

Automatic structures: Ramsey's theorem

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General theme

Classical mathematics

$x_0 < x_1 < x_2 \dots$ in \mathbb{Q} bounded $\implies \exists \rho \in \mathbb{R} : \rho = \sup\{x_n \mid n \geq 0\}$

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Computable mathematics

(restrict objects of interest to computable ones)

$x_0 < x_1 < x_2 \dots$ in \mathbb{Q} bounded and computable

$\not\Rightarrow \exists \rho \in \mathbb{R}$ computable : $\rho = \sup\{x_n \mid n \geq 0\}$

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Automatic mathematics

(restrict Turing machines to finite automata)

$x_0 < x_1 < x_2 \dots$ in \mathbb{Q} bounded and “automatic”

$\implies \exists \rho \in \mathbb{R}$ “ ω -automatic” : $\rho = \sup\{x_n \mid n \geq 0\}$

Ramsey's theorem

Effective version

Even more infinite graphs

Ramsey's theorem for graphs

Theorem (Ramsey '30)

Any infinite graph contains some infinite clique or some infinite anticlique.

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Consequence

$\forall n \exists r(n) : \text{any } r(n)\text{-nodes graph contains a clique or anticlique of size } n.$

Definitions

set of ℓ -elements subsets of V : $[V]^\ell$

(k, ℓ) -partition: (V, E_1, \dots, E_k) with (E_1, \dots, E_k) partition
of $[V]^\ell$

$W \subseteq V$ **homogeneous** if $[W]^\ell \subseteq E_i$ for some $1 \leq i \leq k$

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Theorem (Ramsey '30)

$k, \ell \geq 2$, (V, E_1, \dots, E_k) infinite (k, ℓ) -partition

$\implies \exists W \subseteq V$ homogeneous and infinite

Ramsey's theorem

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The computable content of Ramsey's theorem

shorthand $(k, \ell \geq 2)$

$\aleph_0 \rightarrow (\aleph_0)_k^\ell$ means “ \forall infinite (k, ℓ) -partition (V, E_1, \dots, E_k)
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Theorem

- $\forall k, \ell \geq 2 : \aleph_0 \rightarrow (\aleph_0)_k^\ell$ (*Ramsey '30*)
- *there exists a recursive $(2, 2)$ -partition (\mathbb{N}, E_1, E_2) , such that no infinite recursive set is homogeneous, i.e., $(\aleph_0, \text{REC}) \not\rightarrow (\aleph_0, \text{REC})_k^2$ (*Specker '69*)*

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- $\forall \ell \geq 2 : (\aleph_0, \text{REC}) \not\rightarrow (\aleph_0, \Sigma_\ell^0)_k^\ell$ (*Jockusch '72*)

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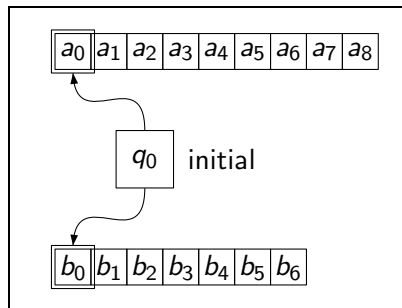
- $\forall k, \ell \geq 2 : \aleph_0 \rightarrow (\aleph_0)_k^\ell$ (*Ramsey '30*)
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- $\forall k, \ell \geq 2 : (\aleph_0, \text{REC}) \rightarrow (\aleph_0, \Pi_\ell^0)_k^\ell$ (*Jockusch '72*)

Summary

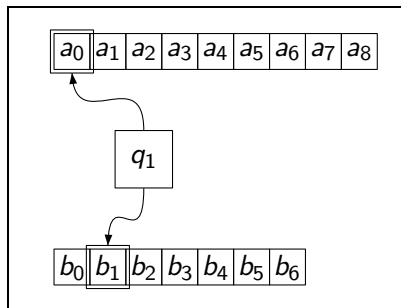
for all $k, \ell \geq 2$:

$$\left. \begin{array}{l} \aleph_0 \rightarrow (\aleph_0)_k^\ell \quad (\text{Ramsey '30}) \\ (\aleph_0, \text{REC}) \not\rightarrow (\aleph_0, \text{REC})_k^\ell \\ \quad \quad \quad (\text{Specker '69}) \end{array} \right\}$$

