Generalised reachability conditions

Let $\Sigma$ be an alphabet and $L \subseteq \Sigma^*$ a (regular) language of finite words.

$\text{Reach}(L) = \{ w \in \Sigma^* : \exists u \in \Sigma^* \text{ a prefix of } w \text{ such that } u \in L \}$

We will suppose that $L$ is suffix-closed (we can always take $L \cdot \Sigma$).

Generalised reachability conditions are exactly topologically open conditions (without regularity assumptions).

Problem

Given a language $L$, what is the optimal memory required by the existential player (Eve) to win games with condition $\text{Reach}(L)$?

That is, what is the number $M_L$ such that:

• If Eve wins a game with condition $\text{Reach}(L)$, she has an strategy using memory of size at most $M_L$.

• For every $L$ there exists a $\text{Reach}(L)$-game won by Eve where she requires at least $M_L$ memory states to win.

Upper bound

If $L$ is a regular language, let $A$ be the minimal deterministic automaton for it. Then, given a $\text{Reach}(L)$-game $G$ won by Eve, the game $G \times A$ is won by her positionally.

Conclusion

The automaton $A$ can be used as a memory structure for every $\text{Reach}(L)$-game and $M_L \leq |A|$.

Lower bound

For $w \in \Sigma^*$, its left quotient is

$$w^{-1}L = \{ u \in \Sigma^* : wu \in L \}$$

The set of left quotients is naturally ordered by inclusion.

An antichain is a subset $A$ of a partially ordered set such that every pair of elements of it are incomparable.

In [CFH14] the memory requirements for the universal player are established (i.e. topologically closed conditions):

**Theorem [CFH14]**

The memory required by the universal player is given by the size of a maximal antichain of left quotients of $L \cdot \Sigma^*$.

(No regularity assumptions).

This lower bound holds for the existential player too, but it is not tight.

**Lower bound**

$M_L$ is at least as big as a maximal antichain of left quotients of $L \cdot \Sigma^*$.

Examples

$|\Sigma| = \{a, b\}$

**Example 1:** $L = aaa \cdot \Sigma^*$

Left quotients: $\varepsilon^{-1}L \subseteq a^{-1}L \subseteq aa^{-1}L \subseteq aaa^{-1}L$.

For every $L$-game, 1 memory state suffices (we meet the lower bound).

**Example 2:** $L = aa'b \cdot \Sigma^*$

Left quotients: $\varepsilon^{-1}L \subseteq a^{-1}L \subseteq ab^{-1}L$.

The lower bound we provide is 1, but we need 2 memory states for a game with one vertex controlled by Eve.

References