A proof of Kummer's theorem

Bruno Gauthier *

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Abstract

Following suggestions of T. H. Koornwinder [3], we give a new proof of Kummer's theorem involving Zeilberger's algorithm, the WZ method and asympotic estimates. In the first section, we recall a classical proof given by L. J. Slater [7]. The second section discusses the new proof, in the third section sketches of similar proofs for Bailey's and Dixon's theorems are given.

1 Slater's proof

In classical hypergeometric theory, formulae concerning summation, transformation and contiguous relations are provided in order to manipulate expressions that represent special functions. One of those is Kummer's non-terminating summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)}\ , \tag{1} $$

in which $Re(b) < 1$ ensures the series to be convergent.

The first step in Slater's proof is to prove the following quadratic transformation, due to Kummer:

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix}; z\right] = (1-z)^{-a}\ {}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{1}{2}+\frac{a}{2}-b \\ 1+a-b \end{matrix}; \frac{-4\,z}{(1-z)^2}\right]\ . \tag{2} $$

The proof of this transformation is as follows.
The right-hand side of (2) can be written:

$$ \begin{aligned} & (1-z)^{-a} \sum_{k=0}^{\infty} \frac{(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2}-b)_k}{(1+a-b)_k\,k!}\,\frac{(-4)^k\,z^k}{(1-z)^{2\,k}} \\ = & \sum_{k=0}^{\infty} \frac{(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2}-b)_k\,(-4)^k}{(1+a-b)_k\,k!}\,z^k\,(1-z)^{-a-2\,k}\ , \end{aligned} $$

*Bruno.Gauthier@univ-mlv.fr, *Institut Gaspard Monge*, Université Marne La Vallée, Cité Descartes, 5 bd Descartes, Champs sur marne, F-77454 Marne La Vallée Cedex 2.

$$= \sum_{k=0}^{\infty} \frac{(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2}-b)_k\,(-4)^k}{(1+a-b)_k\,k!}\,z^k\,{}_1F_0\left[\begin{matrix}a+2\,k\\ -\end{matrix};z\right] \qquad \text{by the binomial theorem,}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2}-b)_k\,(-4)^k}{(1+a-b)_k\,k!}\,\frac{(a+2\,k)_n}{n!}\,z^{k+n}\;.$$

Extracting the coefficient of z^N yields:

$$\sum_{k=0}^{N} \frac{(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2}-b)_k\,(-4)^k\,(a+2\,k)_{N-k}}{(1+a-b)_k\,(N-k)!\,k!} = \frac{(a)_N}{N!}\sum_{k=0}^{N}\frac{(\frac{1}{2}+\frac{a}{2}-b)_k\,(a+N)_k\,(-N)_k}{(1+a-b)_k\,(\frac{1}{2}+\frac{a}{2})_k\,k!} \tag{3}$$

since:

$$\frac{1}{(N-k)!} = \frac{(-1)^k\,(-N)_k}{N!}$$

and

$$(a+2\,k)_{N-k} = \frac{(a)_{N+k}}{(a)_{2\,k}} = \frac{(a)_N\,(a+N)_k}{4^k\,(\frac{a}{2})_k\,(\frac{1}{2}+\frac{a}{2})_k}\;.$$

The right-hand side of (3) is summable by Saalschütz theorem [7, (III.2)]:

$$\frac{(a)_N}{N!}\;\frac{(\frac{1}{2}+\frac{a}{2})_N\,(1-b-N)_N}{(1+a-b)_N\,(\frac{1}{2}-\frac{a}{2}-N)_N}\;.$$

As this is the coefficient of z^N in the right-hand side of (2), we obtain:

$$\sum_{N=0}^{\infty}\frac{(a)_N}{N!}\;\frac{(\frac{1}{2}+\frac{a}{2})_N\,(1-b-N)_N}{(1+a-b)_N\,(\frac{1}{2}-\frac{a}{2}-N)_N}\,z^N = \sum_{N=0}^{\infty}\frac{(a)_N\,(b)_N}{(1+a-b)_N\,N!}\,z^N = {}_2F_1\left[\begin{matrix}a,b\\ 1+a-b\end{matrix};z\right]$$

which proves Kummer's quadratic transformation.

