Defect Theorems for Trees *

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EXTENDED ABSTRACT

1 Introduction

The defect theorem is one of the fundamental results of the combinatorial theory of words (cf. [9]). It states that if a set of n words satisfies a nontrivial relation, then these words can be expressed as a catenation of at most $n-1$ words. Actually there does not exist just one, but several results which formalize the above defect effect (cf. [4, 6]). Also, instead of finite relations, one-way infinite ones can be used (cf. [3, 4]), and, very recently, a defect theorem for two-way infinite relations was proved in [8].

Research on combinatorial problems of node-labeled k-ary trees, viewed as extensions of words, was initiated by Nivat in [12]. One of such natural problems is to ask whether defect theorems hold for trees. This question was answered affirmatively in [11, 10], by showing that the most basic defect theorem – stating that the cardinality of the base of the free hull of a finite noncode is strictly smaller than the cardinality of the set itself – has a counterpart in trees.

We propose a systematical study of the defect theorems for trees. More concretely, we look at which defect theorems on words can be extended to trees. It turns out that in many cases this is possible. However, interestingly, the proofs are not just translations,

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but contain several different features compared to those for words. Moreover, due to the above differences, some results cannot be extended, like, for example, a defect theorem for the prefix rank.

We begin by defining finite and infinite factorizations of trees, and in particular different kinds of "free factorizations". Moreover we introduce algebraic notions characterizing these free factorizations. In particular, we give the definitions of *stable*, *ω-stable*, *left unitary* and *right unitary* set of trees, generalizing the analogous notions for sets of words and we prove that these sets of trees are exactly the ones having as minimal generating set a tree code, an ω-tree code, a suffix tree code and a prefix tree code, respectively. The notions of free, ω-free, suffix and prefix ranks are given as the size of these minimal generating sets. Then we give the different variants of the defect theorem, regarding the stable, the ω-stable and the suffix rank. We show also, by giving an example, that there is no defect effect witnessed by the prefix rank. This shows that when we are dealing with trees we have sometimes a very different behaviour to the one we have in the word case. This is mainly due to the branching structure of trees.

Finally we also show that there is no defect theorem for two-dimensional words, i.e. for pictures.

The full version of this paper, with the complete proofs of the theorems can be found in [7] and will appear in Fundamenta Informaticae.

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