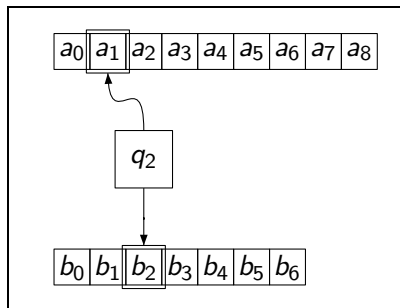
Multitape automata



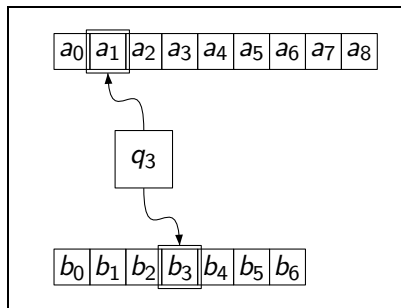
Multitape automata



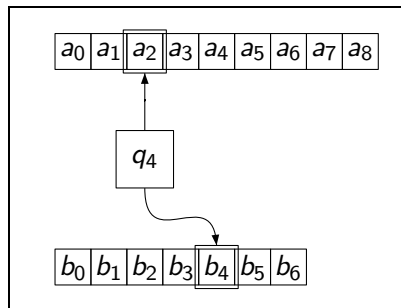
Multitape automata



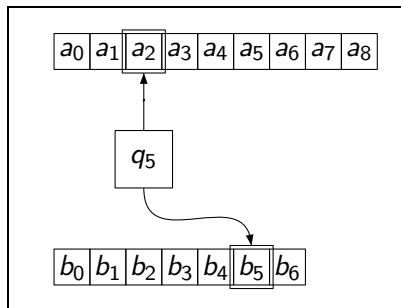
Multitape automata



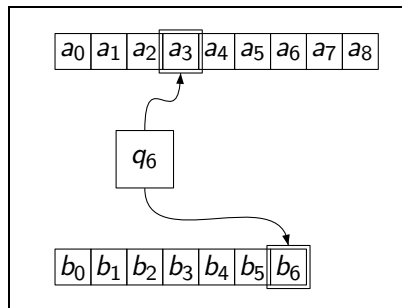
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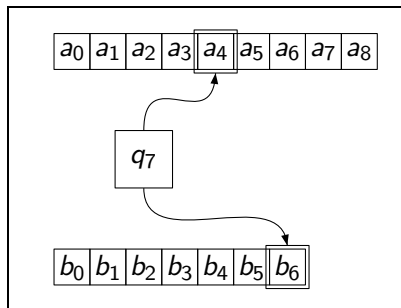
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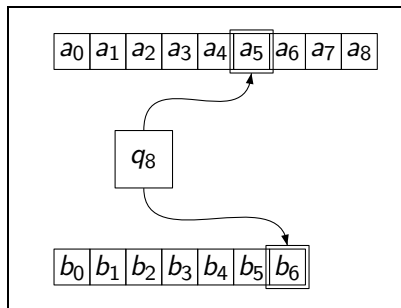
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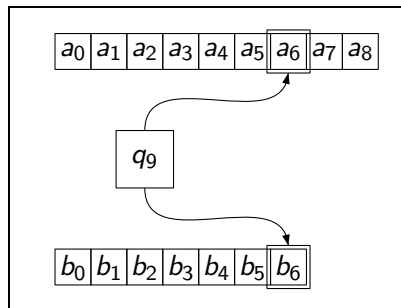
Multitape automata



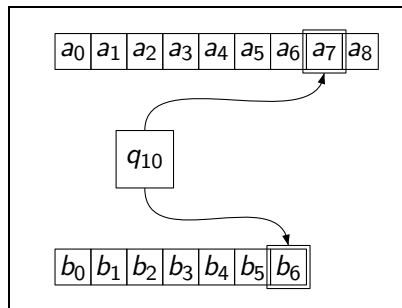
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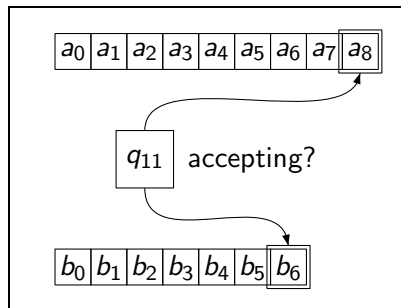
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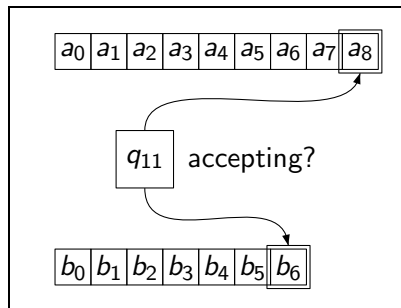
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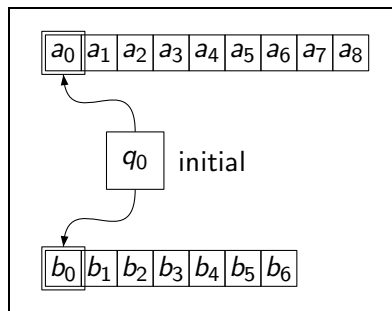
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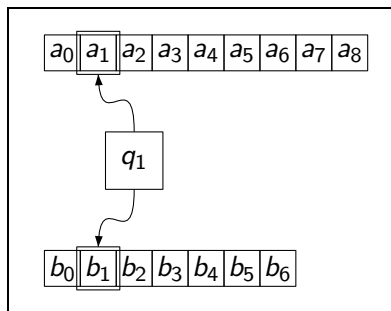
Some properties

