

Graph temporalisation

Filippo Brunelli, Pierluigi Crescenzi and Laurent Viennot

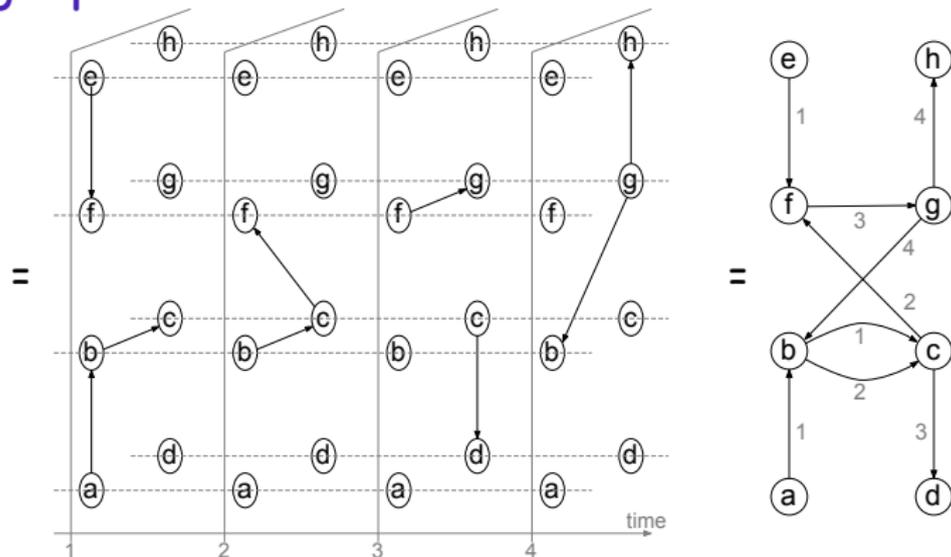


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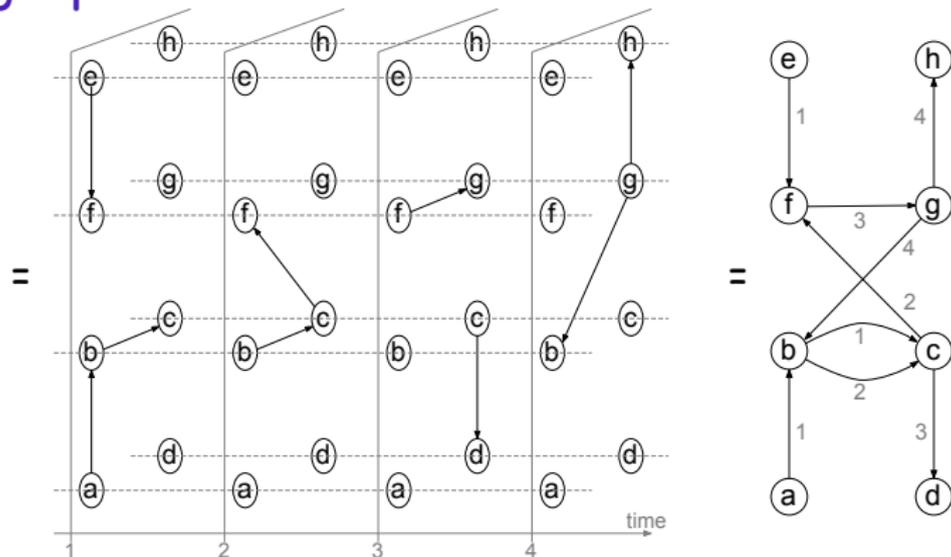
Temporal graphs



Also known as :

- time-dependent networks [Cooke, Halsey 1966],
- edge scheduled networks [Berman 1996], dynamic graphs [Harary, Gupta 1997], temporal networks [Kempe, Kleinberg, Kumar 2002], evolving graphs [Bhadra, Ferreira 2003],
- time-varying graphs [Casteigts, Flocchini, Quattrociocchi, Santoro 2012],
- link streams [Latapy, Viard, Magnien 2018],...

