

Gallai's path decomposition conjecture: the case of planar graphs

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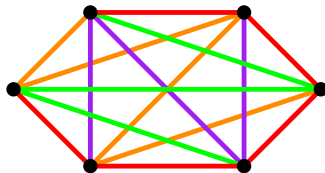
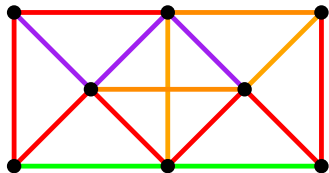
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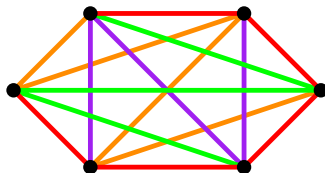
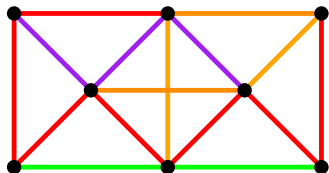
Path decomposition

Path decomposition: a **partition** of the edges into paths



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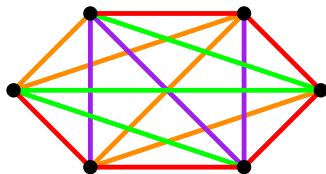
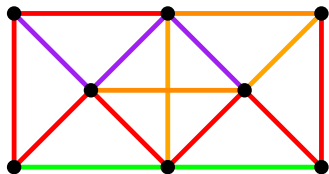


Conjecture (Gallai, 1968)

Any connected graph on n vertices has a decomposition into at most $\left\lceil \frac{n}{2} \right\rceil$ paths.

Path decomposition

Path decomposition: a **partition** of the edges into paths



Conjecture (Gallai, 1968)

Any connected graph on n vertices has a decomposition into at most $\left\lceil \frac{n}{2} \right\rceil$ paths.

Conjecture (Hajós, 1968)

An Eulerian graph on n vertices has a decomposition into at most $\left\lfloor \frac{n}{2} \right\rfloor$ cycles.

Theorem (Lovász, 1968)

A graph on n vertices can be decomposed into at most $\lfloor \frac{n}{2} \rfloor$ paths and cycles.

General bounds

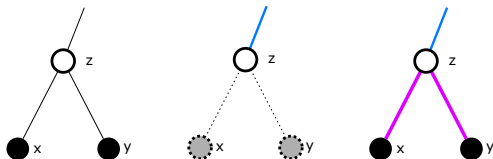
$|odd|, |even|$: number of vertices of odd, even degree

- Lovász, 1968: $\leq \frac{|odd|}{2} + |even| - 1$
- Donald, 1980: $\leq \frac{|odd|}{2} + \left\lfloor \frac{3|even|}{4} \right\rfloor$
- Dean, Kouider, 2000: $\leq \frac{|odd|}{2} + \left\lfloor \frac{2|even|}{3} \right\rfloor$

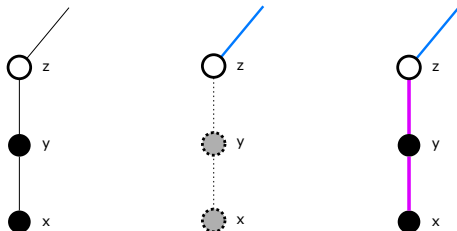
Example for the trees

If a tree is a **minimum counterexample** for Gallai's conjecture:

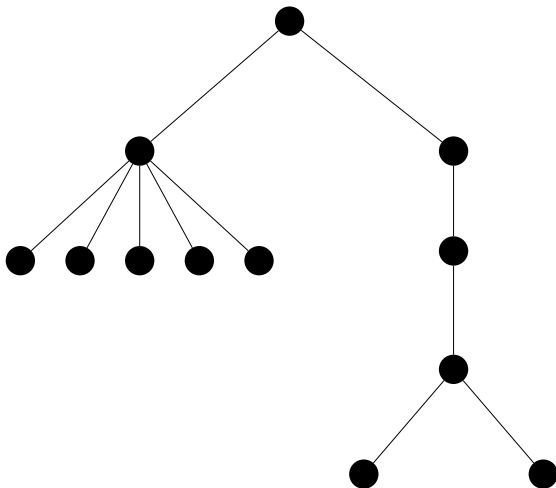
- We are able to deal with a pair of leaves having a common parent:



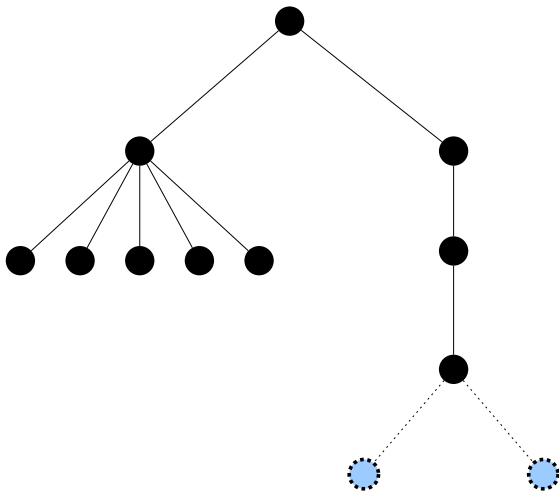
- We are able to deal with single leaves:



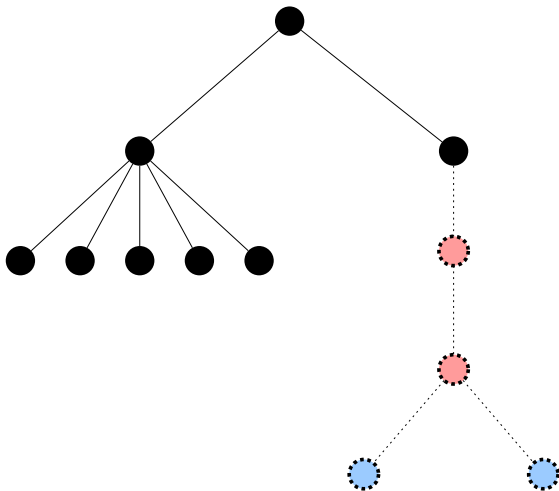
Example for the trees



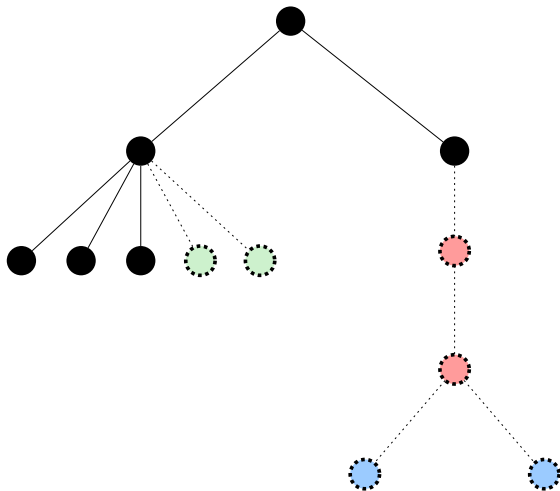
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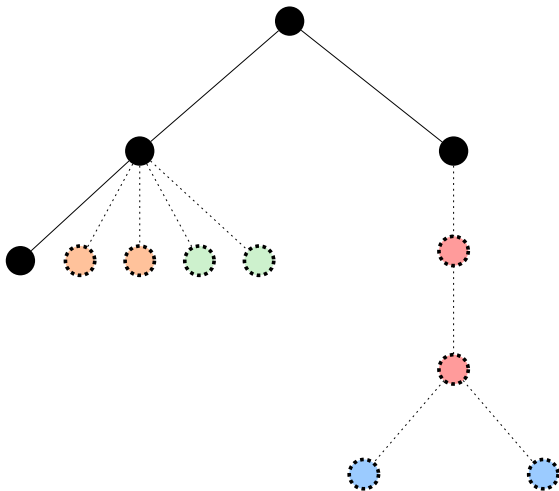
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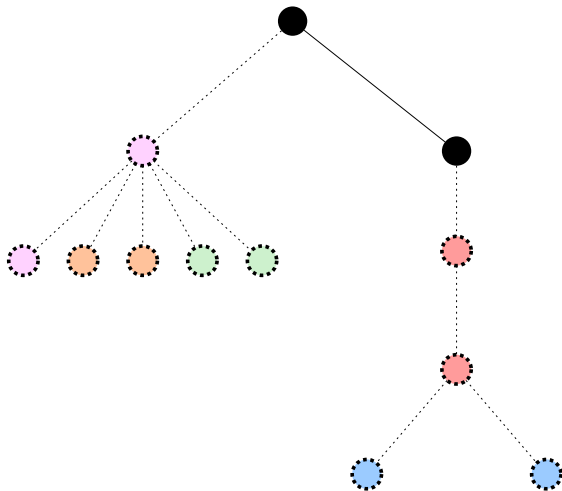
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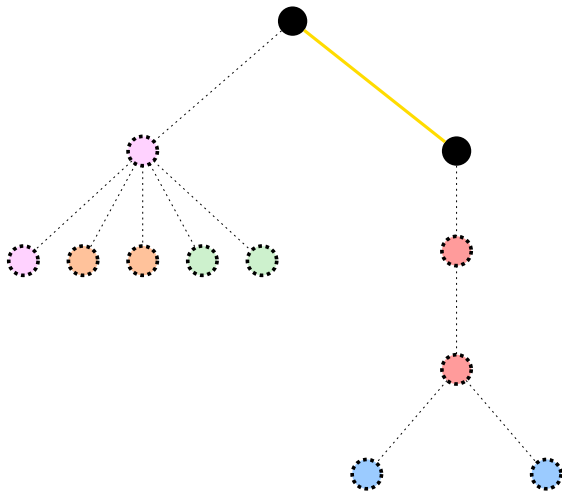
Example for the trees