- accept relations on Γ^* , emptiness decidable
- effective closure under union, projection, cylindrification
- not closed under complementation, intersection; universality undecidable

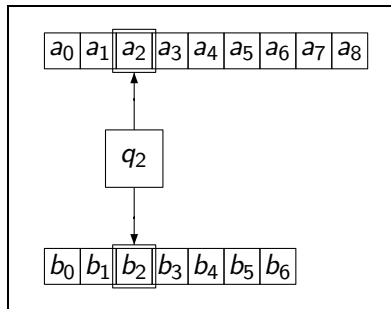
Synchronous multitape automata



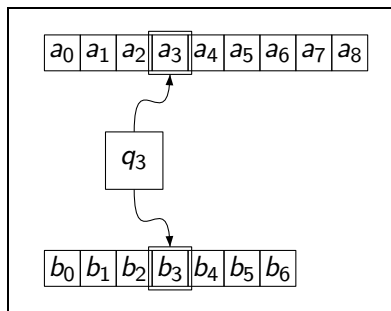
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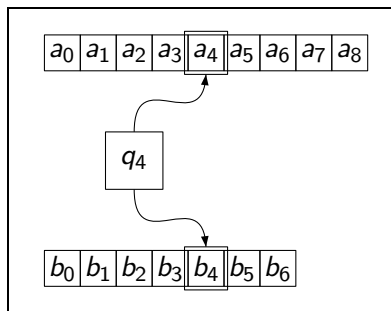
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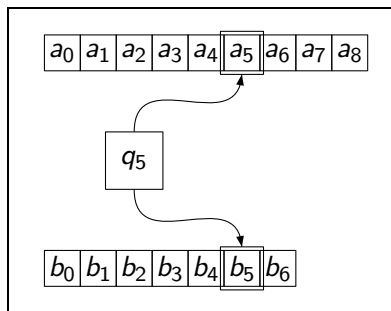
Synchronous multitape automata