The second step is to set z to -1 in (2). Therefore

$${}_2F_1\left[\begin{matrix}a,b\\ 1+a-b\end{matrix};-1\right] = 2^{-a}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{1}{2}+\frac{a}{2}-b\\ 1+a-b\end{matrix};1\right]\;. \tag{4}$$

Note that these series are convergent if $Re(b) < 1$.
We can sum the series on the right-hand side by Gauss's theorem [7, (III.3)]. This yields

$${}_2F_1\left[\begin{matrix}a,b\\ 1+a-b\end{matrix};-1\right] = 2^{-a}\,\frac{\Gamma(1+a-b)\,\Gamma(\frac{1}{2})}{\Gamma(1+\frac{a}{2}-b)\,\Gamma(\frac{1}{2}+\frac{a}{2})}\;. \tag{5}$$

Since $\Gamma(\frac{1}{2}+\frac{a}{2})\,\Gamma(\frac{a}{2}+1) = 2^a\,\Gamma(\frac{1}{2})\,\Gamma(a+1)$, we finally obtain Kummer's theorem:

$${}_2F_1\left[\begin{matrix}a,b\\ 1+a-b\end{matrix};-1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+a)}\;. \tag{6}$$

2 A new proof

In formula (1), let us replace a by $a + 2\,n$ where n is a new variable that stands for a nonnegative integer. The resulting formula

$$
{}_2F_1\left[\begin{matrix} a+2\,n, b \\ 1+a+2\,n-b \end{matrix}; -1\right] = \frac{\Gamma(1+\frac{a}{2}+n)\,\Gamma(1+a+2\,n-b)}{\Gamma(1+a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)} \tag{7}
$$

is clearly equivalent to (1).
The introduction of the free parameter n is the key to this new computer-assisted proof.

Formula (7) can be written as:

$$
\frac{\sum_k f(n,k)}{S(n)} = 1 \;, \tag{8}
$$

where

$$
f(n,k) = \frac{(a+2\,n)_k\,(b)_k}{(1+a+2\,n-b)_k}\,\frac{(-1)^k}{k!} \quad \text{and} \quad S(n) = \frac{\Gamma(1+\frac{a}{2}+n)\,\Gamma(1+a+2\,n-b)}{\Gamma(1+a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)} \;.
$$

(The coefficient 2 in the substitution $a \to a + 2\,n$ makes $S(n)$ hypergeometric in n.)
Formula (8) can be rewritten as:

$$
\sum_k F(n,k) = 1 \qquad \text{where } F(n,k) = \frac{f(n,k)}{S(n)}. \tag{9}
$$

We shall actually prove this last formula. Using Gauthier's Maple package `HYPERG`'s [1], we apply Zeilberger's algorithm to obtain:

$$
F(n,k) - F(n+1,k) = G(n,k+1) - G(n,k) \;, \tag{10}
$$

with $G(n,k) = F(n,k)\,C(n,k)$ and $C(n,k) = -\dfrac{(b-1)\,k}{(1+a+2\,n-b+k)\,(a+2\,n)}$.
This $C(n,k)$ is the so-called certificate of the WZ-pair (F,G).
After summing (10) for k from 0 to K, the right-hand side telescopes:

$$
\begin{aligned}
\sum_{k=0}^{K} F(n,k) - \sum_{k=0}^{K} F(n+1,k) &= \sum_{k=0}^{K} G(n,k+1) - \sum_{k=0}^{K} G(n,k) \;, && (11)\\
&= G(n,K+1) - G(n,0) \;, && (12)\\
&= F(n,K+1)\,C(n,K+1) \;. && (13)
\end{aligned}
$$

We now let K tend to infinity. Clearly,

$$
\lim_{K\to+\infty} C(n,K+1) = -\frac{b-1}{a+2\,n} \;.
$$

To determine $\lim_{K\to+\infty} F(n,K+1)$, we use the well-known estimate (see e.g. [3]):

$$\frac{\Gamma(a+k)}{\Gamma(b+k)} \sim k^{a-b} \quad \text{as } k \to \infty \ . \tag{14}$$

First,

$$\begin{aligned} F(n,k) &= \frac{(a+2\,n)_k\,(b)_k}{(1+a+2\,n-b)_k}\,\frac{(-1)^k}{k!}\,\frac{1}{S(n)}\ , \\ &= \frac{\Gamma(a+2\,n+k)\,\Gamma(b+k)\,(-1)^k}{\Gamma(1+a+2\,n-b+k)\,\Gamma(k+1)}\,T(n)\ , \end{aligned}$$

where $T(n) = \dfrac{(a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)}{\Gamma(b)\,\Gamma(1+\frac{a}{2}+n)}$ does not depend on k.
Then,

$$\begin{aligned} |F(n,k)| &\sim k^{a+2n-(1+a+2n-b)}\,k^{b-1}\,T(n)\ , \\ &\sim k^{2b-2}\,T(n)\ . \end{aligned}$$

