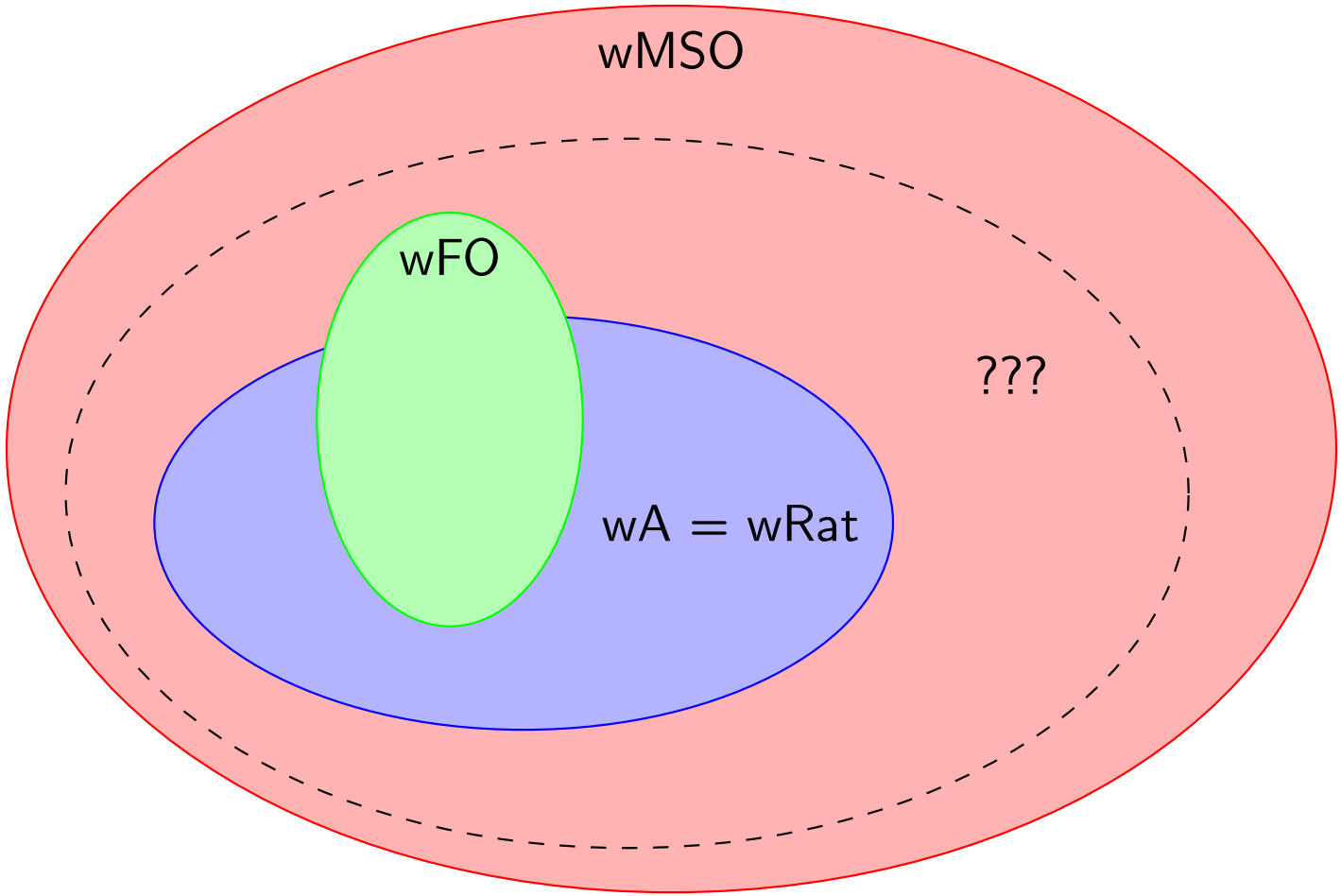

Find a robust class containing both wFO and wAutomata.

Weighted Expressions

Syntax of WE($\mathcal{L}$)

Fix $\mathcal{L}$ a logic (eg, MSO, FO($<$)).

$$E ::= \varphi \mid k \mid E \oplus E \mid E \otimes E \mid \bigoplus_x E \mid \bigotimes_x E$$

where $\varphi \in \mathcal{L}$, $k \in K$, x is a first-order variable.

Semantics

- An expression E without free variables defines a mapping $[\![\varphi]\!] : \Sigma^+ \to K$.
- For $\varphi \in \mathcal{L}$, we have $[\![\varphi]\!](w) \in \{0, 1\}$ (in the chosen semiring).

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- $\bigoplus_x \varphi$ interpreted as a sum over all positions.
- $\bigotimes_x \varphi$ interpreted as a product over all positions.

Weighted expressions: examples

On $(\mathbb{N}, +, \times)$:

- $[\![\bigoplus_x a(x)]\!](u) = \sum_{i \in \mathrm{pos}(u)} [\![a(x)]\!](u, i) = |u|_a$ recognizable by wA

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w-Automata are not closed under $\bigotimes$.