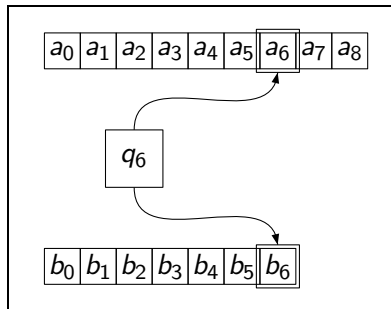
Synchronous multitape automata



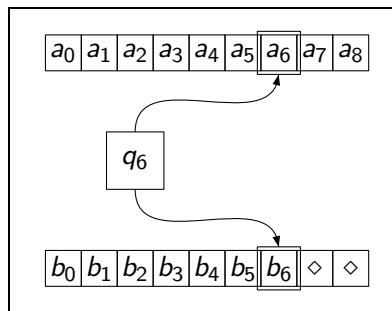
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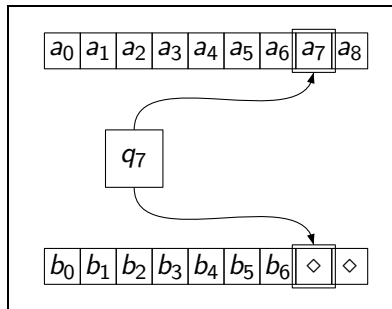
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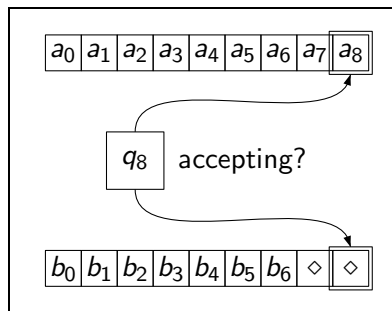
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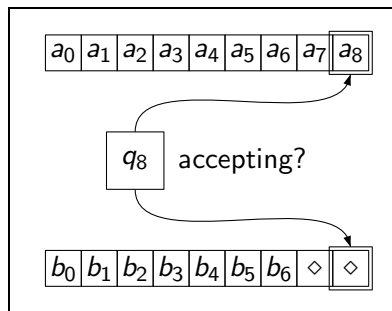
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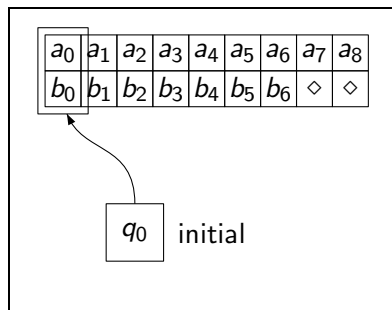
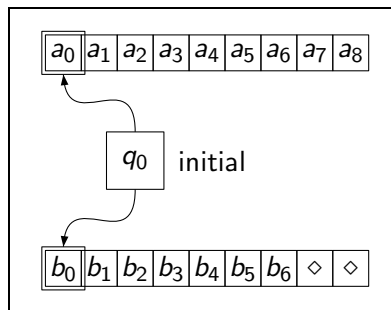


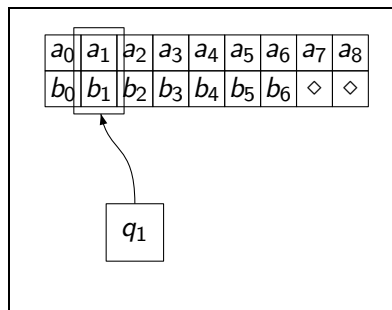
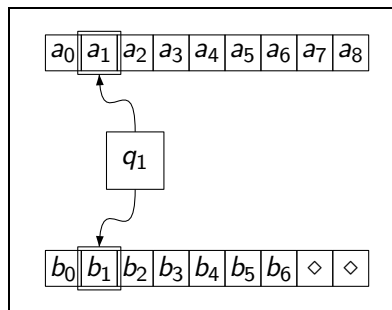
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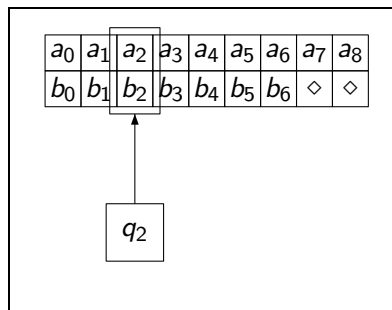
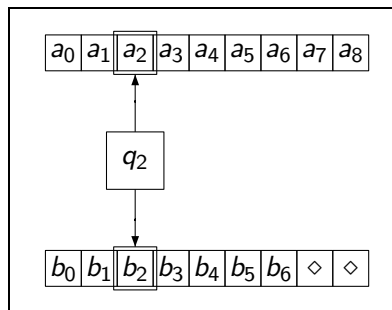


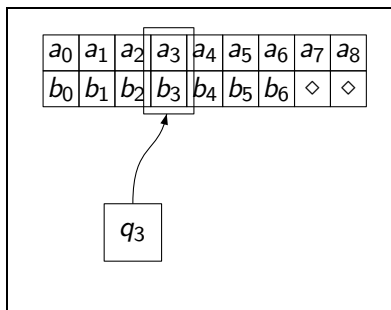
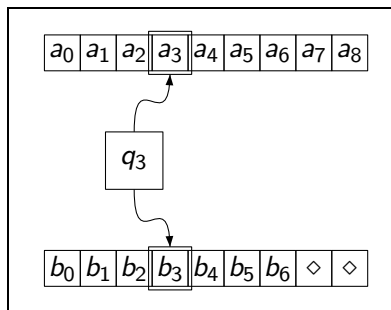
relation accepted by M : $R(M)$

