

# Gallai’s path decomposition for planar graphs

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**Abstract.** In 1968, Gallai conjectured that the edges of any connected graph with  $n$  vertices can be partitioned into  $\lceil \frac{n}{2} \rceil$  paths. Although this conjecture has been tackled and partially solved over the years, it is still open as of today. We prove that the conjecture is true for every planar graph. More precisely, we show that every connected planar graph except  $K_3$  and  $K_5^-$  ( $K_5$  minus one edge) can be decomposed into  $\lfloor \frac{n}{2} \rfloor$  paths.

**Keywords:** graph theory, graph decomposition, paths

## 1 Introduction

Given a graph  $G$ , a  $k$ -path decomposition of  $G$  is a partition of the edges of  $G$  into  $k$  paths. In 1968, Gallai stated this simple but surprising conjecture [8]: every finite undirected connected graph on  $n$  vertices admits a  $\lceil \frac{n}{2} \rceil$ -path decomposition. Gallai’s conjecture is still unsolved as of today, and has only been confirmed on very specific classes of graphs: graphs of degree 2 or 4 [6], graphs whose vertices of even degree induce a forest [9], series-parallel graphs [7], graphs with maximum degree at most 5 [1], or planar 3-trees [4]. Recently, Botler et al. proved that the conjecture is true in the case of triangle-free planar graphs [2]. Chu et al. confirmed the conjecture on graphs of maximum degree 6, under the condition that vertices of degree 6 form an independent set [5].

An *odd semi-clique* is obtained from a clique on  $2k + 1$  vertices by deleting at most  $k - 1$  edges. Bonamy and Perrett asked the following question [1, Question 1.1]: Does every connected graph  $G$  that is not an odd semi-clique admit a  $\lfloor \frac{n}{2} \rfloor$ -path decomposition?

We answer this question positively for planar graphs. Only two odd semi-cliques are planar: the triangle  $K_3$  and  $K_5$  minus one edge, which we denote by  $K_5^-$  (see Figure 1). We can therefore state the result as follows:

**Theorem 11** *Every connected planar graph  $G$  on  $n$  vertices, except  $K_3$  and  $K_5^-$ , can be decomposed into  $\lfloor \frac{n}{2} \rfloor$  paths.*

To prove this result, we proceed with a standard approach for coloring problems, by considering a planar graph that is a counterexample to our theorem and is vertex-minimum with respect to this property. We can prove that such a *minimum counterexample* (MCE) cannot contain a certain set of configurations, by providing for each of these configurations a reduction rule that takes advantage of the properties of the MCE and yields a contradiction. This technique is

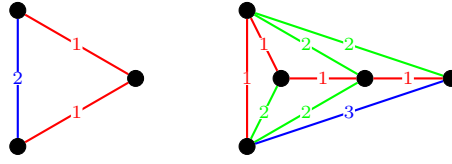


Fig. 1: A 2-path decomposition of  $K_3$  and a 3-path decomposition of  $K_5^-$

widely used in the literature on graph coloring, and especially on Gallai’s conjecture ([1,2,3,5]). More precisely, these reducible configurations deal with vertices of small degree (at most 5), so after showing that our MCE cannot contain any of these configurations, we know that it has mostly vertices of degree at least 6. Finding these reducible configurations and their associated reduction rules makes up the bulk of the proof, and this is the part we develop in the next section. We finally use Euler’s formula for planar graphs and structural arguments to prove that there is no such graph.

## 2 Reducible configurations

We say that a path decomposition of a graph on  $n$  vertices is *good* if it contains at most  $\lfloor \frac{n}{2} \rfloor$  paths. Given a planar graph  $G$ , a *2-family* is a set  $U$  of two vertices of  $G$  of degree at most 4. A *4-family* is a set of four vertices of degree 5. We say that a graph with a 4-family  $U$  is *almost 4-connected* w.r.t.  $U$  if it is 3-connected and does not contain a set  $A = \{a_1, a_2, a_3\}$  of vertices such that  $A$  separates two vertices  $u_1, u_2 \in U$ , or such that there is a vertex  $u \in U \cap A$ , and  $A$  separates two neighbors of  $u$ . The main lemma of the paper is the following.

**Lemma 21** *Let  $G$  be a connected planar graph on  $n$  vertices, other than  $K_3$  and  $K_5^-$ , and assume that  $G$  is a minimum counterexample to Theorem 11, i.e. that  $G$  does not admit a  $\lfloor \frac{n}{2} \rfloor$ -path decomposition and is vertex-minimum for this property. Then  $G$  does not contain any of the following configurations:*

- $(C_I)$ : a 2-family ;
- $(C_{II})$ : an almost 4-connected component with respect to a 4-family.