Capturing Weighted Automata

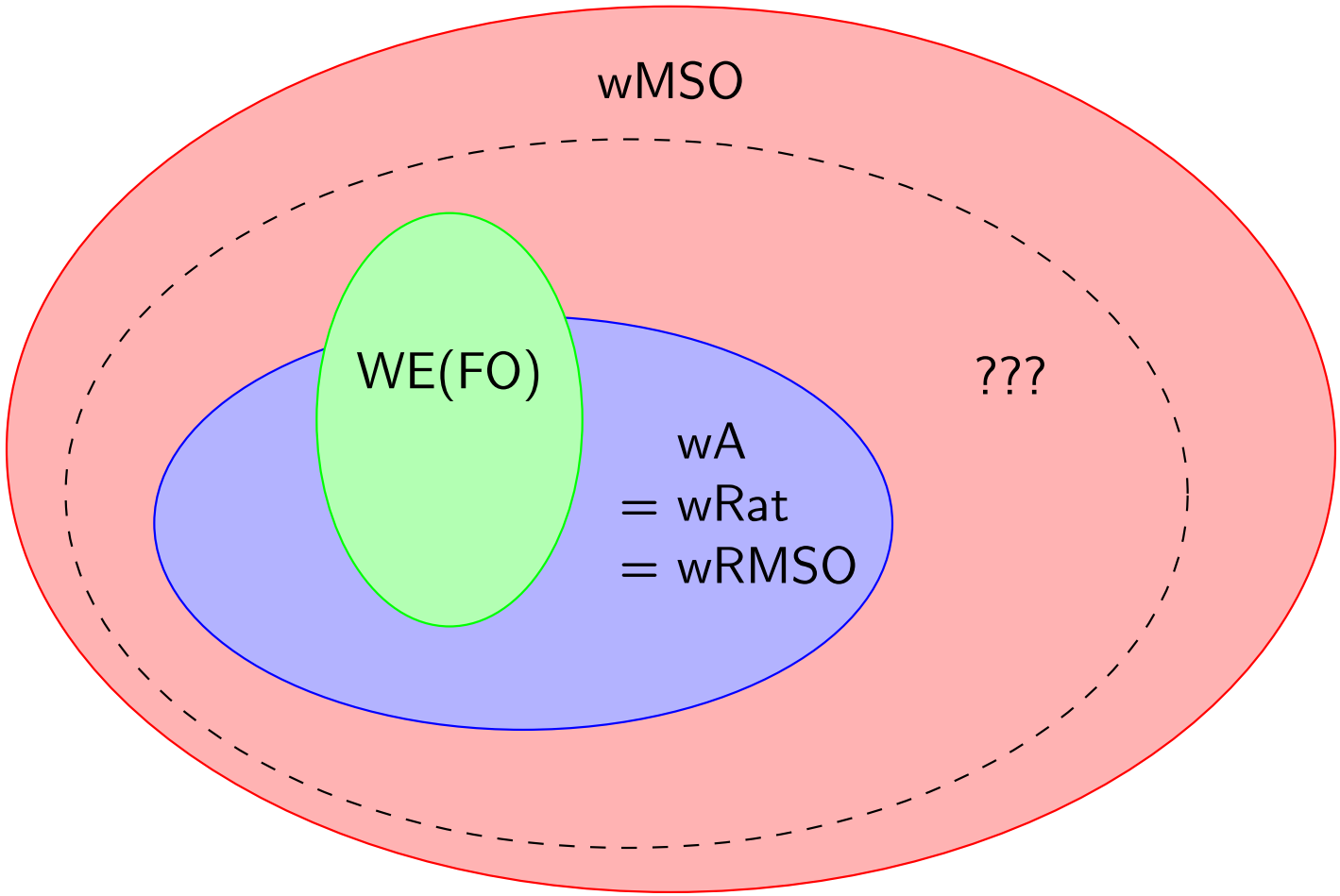
Theorem (Droste & Gastin'05)

$$\text{wAutomata} = \text{wRMSO}$$

wRMSO consists of weighted expressions with

- $\bigotimes_x$ restricted to $\bigvee \bigwedge$ of constants and boolean formulae.
- A new second order weighted operator, $\bigoplus_X$, restricted to boolean formulae.

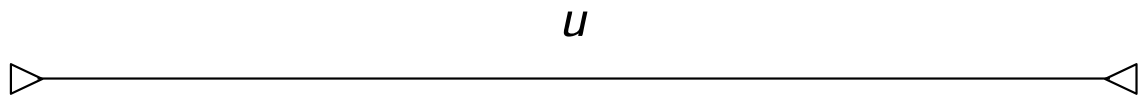
Extending instead of Restricting ?



Aim: robust class extending both WE(FO) and wAutomata.

(2-way) Pebble weighted automata

- Automaton with 2-way mechanism and pebbles $\{1, \ldots, r\}$.



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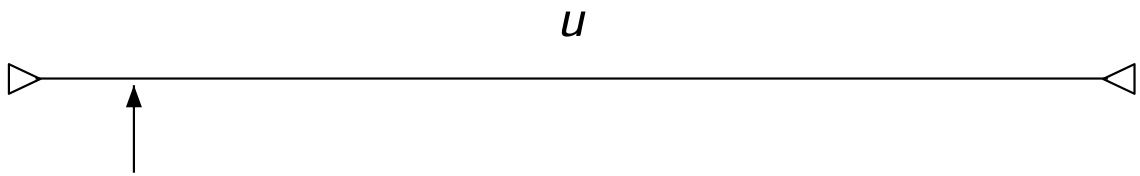

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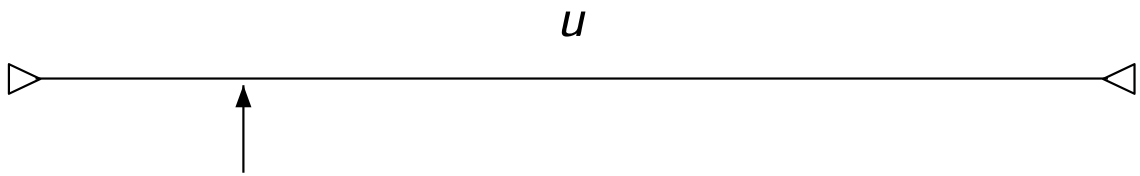