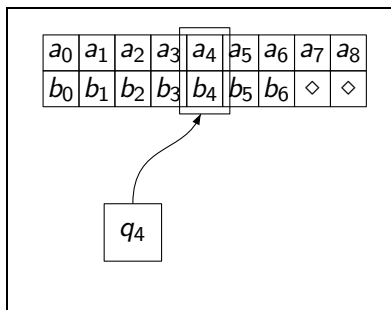
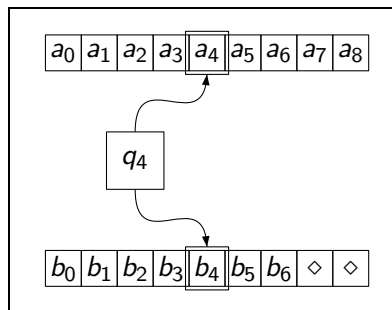
$R \subseteq (\Gamma^*)^k$ **automatic** if it is accepted by some synchronous k -head automaton

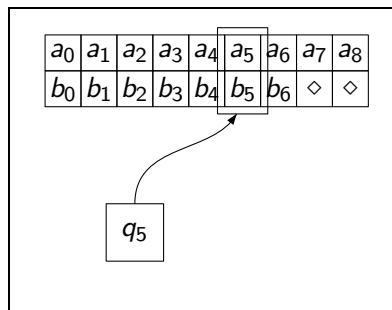
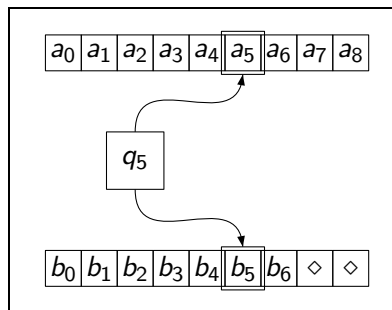
Automatic relations \sim regular languages

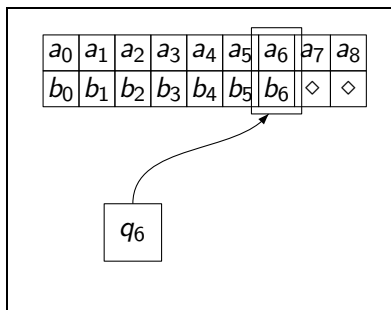
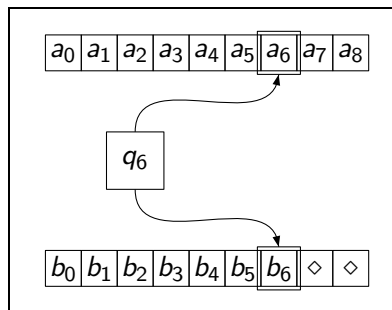
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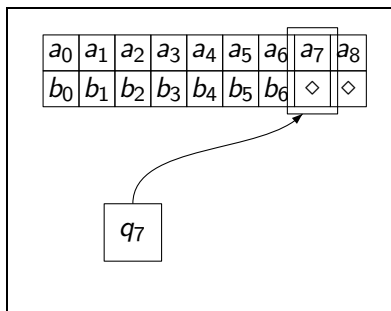
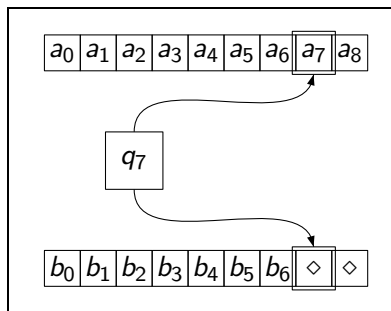
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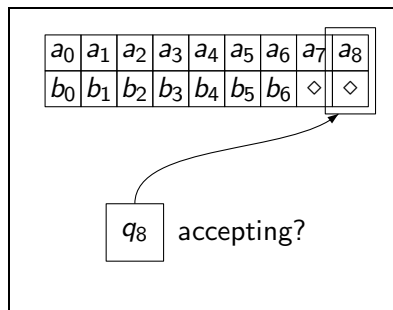
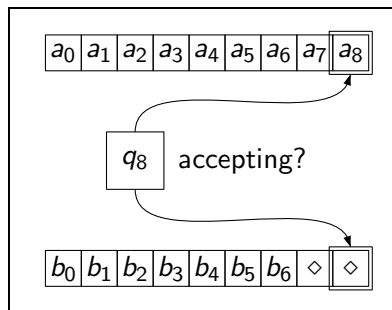
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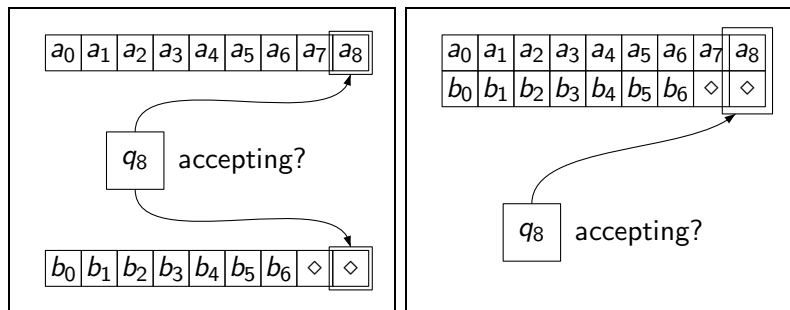
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Some properties of automatic relations