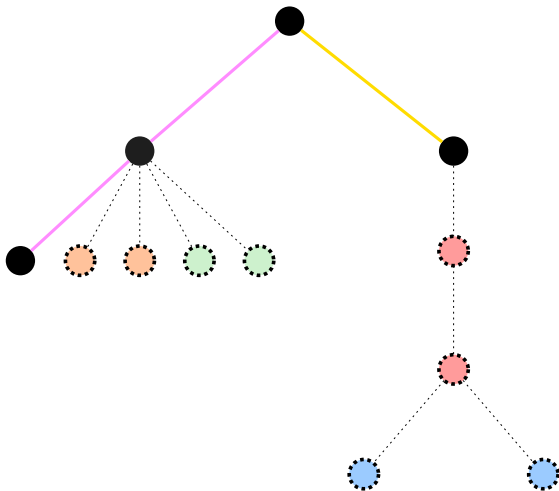
Example for the trees



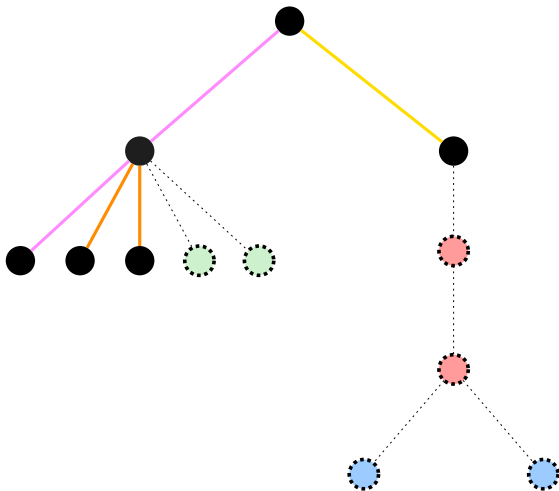
Example for the trees



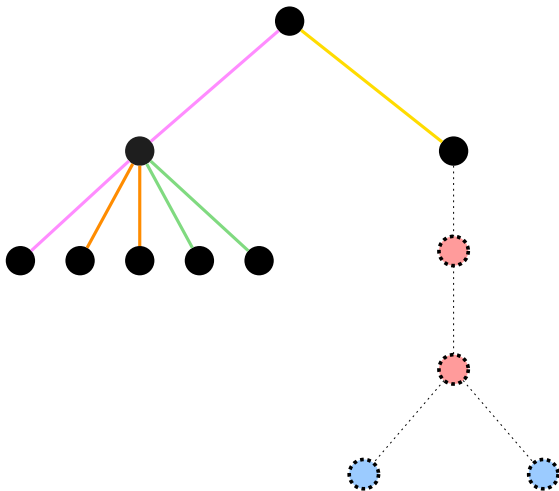
Example for the trees



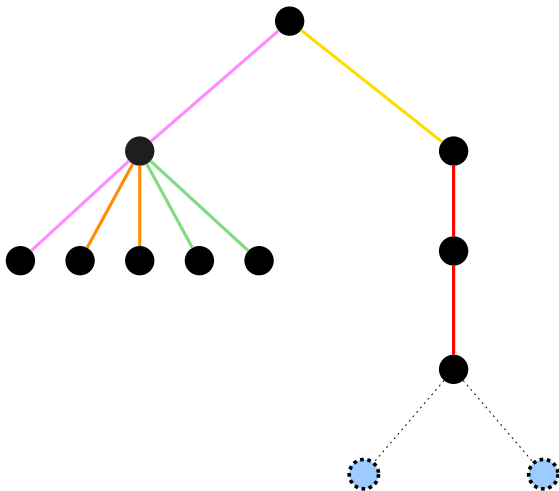
Example for the trees



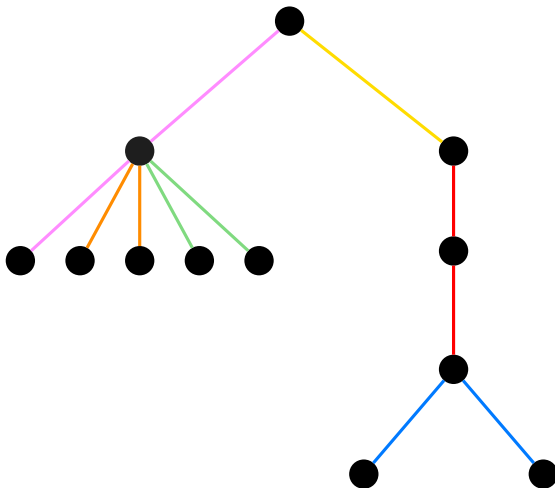
Example for the trees



Example for the trees



Example for the trees



Graph classes on which Gallai's conjecture holds

(G_{even} = graph induced by vertices of even degree)

- Lovász, 1968: $|G_{\text{even}}| \leq 1$
- Favaron, Kouider, 1988: Each vertex has degree 2 or 4
- Pyber, 1996: G_{even} is a forest
- Fan, 2005: Each block of G_{even} is triangle-free with maximum degree ≤ 3
- Bonamy, Perrett, 2016: Maximum degree ≤ 5
- Botler, Sambinelli, 2017: Treewidth ≤ 3
- Botler, Jiménez, Sambinelli, 2018: Triangle-free planar graphs

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The results in red actually show a bound of $\lfloor \frac{n}{2} \rfloor$.

Stronger conjecture

Natural obstructions to the bound $\lfloor \frac{n}{2} \rfloor$:

Odd semi-cliques: cliques on $2k + 1$ vertices minus at most $k - 1$ edges,
i.e. graphs with $> \lfloor \frac{n}{2} \rfloor (n - 1)$ edges

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Strong Gallai conjecture (Bonamy, Perrett, 2016)

Any connected graph on n vertices either has a decomposition into at most $\lfloor \frac{n}{2} \rfloor$ paths or is an odd semi-clique.

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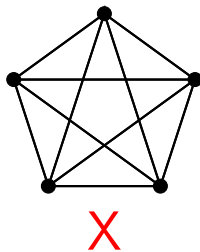
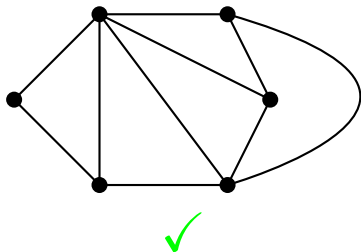
Any connected graph on n vertices either has a decomposition into at most $\lfloor \frac{n}{2} \rfloor$ paths or is an odd semi-clique.