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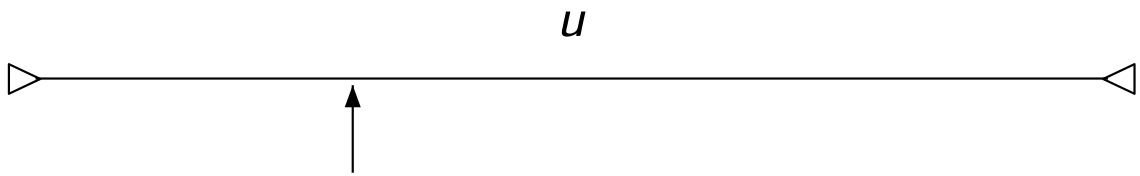

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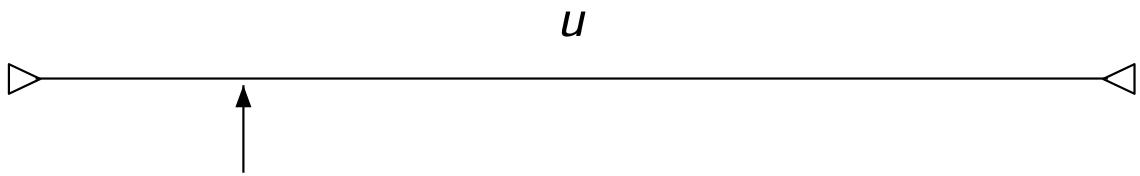

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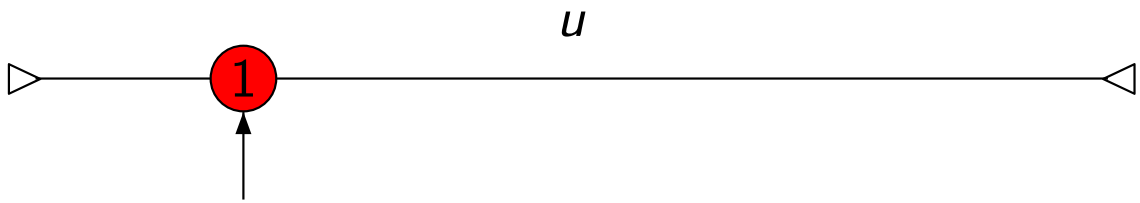

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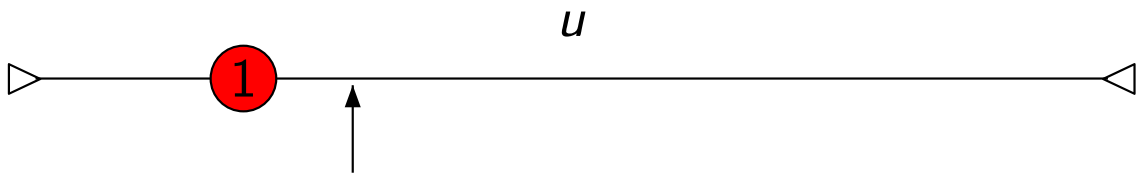

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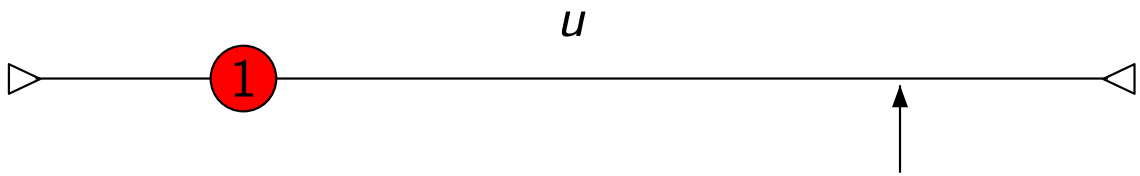

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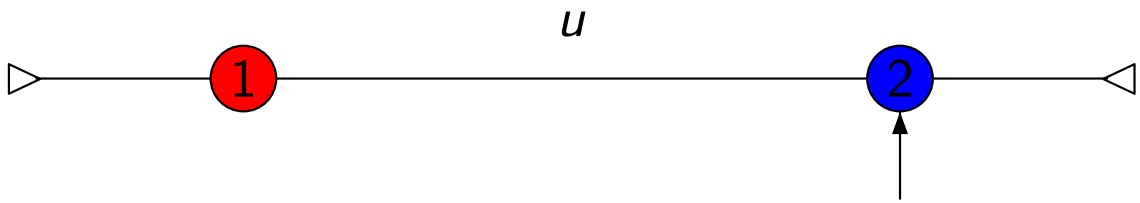

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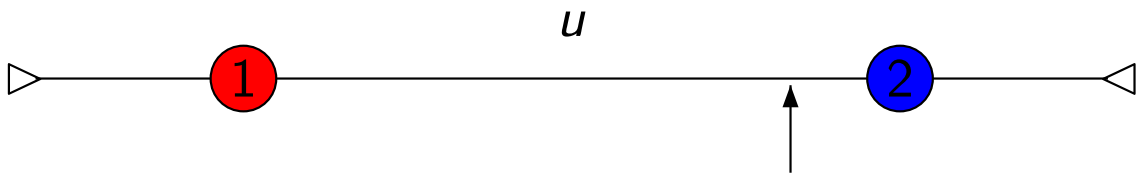

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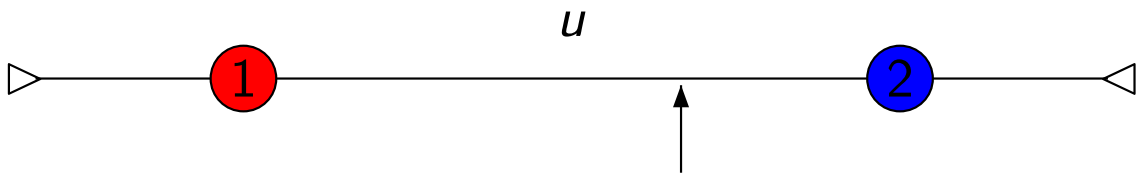