- emptiness and universality decidable
- effective closure under union, projection, cylindrification, complementation, intersection

Automatic partitions

A (k, ℓ) -partition (V, E_1, \dots, E_k) is **automatic** if

- $V \subseteq \Sigma^*$ is regular
- $\{(u_1, u_2, \dots, u_\ell) \in V^\ell \mid \{u_1, \dots, u_\ell\} \in E_i\}$ is automatic for all $1 \leq i \leq k$

Automatic partitions

shorthand $(k, \ell \geq 2)$

$(\aleph_0, \mathbf{A}) \rightarrow (\aleph_0, \mathbf{REG})_k^\ell$ means

“ \forall automatic and infinite (k, ℓ) -partition (V, E_1, \dots, E_k)
 $\exists K \subseteq V$ regular, homogenous and infinite”

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Theorem (Rubin '08)

1. $\forall k, \ell \geq 1 : (\aleph_0, \mathbf{A}) \rightarrow (\aleph_0, \mathbf{REG})_k^\ell$

Automatic partitions

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Theorem (Rubin '08)

1. $\forall k, \ell \geq 1 : (\aleph_0, \mathbf{A}) \rightarrow (\aleph_0, \mathbf{REG})_k^\ell$
2. From an automatic (k, ℓ) -partition (V, E_1, \dots, E_k) and $1 \leq i \leq k$, one can decide whether (V, E_1, \dots, E_k) contains an infinite i -homogeneous set (in which case one can compute an infinite regular i -homogeneous set).

Proof of Rubin's theorem $(\aleph_0, A) \rightarrow (\aleph_0, \text{REG})_k^2$

let $X \subseteq \Sigma^*$ be infinite and $[X]^2 \subseteq E_i$

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build ω -word

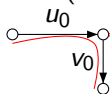
over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:

Proof of Rubin's theorem $(\aleph_0, A) \rightarrow (\aleph_0, \text{REG})_k^2$

let $X \subseteq \Sigma^*$ be infinite and $[X]^2 \subseteq E$;

build ω -word $\left| \begin{array}{c} u_0 \\ \$|u_0| \\ v_0\$ \end{array} \right|$

over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



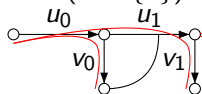
X infinite

Proof of Rubin's theorem $(\aleph_0, A) \rightarrow (\aleph_0, \text{REG})_k^2$

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build ω -word $\left| \begin{array}{c} u_0 \\ \$|u_0| \\ v_0\$ \end{array} \right| \left| \begin{array}{c} u_1 \\ v_0\$ \\ v_1\$ \end{array} \right|$

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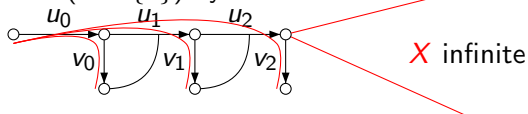
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over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



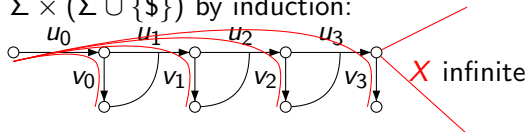
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let $X \subseteq \Sigma^*$ be infinite and $[X]^2 \subseteq E_i$

build ω -word

u_0	u_1	u_2	u_3	$v_3\$$
$\$ u_0 $	$v_0\$$	$v_1\$$	$v_2\$$	$v_3\$$

over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



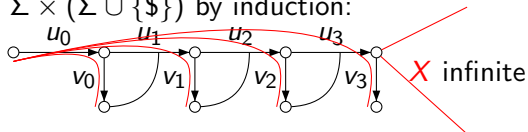
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let $X \subseteq \Sigma^*$ be infinite and $[X]^2 \subseteq E_i$

build ω -word

u_0	u_1	u_2	u_3	u_5	\dots
$\$ u_0 $	$v_0\$$	$v_1\$$	$v_2\$$	$v_3\$$	$v_4\$ \dots$

over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



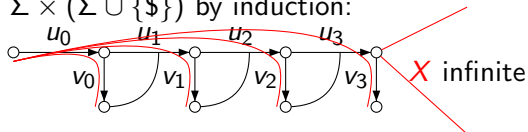
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u_0	u_1	u_2	u_3	u_5	\dots
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over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



Observations

1. set of encodings of combs s.t.