Helpful when dealing with **disconnected** graphs

$$\lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor + \dots + \lfloor \frac{n_k}{2} \rfloor \leq \lfloor \frac{n_1 + n_2 + \dots + n_k}{2} \rfloor$$

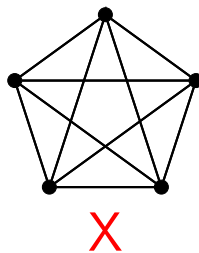
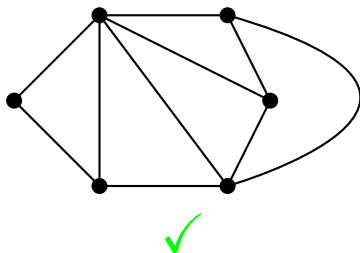
Planar graphs

A graph is **planar** if it can be embedded in the plane, i.e. drawn without edges crossing.



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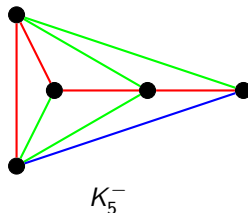
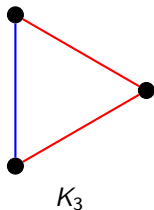


Theorem (B., Bonamy, Bonichon, 2021+)

Gallai's conjecture is true on planar graphs.

Stronger result

Planar odd semi-cliques:

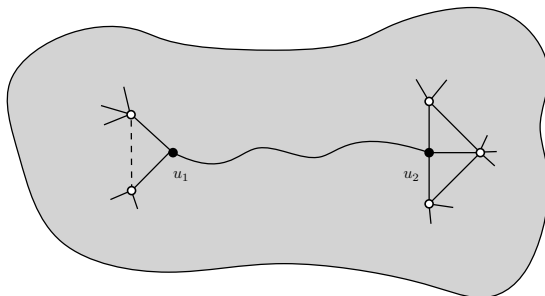


Theorem (B., Bonamy, Bonichon, 2021+)

Every connected planar graph on n vertices, except K_3 and K_5^- , has a decomposition into at most $\lfloor \frac{n}{2} \rfloor$ paths.

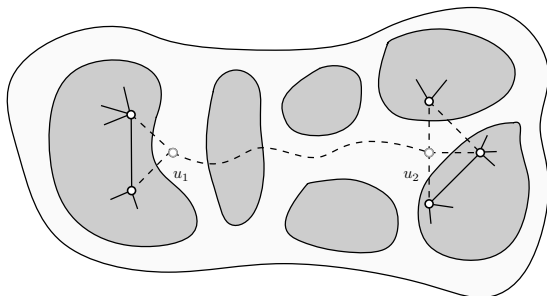
General idea

$G \equiv$ minimum counterexample



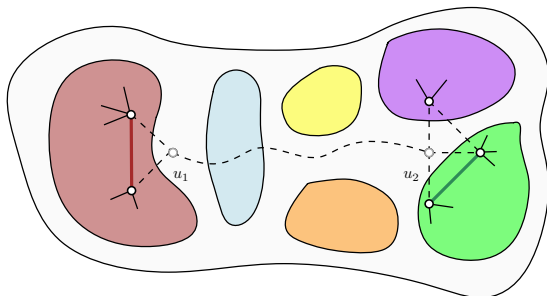
G has n vertices, u_1, u_2 are special vertices, P is a path between u_1, u_2 .

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G has n vertices, u_1, u_2 are special vertices, P is a path between u_1, u_2 .
 $G - \{u_1, u_2\} - P$ has 6 components G_1, \dots, G_6 , with n_1, \dots, n_6 vertices respectively.

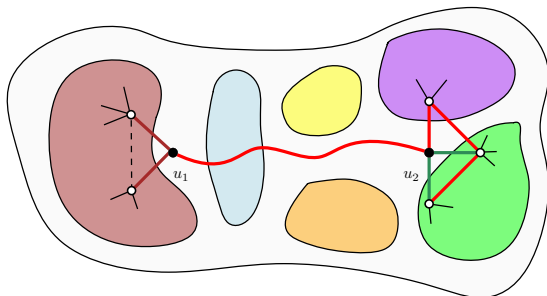
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Assume each component G_i has a path decomposition into $\leq \lfloor \frac{n_i}{2} \rfloor$ paths.

