A characterization of half positional \(\omega\)-regular languages

Antonio Casares, Pierre Ohlmann

HIGHLIGHTS 2023
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White to move.
What is the optimal move?
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What is the optimal move?

Makes sense!

Even if we don't know how we got to this position.
White to move.
What is the optimal move?
Games on Graphs

Players move a token in turns producing an infinite word $w \in \Sigma^\omega$.

$\Sigma = \{a, b, c\}$
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Finite or Infinite

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The winning condition is given by a language $L \subseteq \Sigma^\omega$.

Eve wins if the sequence $w \in \Sigma^\omega$ produced belongs to $L$. 
A positional strategy (for Eve) is a mapping:

$$\sigma : V_{\text{Eve}} \rightarrow E.$$
Positional Strategies

A positional strategy (for Eve) is a mapping:

$$\sigma : V_{Eve} \rightarrow E.$$ 

The positional strategy above ensures the language

$$L = \text{Words containing the factor } bc \text{ infinitely often.}$$
A language $L$ is **half positional** if, for every game $G$ using $L$ as winning condition:

Eve can win $G$ $\implies$ She can win using a positional strategy.
Positional Languages

**Half positionality**
A language $L$ is **half positional** if, for every game $G$ using $L$ as winning condition:

Eve can win $G$ $\implies$ She can win using a positional strategy.

**Bipositionality**
A language $L$ is **bipositional** if both $L$ and $\Sigma^\omega \setminus L$ are half positional.
The class of \( \omega \)-regular languages can be defined equivalently as those that are:

- Recognizable by a Büchi automaton.
- Recognizable by a deterministic parity automaton.
- Definable by \( \omega \)-regular expressions.
- Definable in MSO with successor.
Theoretical Computer Scientist in the Wild
$L \subseteq \Sigma^\omega$
Is it positional?

| L ⊆ Σ^ω |

half / bi
What do we already know?
Known Results About Bipositionality
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Bipositionality over finite graphs

- Characterization of bipositionality over finite graphs (Gimbert, Zielonka 2005).
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- Corollary: $L$ is bipositional if and only if Eve and Adam can win positionally in 1-player games.
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- Characterization of bipositionality over infinite graphs (Colcombet, Niwiński 2006).

For \( L \) prefix-independent:

\[
L \text{ is bipositional} \iff L \text{ is the parity objective.}
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What do we know about half positionality?
What do we know about half positionality?

Not much...*

*This is of course not true. There are many very interesting results about half positionality.
What do we know about half positionality?

Decidability for finite graphs (Kopczyński 2007)

Given an $\omega$-regular language $L$ (represented as a deterministic parity automaton of size $n$) we can decide whether $L$ is half positional over finite graphs in time $O(n^{n^2})$. 
<table>
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<td><strong>Decidability for infinite graphs?</strong></td>
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# Questions About Half Positionality

## Decidability for infinite graphs?

Can we decide whether $L$ is half positional over infinite graphs? Complexity?

## Finite vs Infinite

| Half positional for finite graphs | $\implies$ | Half positional for infinite graphs | ? |
Questions About Half Positionality

Decidability for infinite graphs?
Can we decide whether \( L \) is half positional over infinite graphs?
Complexity?

Finite vs Infinite
Half positional for finite graphs \( \implies \) Half positional for infinite graphs ?

1-to-2-player lift?
Eve can win positionally in Eve-games \( \implies \) \( W \) is half positional ?
Questions About Half Positionality

Kopczyński’s Conjecture (2006)

Let $L_1$ and $L_2$ be two prefix-independent half positional languages. Then:

$L_1 \cup L_2$ is half positional.

\[
\begin{align*}
\text{For all } u \in \Sigma^*: \\
w \in \mathcal{L} & \iff uw \in \mathcal{L}
\end{align*}
\]
Questions About Half Positionality

Kopczyński’s Conjecture (2006)

Let $L_1$ and $L_2$ be two prefix-independent half positional languages. Then:

$$L_1 \cup L_2 \text{ is half positional.}$$

- Does not hold for finite graphs (Kozachinskiy, 2022).
  Non $\omega$-regular counterexample!

- Open for infinite graphs.
Contributions
Main Contribution

Structural characterization of half positional $\omega$-regular languages*

We provide necessary and sufficient syntactic conditions on deterministic parity automata recognizing half positional languages.

*Characterization for deterministic Büchi automata presented in a previous work by Bouyer, C., Randour and Vandenhove (CONCUR 2022).
Corollaries of our Characterization

For $L$ an $\omega$-regular language:

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Corollaries of our Characterization

For $L$ an $\omega$-regular language:

**Finite vs Infinite**

| Half positional for finite graphs | Half positional for infinite graphs |

**Decidability in polynomial time**

Decidability of half positionality of $W$ in *polynomial time*. 
Corollaries of our Characterization

For $L$ an $\omega$-regular language:

### Finite vs Infinite

| Half positional for finite graphs | $\iff$ | Half positional for infinite graphs |

### Decidability in polynomial time

Decidability of half positionality of $W$ in *polynomial time*.

### 1-to-2-player lift

Eve can win positionally in Eve-games $\iff W$ is half positional.
Kopczyński conjecture holds for $\omega$-regular languages

Let $L_1$ and $L_2$ be two prefix-independent, $\omega$-regular, half positional languages. Then:

$$L_1 \cup L_2 \text{ is half positional.}$$
A random fact...

We need to extensively use **history-deterministic** (also called good for games) automata in the proof...

although they have no direct connection to the statement!
A characterization of half positionality
A characterization of half positionality

But... What is this characterization?
A characterization of half positionality

But... What is this characterization?
Take a look at the poster to discover!
Take a look at the poster to discover!

Thanks for your attention!