The general idea is the following: we prove by contradiction that  $(C_I)$  or  $(C_{II})$  configurations cannot occur in an MCE by considering such a graph  $G$  with a 2-family or 4-family  $U$ . We delete the vertices of  $U$ , and add or remove some edges. We call the resulting graph  $G'$  the *reduced graph*.  $G'$  is smaller than  $G$ , so it is made up of connected components that are either  $K_3$ ,  $K_5^-$  or have a good path decomposition. We are able to build a decomposition of  $G'$  into paths and cycles, such that the total number of paths and cycles is  $\lfloor \frac{|V(G')|}{2} \rfloor$ . We describe a specific reduction rule for each case, such that the decomposition of the reduced graph associated with this case can be turned into a good path decomposition of  $G$ . Since  $G$  is a counterexample, it is a contradiction, hence after each case is dealt with we deduce that our MCE does not contain  $(C_I)$  nor  $(C_{II})$  configurations.

2.1 Configurations ( $C_I$ )

The first set of configurations deals with all the cases of our MCE  $G$  containing 2 vertices of degree at most 4, which we call *special vertices*. We start by considering a shortest path  $S$  between  $u_1$  and  $u_2$ . Each special vertex has one neighbor belonging to  $S$ , and up to 3 other *remaining neighbors*. We distinguish between whether or not the special vertices have remaining neighbors in common.

If the special vertices  $u_1, u_2$  have no common remaining neighbor, we deal with the case by considering a *composite rule* made up two *partial rules*. The reduced graph  $G'$  will be created by removing the vertices  $u_1, u_2$  and the edges of  $S$  from  $G$ , as well as adding or removing some edges specified by each partial rule.



Fig. 2: A partial rule for 2 non-adjacent remaining neighbors

Figure 2 features an example of partial rule. The leftmost drawing represents the initial configuration in  $G$ , in this case the special vertex  $u_1$  has degree 3, with a neighbor in the path  $S$  (in red) and two non-adjacent remaining neighbors  $v_1, v_2$ . The middle drawing depicts what edges we add to or remove from the reduced graph  $G'$ , in this case we add the edge  $v_1v_2$ . We consider an arbitrary path decomposition of  $G'$ , and assume the (green) path  $Q$  contains the edge  $v_1v_2$ . The rightmost drawing represents a decomposition of  $G$  that we can build based on the decomposition of  $G'$ : all paths from the decomposition of  $G'$  remain unchanged in  $G$ , except  $Q$  which is deviated on the edges  $v_1u_1$  and  $u_1v_2$ . Since  $|V(G)| - |V(G')| = 2$ , we are allowed one more path, that we use to color the edges of  $S$ . Having one end of such a path on each special vertex is convenient, for example for the rule of Figure 3, where it is extended to help handle edges within the neighborhood of  $u$ .

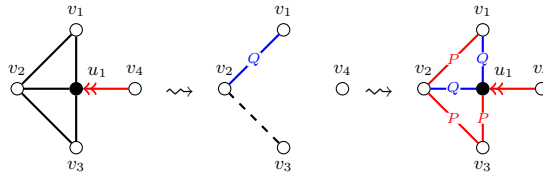


Fig. 3: A partial rule for 3 remaining neighbors

The composite rule is defined by application of the two partial rules, to define the reduced graph  $G'$  and draw a path decomposition of  $G$  from a path decomposition of  $G'$ . Since the special vertices do not have common remaining neighbors, the partial rules do not interfere with each other and can be applied independently.

When the two special vertices have common remaining neighbors, we treat both of them at the same time with a common rule, such as the one depicted in Figure 4. The leftmost drawing depicts the configuration in  $G$ : two adjacent special vertices  $u_1, u_2$  of degree 4 with a common neighbor  $v$ , and such that  $u_1$  has two other neighbors  $v_1, v_3$ ,  $u_2$  has two other neighbors  $v_2, v_4$ , the vertices  $v_1, v_2, v_3, v_4$  are distinct and  $v_1, v_2$  are non-adjacent. The middle drawing depicts the reduced graph  $G'$ : after removing  $u_1, u_2$ , we add the edge  $v_1v_2$ , and this edge belongs to a path  $Q$  in a path decomposition of  $G'$ . The rightmost drawing depicts the path decomposition of  $G$  that we build from the one of  $G'$ : we deviate the path  $Q$  on the edges  $v_1u_1, u_1u_2, u_2v_2$ , and we use an extra path for the edges  $v_3u_1, u_1v, vu_2$  and  $u_2v_4$ .

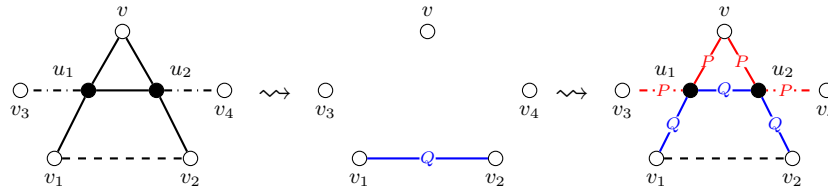


Fig. 4: A reduction rule for special vertices with 1 common neighbor

We find a set of about 30 rules that covers all possible cases of  $(C_I)$  configurations. We thus deduce a contradiction with the nature of  $G$ , which proves that such a minimum counterexample does not contain a configuration  $(C_I)$ .