We know that $Re(b) < 1$ (necessary condition for the convergence of the series in (1)). Therefore

$$\lim_{K\to+\infty} F(n,K+1) = 0\ .$$

Expressing as before $F(n,k)$ in terms of Gamma functions and using the same aymptotics, it is easy to explicit the limit of $F(n,k)$ when n tends to infinity:

$$\begin{aligned} F(n,k) &= \frac{(a+2\,n)_k\,(b)_k}{(1+a+2\,n-b)_k}\,\frac{(-1)^k}{k!}\,\frac{\Gamma(1+a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)}{\Gamma(1+\frac{a}{2}+n)\,\Gamma(1+a+2\,n-b)}\ , \\ &= \frac{\Gamma(a+2\,n+k)\,\Gamma(1+a+2\,n-b)}{\Gamma(a+2\,n)\,\Gamma(1+a+2\,n-b+k)}\,\frac{(b)_k\,(-1)^k}{k!}\,\frac{\Gamma(1+a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)}{\Gamma(1+a+2\,n-b)\,\Gamma(1+\frac{a}{2}+n)}\ , \\ |F(n,k)| &\sim \left|(2\,n)^k\,(2\,n)^{-k}\,\frac{(b)_k\,(-1)^k}{k!}\,(2\,n)^b\,n^{-b}\right|\ . \end{aligned}$$

So,

$$\lim_{n\to+\infty} F(n,k) = \frac{(b)_k\,(-1)^k}{k!}\,2^b\ . \tag{15}$$

By (14), we have $1/S(n) \sim 2^b$ when n tends to infinity, so for all $n \geq n_0$, $1/S(n) < A$ where A is a positive constant. Therefore,

$$|F(n,k)| = \left|\frac{f(n,k)}{S(n)}\right| \leq A\,|f(n,k)|\ . \tag{16}$$

Moreover,

$$|f(n,k)| = \frac{|(a+2\,n)_k|\,|(b)_k|}{|(1+a+2\,n-b)_k|\,k!} \leq \frac{|(b)_k|}{k!}\ , \tag{17}$$

because

$$\forall i \geq 0 \;, \qquad \frac{|a+2\,n+i|}{|1+a+2\,n-b+i|} < 1 \quad \text{when } n \text{ is big enough,}$$

and

$$\begin{aligned}
\frac{|(a+2\,n)_k|}{|(1+a+2\,n-b)_k|} &= \frac{|(a+2\,n)\,(a+2\,n+1)\ldots(a+2\,n+k-1)|}{|(1+a+2\,n-b)\,(1+a+2\,n-b+1)\ldots(1+a+2\,n+b+k-1)|} \;, \\
&\leq \frac{|a+2\,n|\,|a+2\,n+1|\ldots|a+2\,n+k-1|}{|1+a+2\,n-b|\,|1+a+2\,n-b+1|\ldots|1+a+2\,n+b+k-1|} \;, \\
&\leq 1 \;.
\end{aligned}$$

We can finally establish (use (16), (17) and (14)) that

$$|F(n,k)| \leq A \left|\frac{(b)_k}{k!}\right| \sim A \left|\frac{k^{b-1}}{\Gamma(b)}\right| . \tag{18}$$

For $Re(b) < 0$, this last expression is the general term of a convergent series, so $\sum_{k\geq 0} F(n,k)$ is dominated by a convergent series.

The limit of formula (13) when K tends to infinity gives

$$\sum_{k=0}^{\infty} F(n,k) - \sum_{k=0}^{\infty} F(n+1,k) = 0 \;.$$

Hence, the sum $\sum_{k=0}^{\infty} F(n,k)$ is independent of n. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} F(n,k) &= \lim_{n\to+\infty} \sum_{k=0}^{\infty} F(n,k) \;, \\
&= \sum_{k=0}^{\infty} \lim_{n\to+\infty} F(n,k) \;,
\end{aligned}$$

where the last equality is justified by dominated convergence. Moreover,

$$\begin{aligned}
\sum_{k=0}^{\infty} F(n,k) &= \sum_{k=0}^{\infty} \lim_{n\to+\infty} (F(n,k) \;, \\
&= \sum_{k=0}^{\infty} 2^b \, \frac{(b)_k\,(-1)^k}{(k)!} \\
&= 2^b \sum_{k=0}^{\infty} \frac{(b)_k\,(-1)^k}{k!} \;, \\
&= 2^b\,(1-(-1))^b = 1 \;, \qquad \text{by binomial theorem.}
\end{aligned}$$

This is precisely formula (9). So, we proved — with the restriction $Re(b) < 0$ — the validity of Kummer's theorem (specializing n by 0 in (7)).