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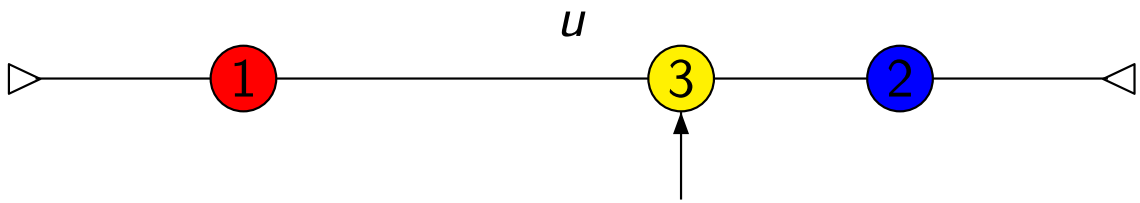

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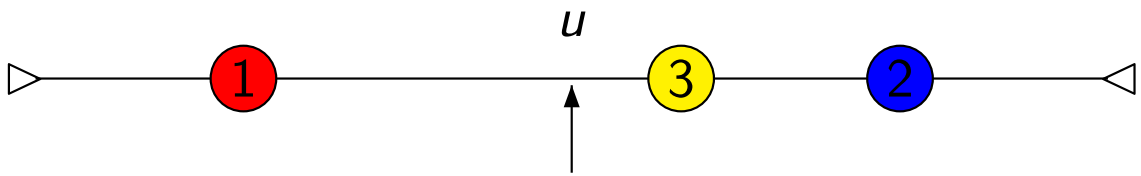

- Applicable transitions depend on current (state,letter,pebbles).

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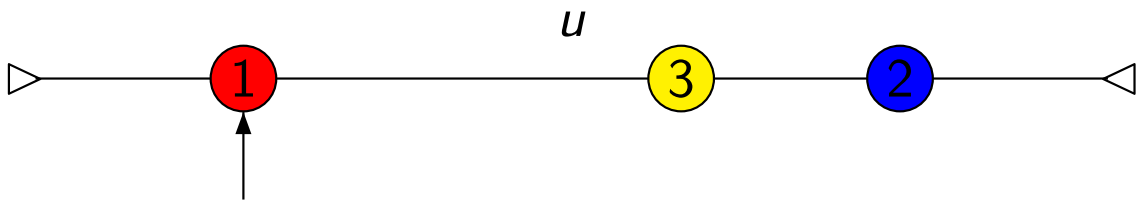

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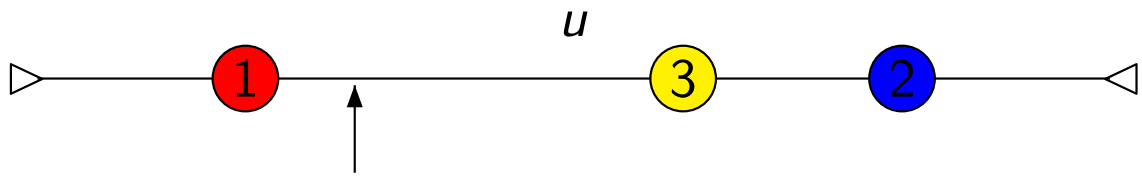

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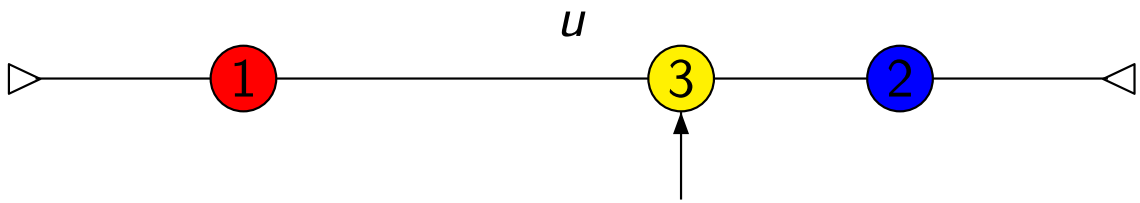

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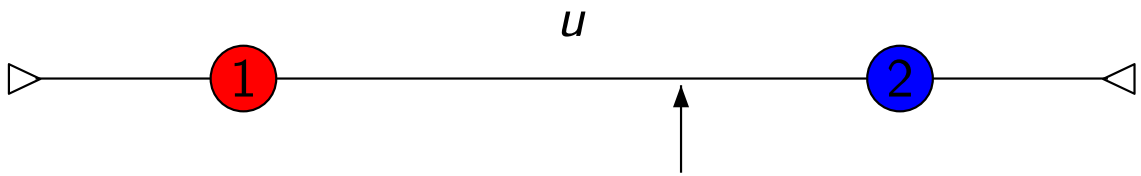

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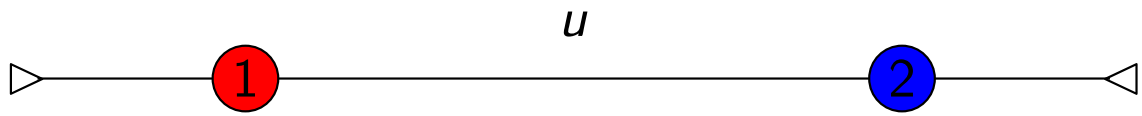

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- Note. For Boolean word automata, this does not add expressive power.