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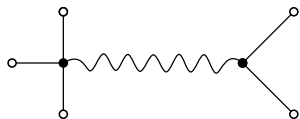
We may find a path decomposition of G into

$$\leq 1 + \lfloor \frac{n_1}{2} \rfloor + \dots + \lfloor \frac{n_6}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor \text{ paths.}$$

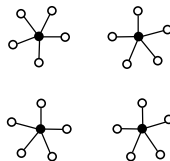
Outline of the proof

- 1 Prove that a minimum counterexample does not contain some configurations (*the hard part*)
- 2 Show a contradiction using Euler's formula (*“easy”*)

2 kinds of configurations to eliminate: C_I and C_{II} configurations



C_I

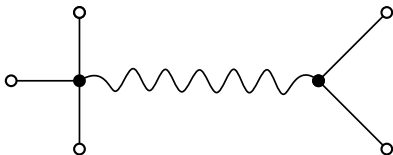


C_{II}

C_I configurations

C_I configuration: 2 vertices of degree at most 4

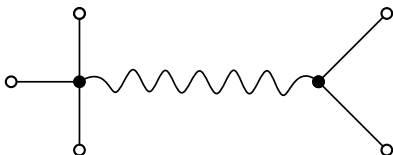
Since the graph is connected, they are associated with a shortest path.



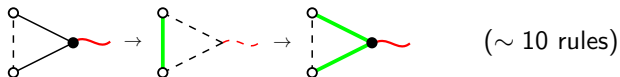
C_1 configurations

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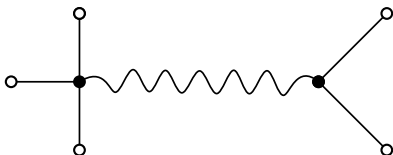
- **Half-rules:** when “far” from each other, at distance ≥ 3



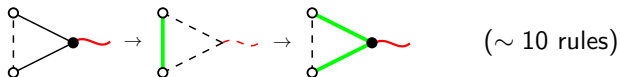
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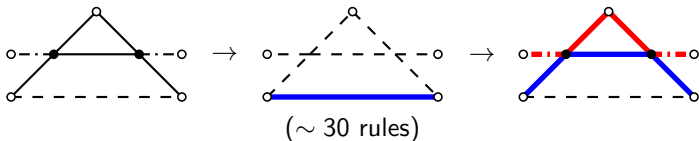
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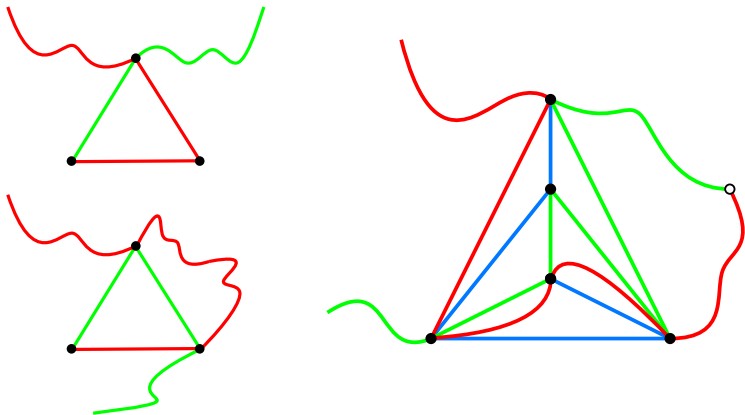


- **Resolution rules:** when the vertices are at distance ≤ 2



K_3/K_5^- strategy

We take care of each K_3 and K_5^- component generated by combining them with a path of the decomposition.



C_{II} configurations

C_{II} configuration: 4 vertices of degree 5 (“special vertices”)

And we now assume the graph is **almost 4-connected**:

- 3-connected (no 2-cut)
- No 3-cut separates two special vertices
- No 3-cut contains a special vertex and separates two of its neighbors

C_{II} configurations

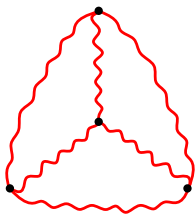
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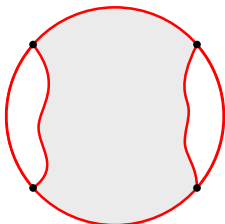
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Corollary of (Yu, 1994)

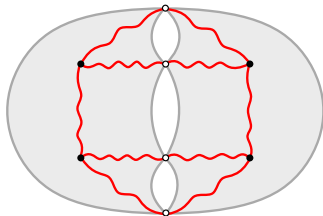
An almost 4-connected planar graph has a K_4 -, C_{4+}^1 - or C_{4+}^2 -subdivision.