$\{u_0 u_1 \dots u_a v_a, u_0 u_1 \dots u_b v_b\} \in E_i$ f.a. $a \neq b$ is ω -regular

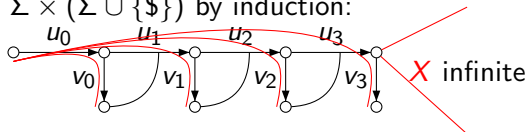
Proof of Rubin's theorem $(\aleph_0, A) \rightarrow (\aleph_0, \text{REG})_k^2$

let $X \subseteq \Sigma^*$ be infinite and $[X]^2 \subseteq E_i$

build ω -word

u_0	u_1	u_2	u_3	u_5	\dots	
$\$ u_0 $	$v_0\$$	$v_1\$$	$v_2\$$	$v_3\$$	$v_4\$$	\dots

over $\Sigma \times (\Sigma \cup \{\$\})$ by induction:



Observations

- set of encodings of combs s.t.
 $\{u_0 u_1 \dots u_a v_a, u_0 u_1 \dots u_b v_b\} \in E_i$ f.a. $a \neq b$ is ω -regular
- hence there exists an encoding of a comb of form

u_0	u_1	u_2	u_3	u_3	u_3	\dots
$\$ u_0 $	$v_0\$$	$v_1\$$	$v_2\$$	$v_2\$$	$v_2\$$	\dots

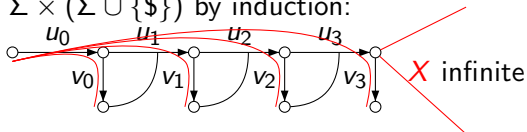
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- hence $K = u_0 u_1 u_2 (u_3)^* v_2 \subseteq X$ is contained in some infinite homogeneous set, i.e., $[K]^2 \subseteq E_i$ □

Ramsey's theorem

Effective version

Even more infinite graphs

Generalization

shorthand ($k, l \geq 2, \alpha, \beta$ cardinals)

$\alpha \rightarrow (\beta)_k^l$ means " $\forall (k, l)$ -partition $(V, E_1, \dots, E_k), |V| = \alpha$
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- $\forall k, l \geq 2, \alpha \geq \aleph_0 : \alpha \rightarrow (\aleph_0)_k^l$ (*Ramsey '30*)
- $\forall k, l \geq 2 : 2^{\aleph_0} \not\rightarrow (2^{\aleph_0})_k^l$ (*Sierpinski '33*)

Proof of Sierpinski's theorem $2^{\aleph_0} \not\rightarrow (\aleph_1)_2^2$

Construction

$V = \mathbb{R}$, \sqsubseteq arbitrary well-order on V

$E_1 = \{\{x, y\} \in [V]^2 \mid x < y \iff x \sqsubset y\}$

$E_2 = [V]^2 \setminus E_1$

Proof of Sierpinski's theorem $2^{\aleph_0} \not\rightarrow (\aleph_1)_2^2$

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suppose $X \subseteq \mathbb{R}$ is homogeneous, e.g., $[X]^2 \subseteq E_1$

then $(X, \sqsubseteq) \cong (X, \leq) \hookrightarrow (\mathbb{R}, \leq)$

hence $|X| \leq \aleph_0$ since no uncountable ordinal embeds into \mathbb{R}

Generalization

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- $\forall k, \ell \geq 2, \alpha > 2^{\aleph_0} : \alpha \rightarrow (2^{\aleph_0})_k^\ell$ (*Erdős, Rado '56*)

Summary

for all $k, \ell \geq 2$:

$$\left. \begin{array}{l} \aleph_0 \rightarrow (\aleph_0)_k^\ell \quad (\text{Ramsey '30}) \\ (\aleph_0, \text{REC}) \not\rightarrow (\aleph_0, \text{REC})_k^\ell \\ \quad \quad \quad (\text{Specker '69}) \end{array} \right\} \text{vs.} \left\{ \begin{array}{l} (\aleph_0, \mathbf{A}) \rightarrow (\aleph_0, \text{REG})_k^\ell \\ \quad \quad \quad (\text{Rubin '08}) \end{array} \right.$$

$$2^{\aleph_0} \not\rightarrow (2^{\aleph_0})_k^\ell \quad (\text{Sierpinski '33})$$