Pebble weighted automata: semantics

Recall from the classical setting:

- Value of a word: sum of all weights of runs on this word.

$$[\![\mathcal{A}]\!](w) = \sum_{\rho \text{ run of } \mathcal{A} \text{ on } w} \text{weight}(\rho)$$

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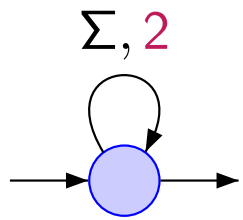
(Other solution would be to restrict to suitable semirings)

Why pebbles?

- Pebbles can be used to encode a first-order variable (its position).



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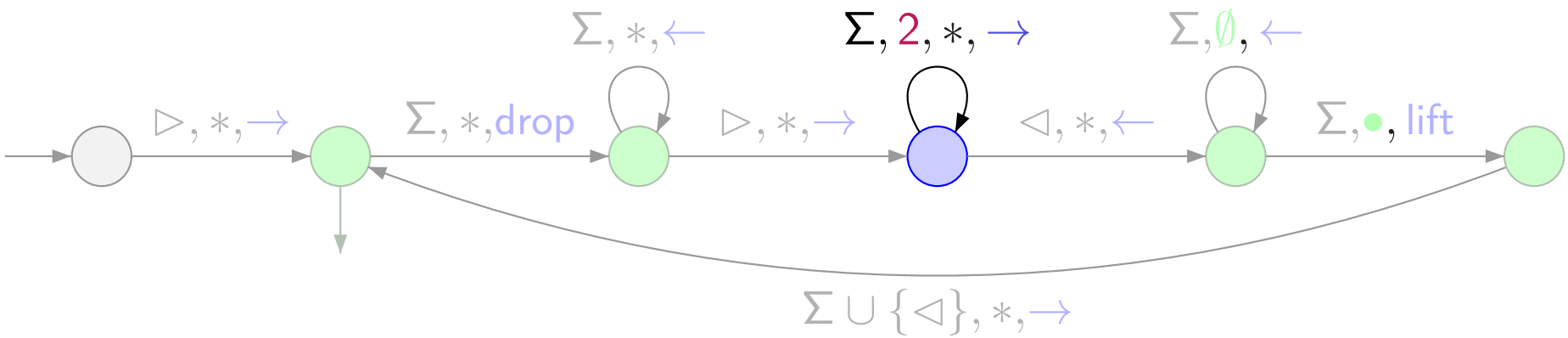

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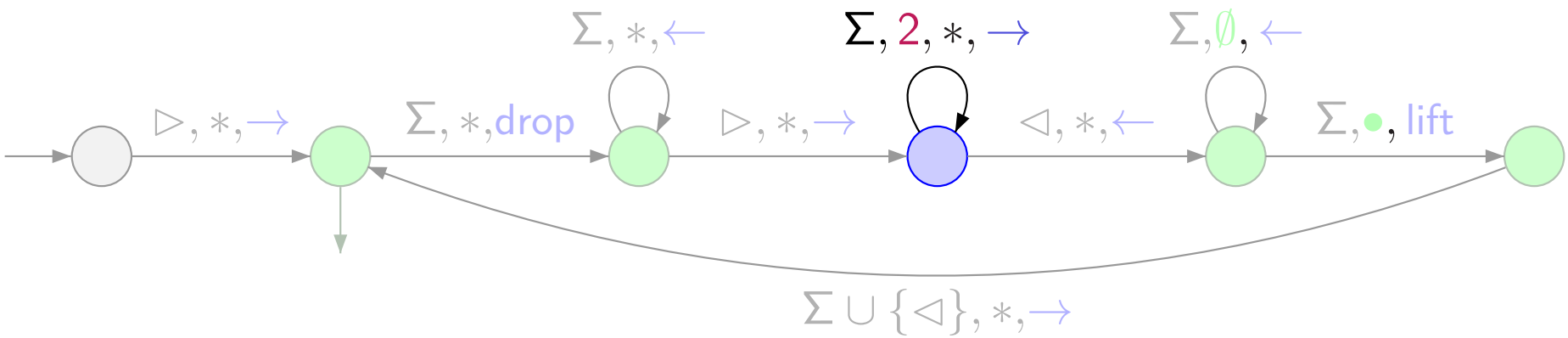

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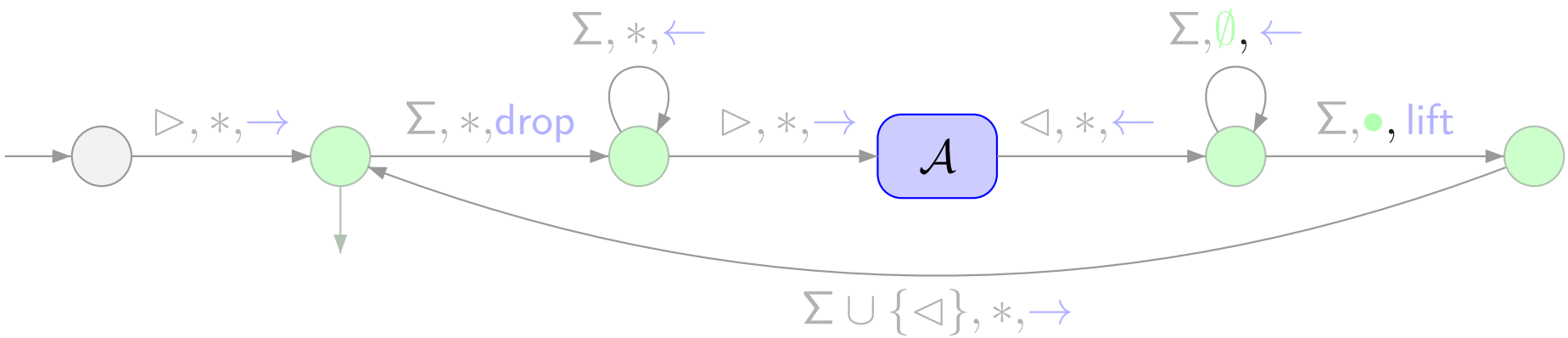

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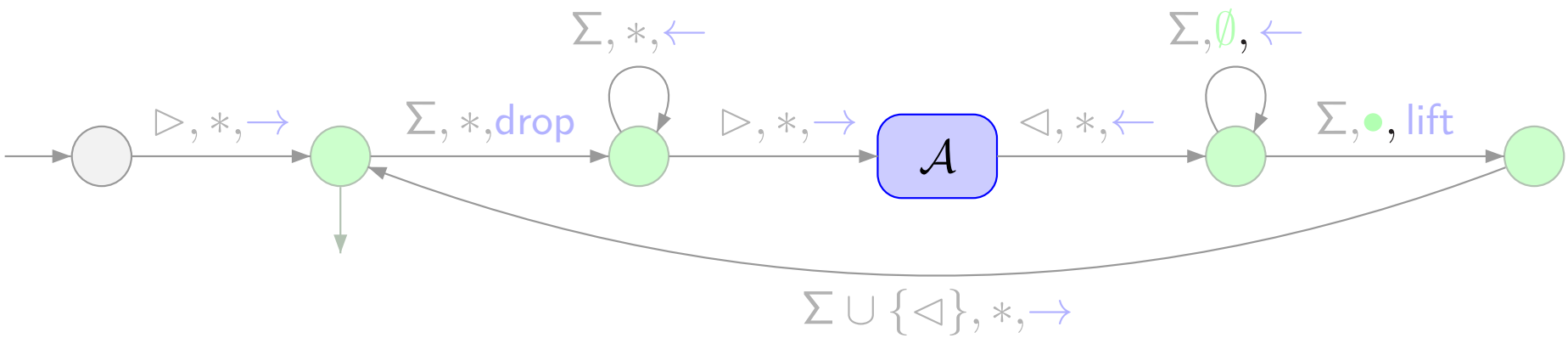

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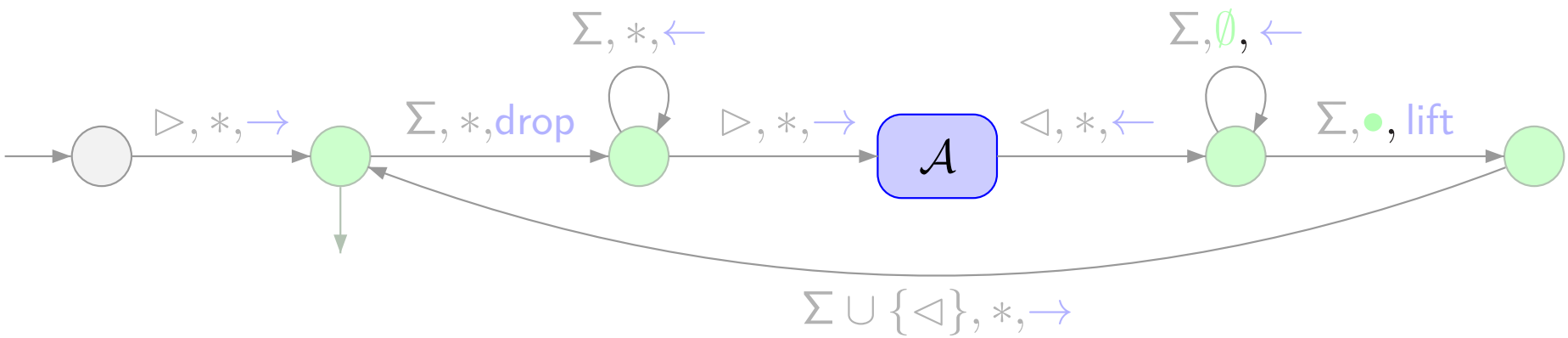

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- Very same idea: pebble weighted automata are closed under $\bigoplus_x$.
  by dropping nondeterministically the pebble instead.
- Summary: pebble wA easily bring closure under $\bigotimes_x$ and $\bigoplus_x$.

1-way pebble weighted automata

The construction for closure under $\bigoplus_x$ and $\bigotimes_x$ uses specific automata.

- After a drop, go to the end of the word and reset.
- No other use of $\leftarrow$ move.

1-way pebble automata with ℓ-resets

A 1-way pebble automaton is a 2-way pebble automaton st.

- no $\leftarrow$ move. Replaced by new move: reset.
- no lift can be immediately followed by a drop,
- each time a pebble is dropped, it gets a credit for ℓ resets (recursively).

Similar to 1-way automata used by Neven, Schwentick, Vianu 04 for data words.