K_4 -subdivision

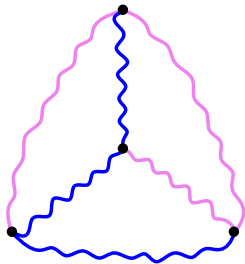


C_{4+}^1 -subdivision

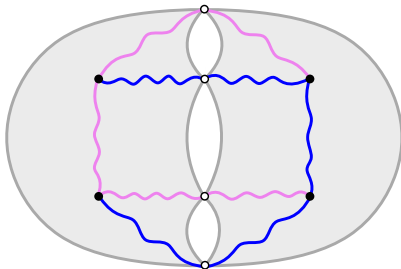
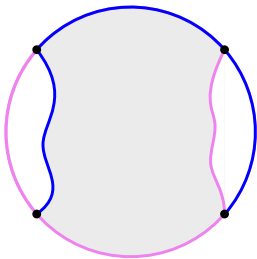


C_{4+}^2 -subdivision

Subdivisions

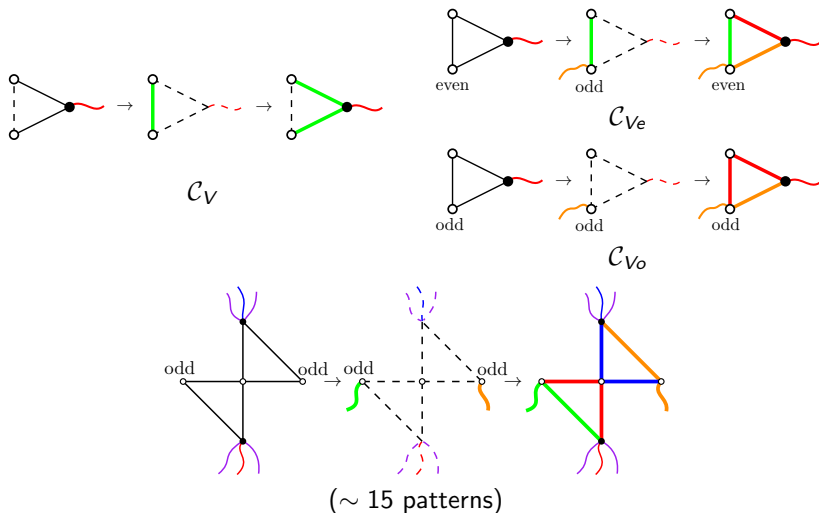


- Decomposable into 2 paths
- 1 path ends on each special vertex



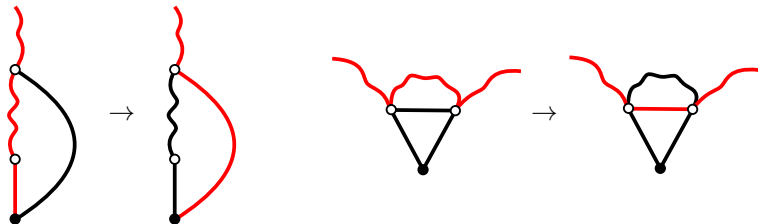
Remaining neighbors

Patterns: to take care of the 1 or 2 remaining neighbors for each special vertex

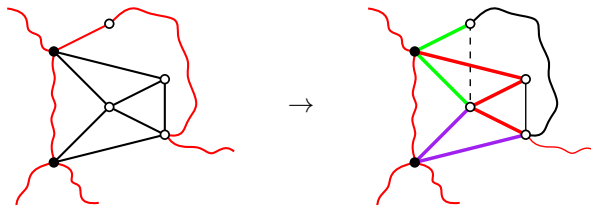


Pre-processing

Step 1 :



Step 2 :



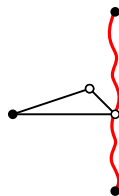
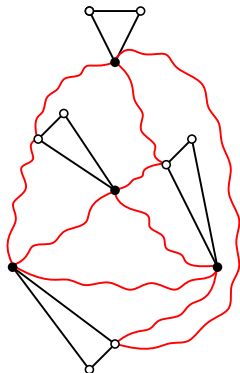
(6 rules)