Kummer's theorem (1) can be rewritten as:

$$\sum_k M(a,b,k) = \frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)}\ ,$$

with $M(a,b,k) = \dfrac{(a)_k\,(b)_k}{(1+a-b)_k\,k!}\,(-1)^k$.

We derive by Zeilberger's algorithm, a recurrence for M with respect to the parameter b:

$$(a-2\,b)\,M(a,b,k) + (-2\,a+2\,b)\,M(a,b+1,k) = G'(a,b,k+1) - G'(a,b,k)$$

with $G'(a,b,k) = M(a,b,k)\frac{(a-b+k)\,k}{b}$ and $Re(b) < 0$. After summing both sides with respect to k from 0 to $K-1$, it gives

$$(a-2\,b)\sum_{k=0}^{K-1} M(a,b,k) + (-2\,a+2\,b)\sum_{k=0}^{K-1} M(a,b+1,k) = G'(a,b,K) - G'(a,b,0)\ ,$$

and then we let K tend to infinity to obtain

$$(a-2\,b)\sum_{k=0}^{\infty} M(a,b,k) + (-2\,a+2\,b)\sum_{k=0}^{\infty} M(a,b+1,k) = 0\ ,$$

because $G'(a,b,0) = 0$ and $\lim\limits_{K\to+\infty} G'(a,b,K) = 0$.

We have now for any b with $Re(b) < 0$

$$\begin{aligned}
\sum_{k=0}^{\infty} M(a,b+1,k) &= \frac{\frac{a}{2}-b}{a-b}\sum_{k=0}^{\infty} M(a,b,k)\ ,\\
&= \frac{\frac{a}{2}-b}{a-b}\,\frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)}\ ,\\
&= \frac{\Gamma(1+\frac{a}{2})\,\Gamma(a-b)}{\Gamma(1+a)\,\Gamma(\frac{a}{2}-b)}\ .
\end{aligned}$$

Let us replace $b+1$ by B, we finally conclude that

$${}_2F_1\left[\begin{matrix} a, B \\ 1+a-B \end{matrix}; -1\right] = \frac{\Gamma(1+\frac{a}{2})\,\Gamma(1-a-B)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-B)}\ ,$$

for all values of B such that $Re(B) < 1$.

Remark: We can also derive this result directly from (18) by a convergence-acceleration argument. For this, we define a new sequence $H(n,k) := F(n,2\,k) + F(n,2\,k+1)$, and we find that $\sum_{k\geq 0} H(n,k)$ is dominated by a convergent series. Hence, we have

$$\sum_{k=0}^{\infty} F(n,k) = \lim_{n\to+\infty}\sum_{k=0}^{\infty} F(n,k)\ ,$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} \sum_{k=0}^{\infty} (F(n,2\,k) + F(n,2\,k+1)) \ , \\
&= \lim_{n\to+\infty} \sum_{k=0}^{\infty} H(n,k) = \sum_{k=0}^{\infty} \lim_{n\to+\infty} H(n,k) \ , \\
&= \sum_{k=0}^{\infty} \lim_{n\to+\infty} (F(n,2\,k) + F(n,2\,k+1)) \ , \\
&= \sum_{k=0}^{\infty} 2^b \left(\frac{(b)_{2\,k}\,(-1)^{2\,k}}{(2\,k)!} + \frac{(b)_{2\,k+1}\,(-1)^{2\,k+1}}{(2\,k+1)!} \right) \\
&= 2^b \sum_{k=0}^{\infty} \frac{(b)_k\,(-1)^k}{k!} \ , \\
&= 1 \ .
\end{aligned}
$$

3 Others summation theorems

The combination of Zeilberger's algorithm and asymptotic estimates suggests an approach to proving automatically identities involving hypergeometric series. It is well suited to summation theorems. As an example, we now give two apparently new proofs of classical hypergeometric identities.

3.1 Bailey's theorem

$$
{}_2F_1 \left[\begin{matrix} a, 1-a \\ b \end{matrix} ; \frac{1}{2} \right] = \frac{\Gamma(\frac{b}{2})\,\Gamma(\frac{1+b}{2})}{\Gamma(\frac{a+b}{2})\,\Gamma(\frac{1-a+b}{2})} \ , \tag{19}
$$

Following the same scheme, let $b \to b + 2\,n$. Formula (19) can be written:

$$
\sum_k F(n,k) = 1 \ , \tag{20}
$$

where $F(n,k) = \dfrac{f(n,k)}{S(n)}$,

with $f(n,k) = \displaystyle\sum_k \frac{(a)_k\,(1-a)_k}{(b+2\,n)_k} \left(\frac{1}{2}\right)^k$ and $S(n) = \dfrac{\Gamma(\frac{b}{2}+n)\,\Gamma(\frac{1}{2}+\frac{b}{2}+n)}{\Gamma(\frac{a+b}{2}+n)\,\Gamma(\frac{1-a+b}{2}+n)}$.