Closure properties of 1 way pwA

- 1-way pebble automata are nonlooping (pebble are "progressing").
- Semantics well-defined as $[\![\mathcal{A}]\!](w) = \sum_{\rho \text{ run of } \mathcal{A} \text{ on } w} \text{weight}(\rho)$.
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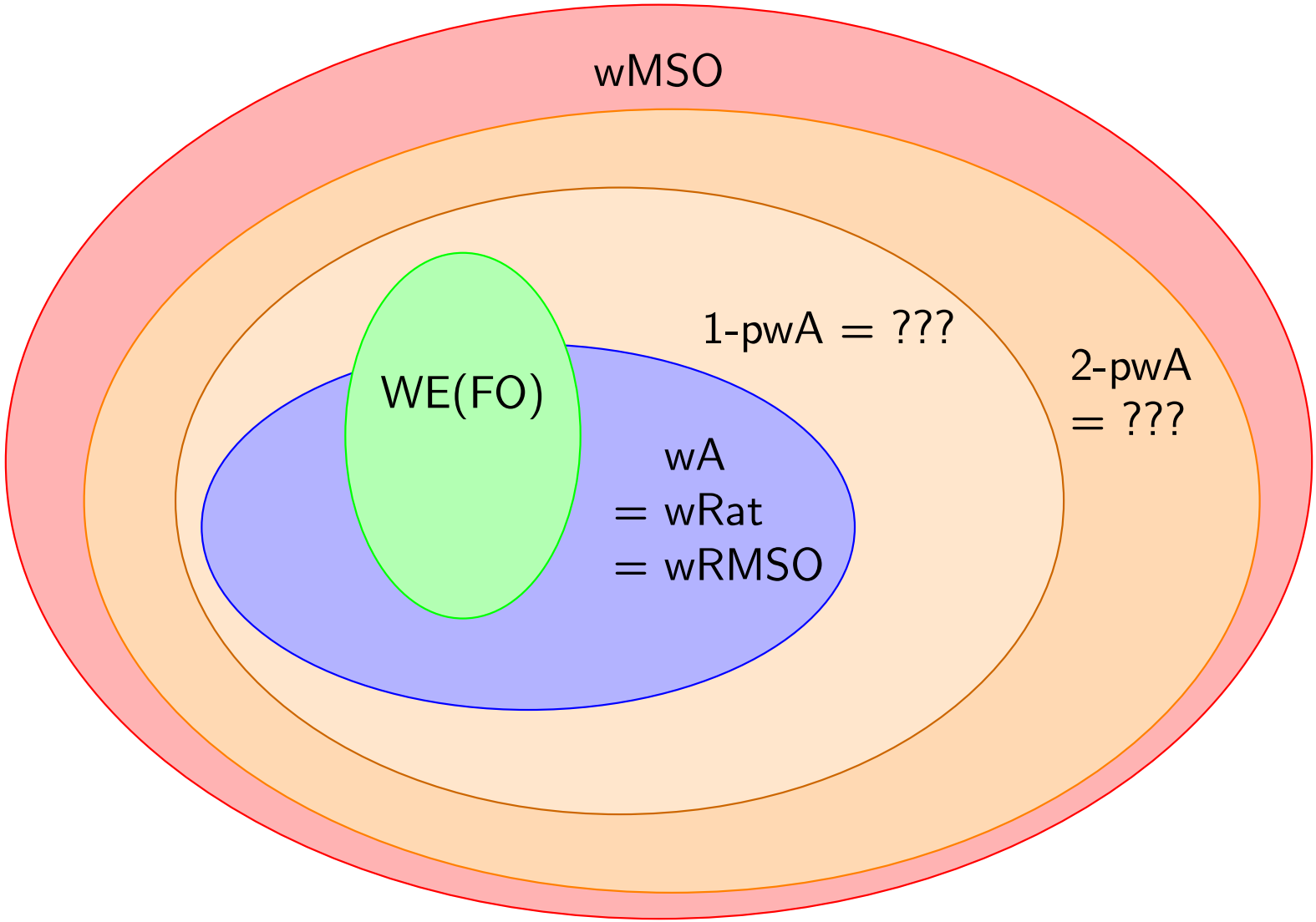
Corollary

1-way pebble weighted automata capture WE(MSO).

For the converse, need to enrich the weighted expressions

Pebble weighted Automata vs WE(MSO)



Characterization of 1-way and 2-way pebble wA in terms of expressions?

Weighted transitive closure: TC and BTC

- Boolean setting: MSO $\equiv$ FO($<$) + Transitive closure.

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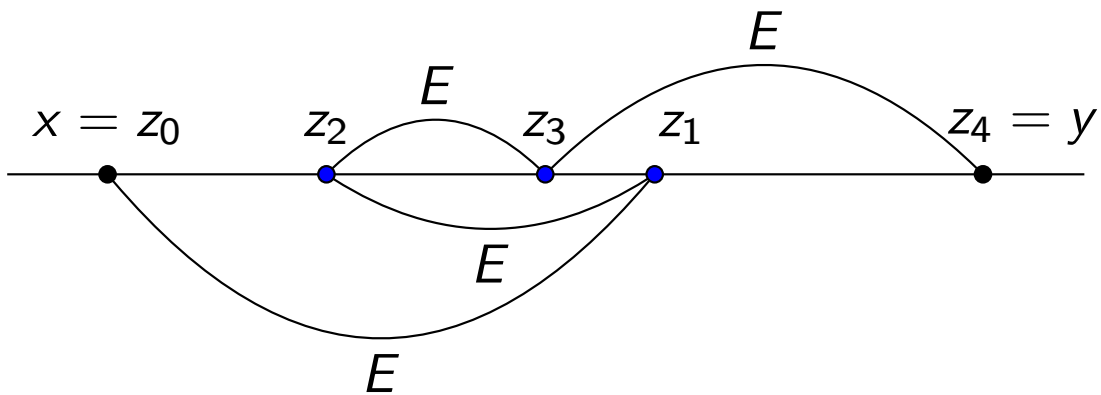
- Weighted: for $E(x, y)$ with (at least) two first order free variables, let

$$E^n(x, y) = \bigoplus_{x=z_0, z_1, \ldots, z_n=y} \left(\bigotimes_{1 \leq \ell \leq n} E(z_{\ell-1}, z_\ell) \right)$$

where the sum ranges over seq. of pairwise distinct positions $z_0, \ldots, z_n$.