Problems

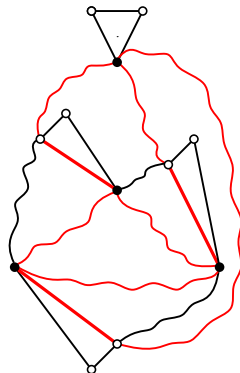
2 kinds of problems occur:

- **Distant** problems

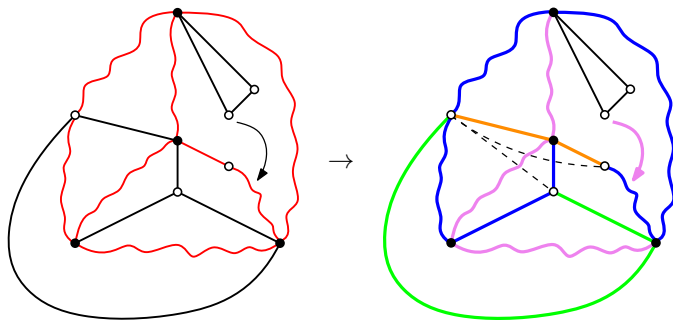
Example:



(10 rules)



- **Close problems**



(10 rules)

Proof

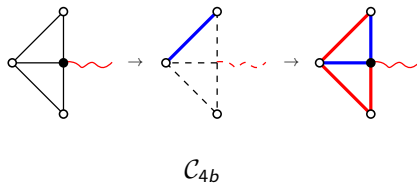
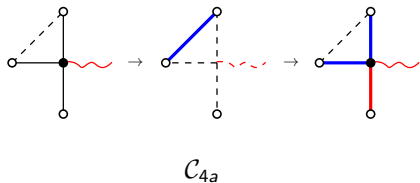
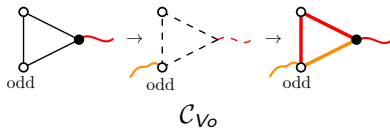
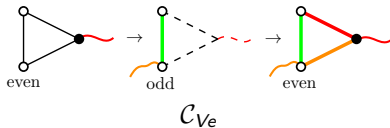
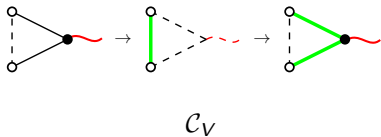
- Eliminating C_I configurations
 - Half-rules
 - Resolution rules for close cases
 - Safety: combining K_3, K_5^- with a path
- Eliminating C_{II} configurations
 - $K_{4-}, C_{4+}^1, C_{4+}^2$ -subdivisions
 - Patterns for each special vertex
 - Elimination of problems
 - Distant problems with routing
 - Close problems by hand
- Contradiction using Euler's formula

Proof

- Eliminating C_I configurations
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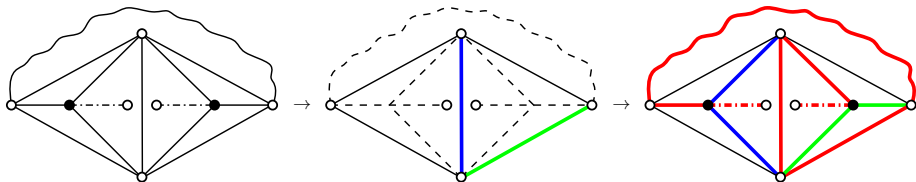
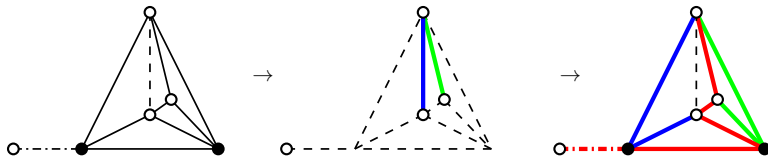
Thanks for your attention.

Half-rules (examples)



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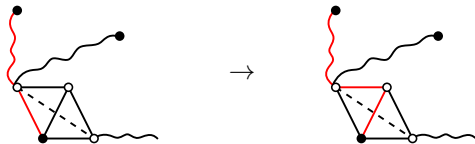
Resolution rules (examples)



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Elimination of distant problems

Routing operation



Ensures that each distant problem is turned into a \mathcal{C}_V pattern

Example:

