

# Mathematical Structures for Reachability Sets and Relations

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## Summary

Given a vector addition system with states (VASS) and an initial configuration, as usual, the reachability set is defined as the set of all configurations that can be reached from the initial configuration by using the transitions of the VASS. The reachability problem for VASS amounts to checking whether a given configuration belongs to a reachability set. An appealing approach for solving the problem consists in characterizing reachability sets by formulæ from a decidable logical formalism. Presburger arithmetic would be a serious candidate for such a logic, but we know that it is not expressive enough. Indeed, in (Hopcroft and Pansiot, 1979), Hopcroft and Pansiot proved that the reachability sets of any VASS of dimension 2 are effectively definable in Presburger arithmetic, whereas there exist VASS of dimension 3 with reachability sets that are not definable in this logic. Hence, characterizing VASS reachability sets by formulæ in a decidable logic is a difficult task, which is also witnessed by the fact that in (Hack, 1976), Hack proved that the problem of checking whether two reachability sets are equal is undecidable (also known as the equivalence problem). Consequently, undecidability entails that it is not possible to characterize effectively reachability sets with formulæ in a decidable logic.

Recently, several classes of sets of tuple of integers have been introduced (Leroux, 2011) and it has been proved that VASS reachability sets are almost semilinear sets and VASS reachability relations are almost semilinear relations. Actually the classes of almost semilinear sets and almost semilinear relations are defined to be well suited for deciding the VASS reachability problem. More precisely, in (Leroux, 2011), we proved that if the reflexive and transitive closure  $R^*$  of a binary relation  $R$  is a almost semilinear relation, there exist inductive invariants for  $R$  in the Presburger arithmetic proving the non-membership for  $R^*$ . However few results are known about the almost semilinear sets and the almost semilinear relations. For instance whereas the class of VASS reachability relations is known to be stable by composition (just by merging the final states of one VASS to the initial state of another one) we do not know if the class of almost semilinear relations is stable by composition. One way for solving this stability property should consist to prove the stability by intersection and by component projections.

Another important property is the stability by transitive closure. In general, the transitive closure of a binary relation definable in the Presburger arithmetic is not recursive (the membership is undecidable). We say that a binary relation  $R$  over  $\mathbb{N}^d$  is *monotonic* if  $(\vec{x}, \vec{y}) \in R$  implies that  $(\vec{x} + \vec{v}, \vec{y} + \vec{v}) \in R$  for every vector  $\vec{v} \in \mathbb{N}^d$ . If a Presburger relation is monotonic, its transitive closure can be denoted as the reachability relation of a VASS. This property shows that transitive closure of monotonic Presburger relations are almost semilinear relations. We are interested in studying the transitive closure of monotonic

almost semilinear relations. In fact, the reachability relation of a VASS with one zero test can be seen as the transitive closure of a monotonic almost semilinear relation (the relation  $R$  defined by  $(\vec{x}, \vec{y}) \in R$  if  $(0, \vec{y})$  is reachable from  $(0, \vec{x})$  in the VASS without the zero test, assuming that the component tested to zero is the first one). Following (Leroux, 2011), we deduce that if the transitive closure of a monotonic almost semilinear relation is still a almost semilinear relation, the reachability problem for VASS with one zero test can be solved with invariants in the Presburger arithmetic.

## Qualifications

Ideally, the candidate holds a Master degree in Computer Science (with courses in formal verification, theoretical computer science and mathematical structures for CS) or equivalently is graduated from a Computer Science Engineering School with a strong background in theoretical computer science.

## References

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