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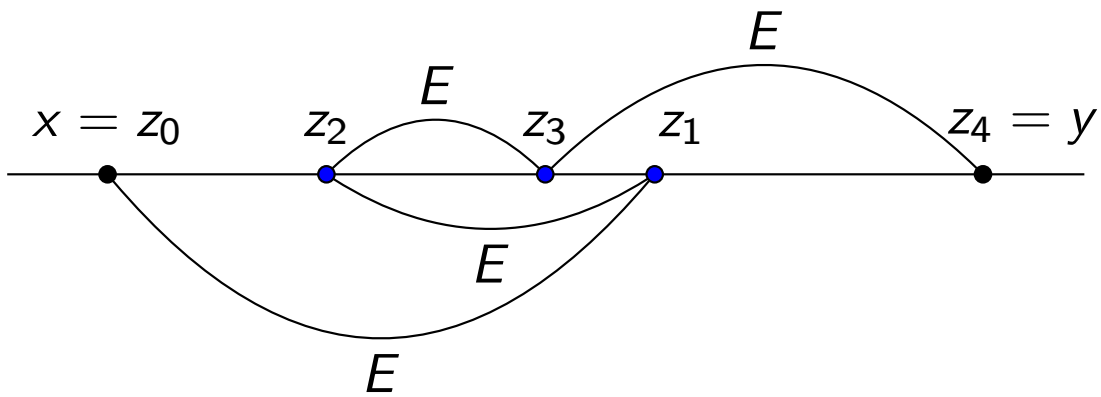

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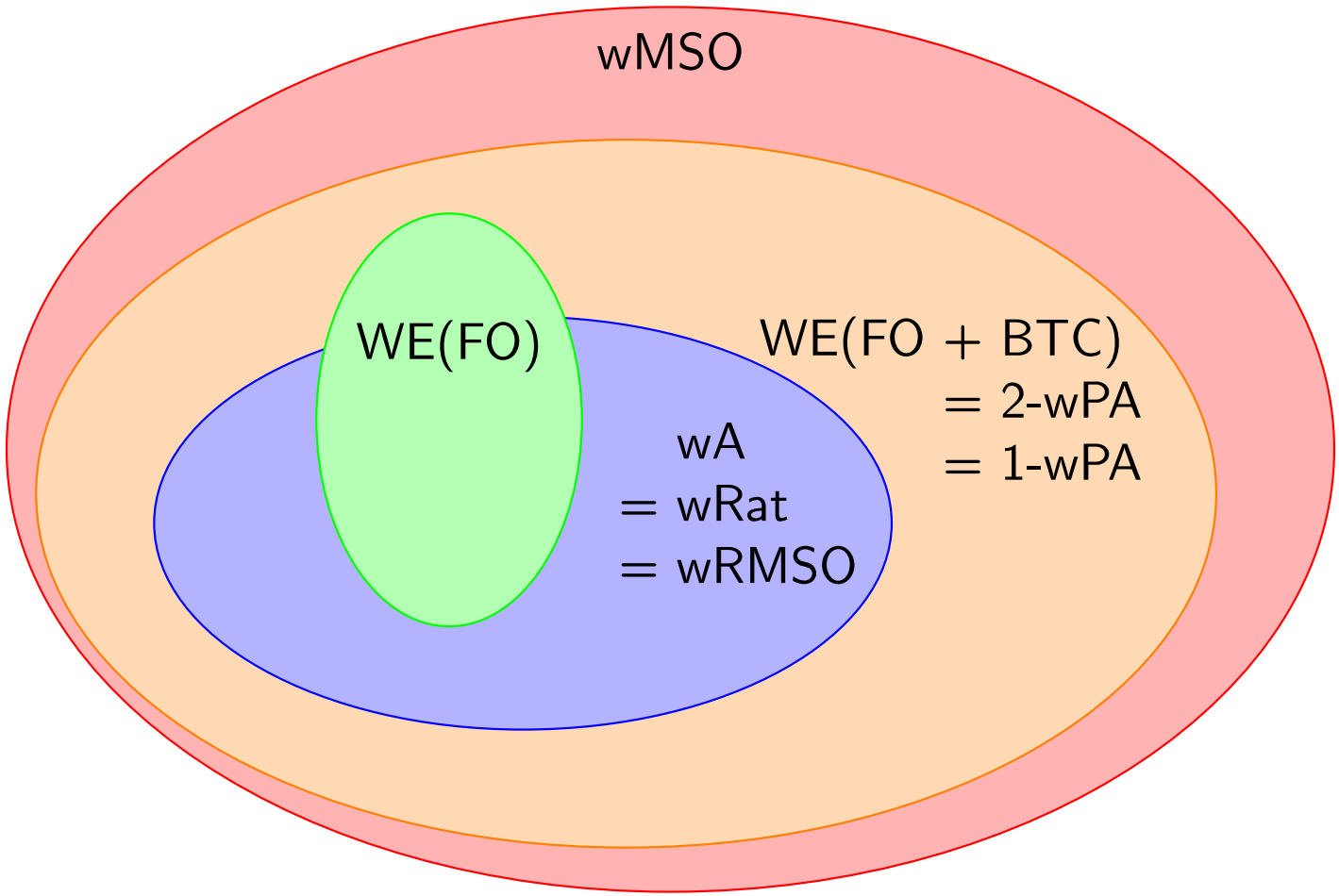
- Bounded transitive closure BTC: $N\text{-}\mathrm{TC}_{xy} E = \mathrm{TC}_{xy}(E \wedge |x - y| \leq N)$

Expressiveness

Theorem: WE(FO + BTC) = 2-way pebble wA = 1-way pebble wA.

(On commutative semirings)



Related and Further Work

- Other nice framework in min-+: cost functions, S/B-automata
  [Th. Colcombet, M. Bojańczyk]

$$\text{Logics} \longleftrightarrow \text{Automata}$$

- Lack of results for proving non-expressiveness.
- Unbounded steps in transitive closure?
- Weak pebbles vs. strong pebbles?
- Extensions to other structures: Trees ($\rightsquigarrow$ Benjamin's talk)