## 2.2 Configurations $(C_{II})$

We proceed in the same way for the configurations  $(C_{II})$ , by generalizing the concepts of the previous section for our set of 4 special vertices of degree 5. We again consider an MCE  $G$  and assume it contains a configuration  $(C_{II})$ . Instead of considering a path  $S$ , we consider a subdivision of a certain graph, rooted on our 4 special vertices. The goal of this part is again to define valid reduction rules that cover each possible case. The rules operate like in the previous section: we remove the special vertices and the edges of the subdivision, and add or remove some edges in the neighborhood of the special vertices, as specified in each rule. The resulting reduced graph can be decomposed into the right number of paths and cycles, and we are able to deduce from the rule a good path decomposition of our MCE  $G$ .

The subdivisions we consider are  $K_4$ -subdivisions (Figure 5a) or  $C_{4+}$ -subdivisions (Figure 5b) rooted on our 4 special vertices (where  $C_{4+}$  is the graph made up of a cycle on 4 vertices with two parallel edges doubled).

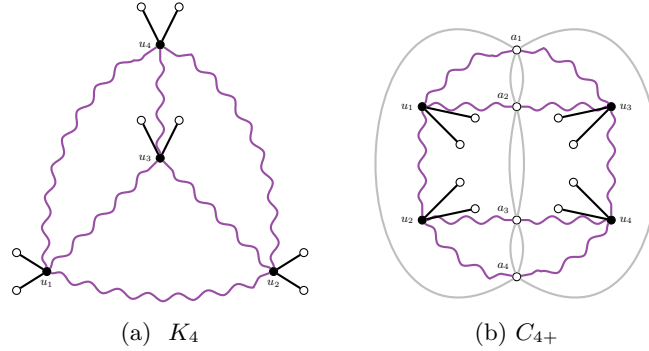


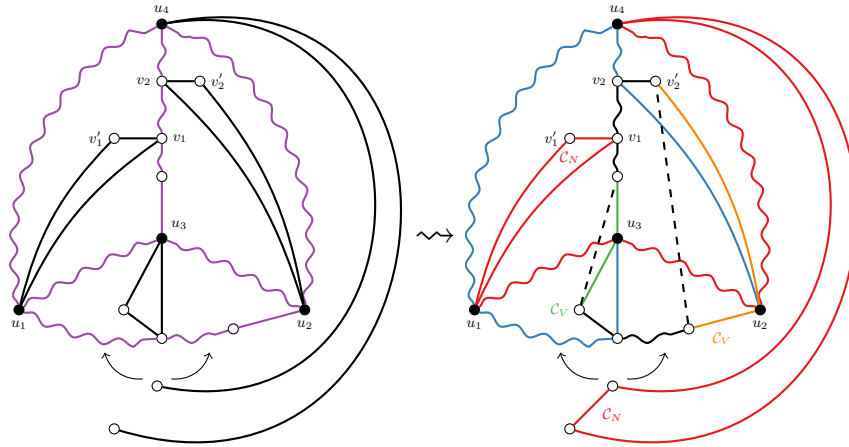
Fig. 5: A  $K_4$ - and a  $C_{4+}$ -subdivision

We use a result by Yu [10] to show that in an almost 4-connected configuration, there exists indeed a  $K_4$ - or  $C_{4+}$ -subdivision rooted on the four special vertices. These subdivisions can be decomposed into 2 paths, which corresponds to the number of extra paths we are allowed to use in our reductions. Additionally, these paths have their four ends on each of the four special vertices, which allows us once again to extend those paths if needed in order to cover all edges within the neighborhood of our special vertices. Each special vertex  $u$  has three incident edges that belong to the subdivision, and two other *remaining neighbors*  $v, v'$ . The edges  $uv, uv'$  are the ones we need to cover in the good path decomposition of  $G$  that we want to build, for each special vertex  $u$ .

We generalize our concept of partial rules for our 4 special vertices. Each partial rule treats the neighborhood of one or two special vertices at once, so we now consider composite rules made up of a subdivision and between 2 and 4 partial rules. If these partial rules affect disjoint areas of the graph and are disjoint from the subdivision, they can be applied independently, but it is not always the case.

In order to find a composite rule made up of compatible partial rules, we apply a series of alterations to the subdivision, to eliminate some unwanted configurations. When all modifications have been applied, we do a careful case analysis and find a composite rule for each case, such as the one in Figure 6.

The configuration depicted on the first drawing consists in a  $K_4$ -subdivision  $S$ , such that some of the remaining neighbors of the special vertices belong to  $S$ , which prevents partial rules from being applied as-is. The reduction rule consists in finding an alternate subdivision  $S'$  (in this case the  $K_4$ -subdivision is turned into a  $C_{4+}$ -subdivision), describing a 2-path decomposition of  $S'$  (in red and blue on the second drawing) and describing a set of partial rules (in this case

Fig. 6: Example of a composite rule on a  $K_4$ -subdivision

$\mathcal{C}_V$  and  $\mathcal{C}_N$ , which deals with two adjacent remaining neighbors) for each special vertex, while making sure these partial rules are compatible with one another and with the subdivision. We find a set of around 20 composite rules that covers all cases of  $(C_{II})$  configurations.

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