Zeilberger's algorithm applied to $F(n,k)$ succeeds to finding a WZ-pair:

$$
F(n,k) - F(n+1,k) = G(n,k+1) - G(n,k) \ , \tag{21}
$$

where $G(n,k) = F(n,k) \left(-2\,\dfrac{k}{b+2\,n+k} \right)$.

As with Kummer's theorem, we have:

$$
G(n,0) = 0 \ , \tag{22}
$$

$$
\lim_{k\to+\infty} G(n,k) = 0 \ , \tag{23}
$$

and so, $\sum_k F(n,k)$ is independent of n.
In this case, $\lim_{n\to+\infty} F(n,k) = \delta_{k,0}$, so that:

$$\sum_{k=0}^{\infty} F(n,k) = \lim_{n\to+\infty} \sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} \lim_{n\to+\infty} F(n,k) = \sum_{k=0}^{\infty} \delta_{k,0} = 1 \ , \tag{24}$$

which proves Bailey's theorem.

3.2 Dixon's theorem

$${}_3F_2\left[\begin{matrix} a,b,c \\ 1+a-b,1+a-c \end{matrix};1\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{a}{2})\,\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+\frac{a}{2}-c)\,\Gamma(1+a-b-c)} \ , \tag{25}$$

where $Re(2+a-2\,b-2\,c) > 0$.
Once again, we substitute a by $a+2\,n$. With the same notation (20), we now have:

$$\begin{aligned} f(n,k) &= \sum_k \frac{(a+2\,n)_k\,(b)_k\,(c)_k}{(1+a+2\,n-b)_k\,(1+a+2\,n-c)_k\,k!} \ , \\ \text{and} \quad S(n) &= \frac{\Gamma(1+a+2\,n-b)\,\Gamma(1+a+2\,n-c)\,\Gamma(1+\frac{a}{2}+n)\,\Gamma(1+\frac{a}{2}+n-b-c)}{\Gamma(1+a+2\,n)\,\Gamma(1+\frac{a}{2}+n-b)\,\Gamma(1+\frac{a}{2}+n-c)\,\Gamma(1+a+2\,n-b-c)} \ . \end{aligned}$$

The WZ-pair given by Zeilberger's algorithm is

$$F(n,k) - F(n+1,k) = G(n,k+1) - G(n,k) \ , \tag{26}$$

with $G(n,k) = F(n,k)\,C(n,k)$, where the computed certificat $C(n,k)$ is:

```
-(-2-8*a*n*c-a+4*b+4*c-2*n-2*n*b*c-2*a*b*k-2*a*k*c-a*b*c-8*a*n*b-4*n*k*c-4*n*
b*k+a*b+2*n*b-2*a^2*c-8*n^2*c+a*c+2*n*c-2*b^2+3*a^2+12*a*n+12*n^2+12*a^2*n-2*a^
2*b+24*a*n^2+a*k^2-8*n^2*b+2*n*k^2+2*a^3+16*n^3-6*c*b+3*k*a^2+3*k*a+6*k*n+12*k*
n^2+12*k*a*n+2*b^2*c+2*b*c^2-2*c^2)*k/(2+a+2*n-2*b-2*c)/(1+a+2*n-c+k)/(1+a+2*n-
b+k)/(a+2*n)
```

It is easy to establish the following limits:

$$\begin{aligned} G(n,0) &= 0 \ , \\ \lim_{k\to+\infty} G(n,k) &= \lim_{k\to+\infty} k^{2\,b+2\,c-2\,n-4-a}\,T(n) \\ &= 0 \ . \end{aligned}$$

Again, $\sum_k F(n,k)$ is independent of n.
Since $\lim_{n\to+\infty} F(n,k) = \delta_{k,0}$, we conclude again that:

$$\sum_{k=0}^{\infty} F(n,k) = \lim_{n\to+\infty} \sum_{k=0}^{\infty} F(n,k) = \sum_{k=0}^{\infty} \lim_{n\to+\infty} F(n,k) = \sum_{k=0}^{\infty} \delta_{k,0} = 1 \ . \tag{27}$$

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