Deterministic Leader Election in $O(D + \log n)$ Rounds with Messages of size $O(1)$

A. Casteigts, Y. Métivier, J.M. Robson, A. Zemmari

LaBRI - University of Bordeaux

DISC 2016, Paris
“[ Likely one of] the two most studied tasks in distributed computing literature” (Dinitz et al. JACM). More than 1500 papers published on this problem.

**Def.:** A distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one process is marked as *elected* and all others are *non-elected*.
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- Feasibility of deterministic LE in **anonymous** graphs
- Complexity of deterministic LE with unique **identifiers**
- Complexity of randomized LE (in anonymous graphs)
Leader election

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This talk.
Model and complexity measures

The model

- Synchronous rounds (and simultaneous wakeup)
  1. Send messages to neighbors
  2. Receive messages from neighbors
  3. Perform computation
- Possibility to send different messages to different neighbors
- Identifiers of size $O(\log n)$
- Messages of size $O(1)$
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### Bit complexity

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→ Measure 3 captures both time and communication complexities (silence is not free)
## Best bounds

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Lower bound

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<td>$2[\log_2((Id_{max} + 2)/3.5)] = \Omega(\log n)$ bits</td>
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**Algorithm STT**

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This general principle is folklore. Takes $O(D)$ time in the CONGEST model. Trivial adaptation in $O(D \log n)$ bit rounds. → **STT** takes it down to $O(D + \log n)$ bit rounds.
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![Diagram](attachment:image.png)
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![Graph diagram]

Trivial adaptation in $O(D \log n)$ bit rounds.

$\rightarrow STT$ takes it down to $O(D + \log n)$ bit rounds.
The main trick...

**α-encoding of the identifiers**

Given \(ld\), we define \(\alpha(ld)\) as the unary representation of the binary length of \(ld\), followed by 0, followed by the binary representation of \(ld\), i.e.:

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\alpha(ld) = \text{base1}(\text{base2}(ld)) \cdot 0 \cdot \text{base2}(ld).
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Ex. \(ld = 25 = 11001\), then \(\alpha(ld) = 11111011001\)
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Order induced by the first distinct letter.
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→ Decidable based on prefixes, no need to wait...
The Spreading Algorithm $S$

**Pipelining the identifiers**

Each node maintains the largest prefix of (encoded) identifier known so far, based on that of neighbors. Here are the round actions:

1. Update local largest prefix (e.g. based on that of neighbors)
2. Send signals indicating how the prefix was updated (or not)
3. Receive signals from neighbors
4. Update local copy of each neighbor’s largest prefix

Exchanged signals (constant-size):
- `append 0` or `append 1`
- `delete 1`, `delete 2`, `delete 3`
- `change`, `null`

These signals mean that the largest known prefix was updated by:
- appending `0` or `1`
- deleting one, two or three letters
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▶ deleting one, two or three letters
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▶ leaving it unchanged
Update rules for prefix

The largest known prefix, denoted $p$ here, is updated in each round based on current status (active or follower) and neighbors largest known prefixes (denoted $p_n$ here) learnt through signals.

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<tr>
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7. null action
Resulting properties

Lemma 8

Let \( u \) and \( v \) be two neighbors and \( p_u \) and \( p_v \) their prefixes. The only possible cases after a round (up to renaming) are:

1. \( p_u = p_v \)
2. \( p_u = p \) and \( p_v = pw \) with \( 1 \leq |w| \leq 2 \)
3. \( p_u = p0 \) and \( p_v = p1a \)
4. \( p_u = p1 \) and \( p_v = p0w \) and \( |w| \leq 3 \)
5. \( p_u = p \) and \( p_v = pw \) and \( 3 \leq |w| \leq 6 \) and \( u \) performed a delete

where \( p \) and \( w \) are words and \( a \) is either 0 or 1.
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After at most $6D + |\alpha(Id_{max})|$ rounds, the spreading is complete and all nodes know $\alpha(Id_{max})$. 
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$\rightarrow O(D + \log n)$ bit rounds.
Spanning tree construction ($\rightarrow ST$)

**Occurs in parallel of spreading**

- If $u$ applies rule 2 relative to $v$, then $v$ becomes $u$’s parent.
- If $u$ applies rule 3 relative to $v$, then $v$ becomes (or remains) $u$’s parent.
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$\rightarrow$ Additional signals (constant-size).
Termination ($\rightarrow S\bar{T}T$)

Recursive termination proceeds from the leaves up to the root, according to the following rule.

**Termination rule**

If it holds that

1. $u$ is a follower
2. $u$’s prefix is well-formed and identical to that of neighbors
3. for every child $v$ of $u$, $\text{term}_v = \text{true}$
4. $\text{term}_u = \text{false}$

then $\text{term}_u \leftarrow \text{true}$ and $u$ notifies its parent
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**Remark:** $\text{term}_u$ may wrongly be $\text{true}$ due to local maxima and then re-become $\text{false}$ (this is fine...). However, when an active node is notified by all its children, it becomes elected and correctly decides termination.
Conclusion

\( STT \) runs in \( \Theta(D + \log n) \) bit rounds

1. Spreading phase in \( O(D + \log n) \)
2. Spanning tree in parallel of spreading
3. Termination detection costs \( O(D) \) additional rounds

+ Lower bound \( \Omega(D + \log n) \)

Open questions

▶ How about weaker models? E.g. beeping models?
▶ Is the technique useful for other problems?
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