ω -automatic partitions

A (k, ℓ) -partition (V, E_1, \dots, E_k) is **ω -automatic** if

- $V \subseteq \Sigma^\omega$ is ω -regular
- $\bar{E}_i = \{(u_1, u_2, \dots, u_\ell) \in V^\ell \mid \{u_1, \dots, u_\ell\} \in E_i\}$ is ω -automatic for all $1 \leq i \leq k$

Remark

If (V, E_1, \dots, E_ℓ) is an ω -automatic presentation, then $|V| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$, i.e., continuum hypothesis holds in this context

ω -automatic partitions

shorthand ($k, \ell \geq 2, \alpha, \beta$ cardinals, \mathcal{C} class of ω -languages)

$(\alpha, \omega A) \rightarrow (\beta, \mathcal{C})_k^\ell$ means

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Theorem

- $\forall k \geq 2: (2^{\aleph_0}, \omega A) \rightarrow (2^{\aleph_0}, co\text{-}\omega CF \cap \Lambda)_k^2$

Proof of $(2^{\aleph_0}, \omega A) \rightarrow (2^{\aleph_0}, co-\omega CF \cap \Lambda)_2^2$ by example

$$V = (1+0^+)^\omega$$

$$E_1 = \{\{x, y\} \in [V]^2 \mid \text{supp}(x) \setminus \text{supp}(y), \text{supp}(y) \setminus \text{supp}(x) \text{ infinite}\}$$

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let K be set of branches in infinite binary tree:

- $|K| = 2^{\aleph_0}$
- $[K]^2 \subseteq E_1$
- complement of K is ω -context-free, i.e., $K \in co-\omega CF$
- $(\mathbb{R}, \leq) \hookrightarrow (K, \leq_{lex})$, i.e., $K \in \Lambda$

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Remark (Bárány, Kaiser & Rubin '08)

Any ω -automatic $(k, 2)$ -partition “contains” above example.

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Consequence

(V, \leq) ω -automatic partial order, $|V| = 2^{\aleph_0}$
 $\implies (\mathbb{R}, \leq) \hookrightarrow (V, \leq)$ or $(\mathbb{R}, =) \hookrightarrow (V, \leq)$

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4. Hence $[K]^2 \subseteq E_2$ since K is homogeneous.

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5. Hence $|K| \leq \aleph_0$. □

ω -automatic partitions

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- $\forall k \geq 2 \forall \ell \geq 3: (2^{\aleph_0}, \omega\mathbf{A}) \not\rightarrow (2^{\aleph_0}, \omega\mathbf{LANG})_k^\ell$

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$$V = \{0, 1\}^\omega$$

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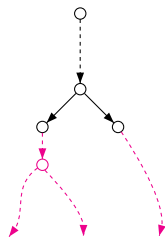
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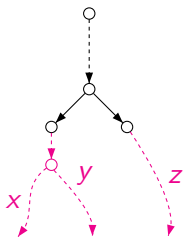
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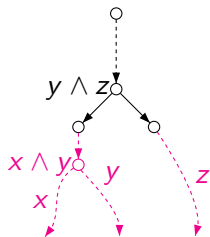
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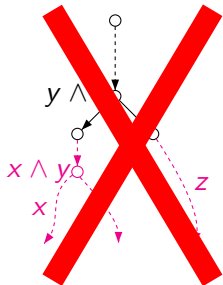
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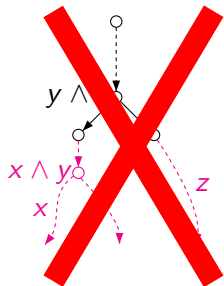
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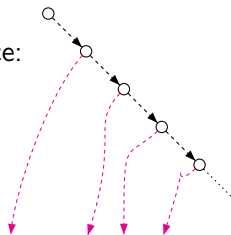
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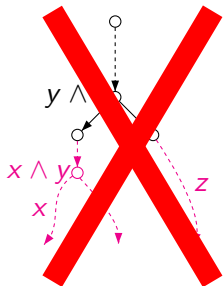
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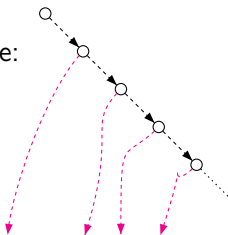
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Hence K is countable. □