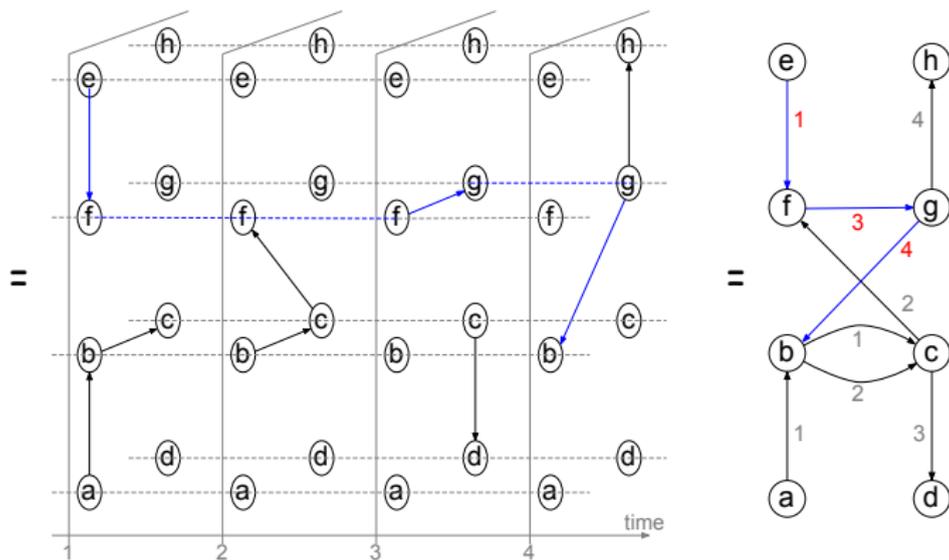
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Temporal path



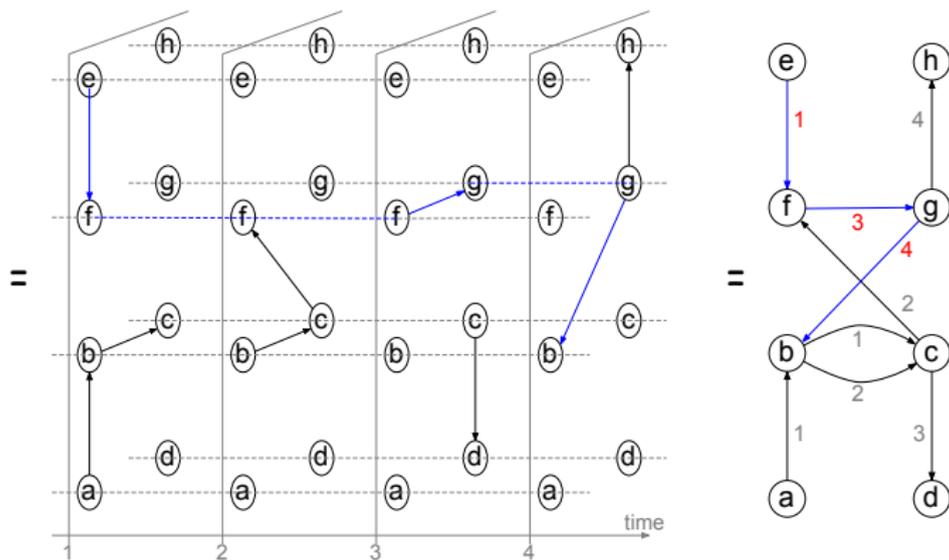
A path with increasing time labels.

Temporal paths are strict and waiting is allowed.

Reachable set $\mathcal{R}(u)$: the set of nodes v that u can reach by a temporal path (we let $u\mathcal{R}v$ denote $v \in \mathcal{R}(u)$).

Above, we have : $e\mathcal{R}b, b\mathcal{R}h, \neg a\mathcal{R}f, \neg a\mathcal{R}h$

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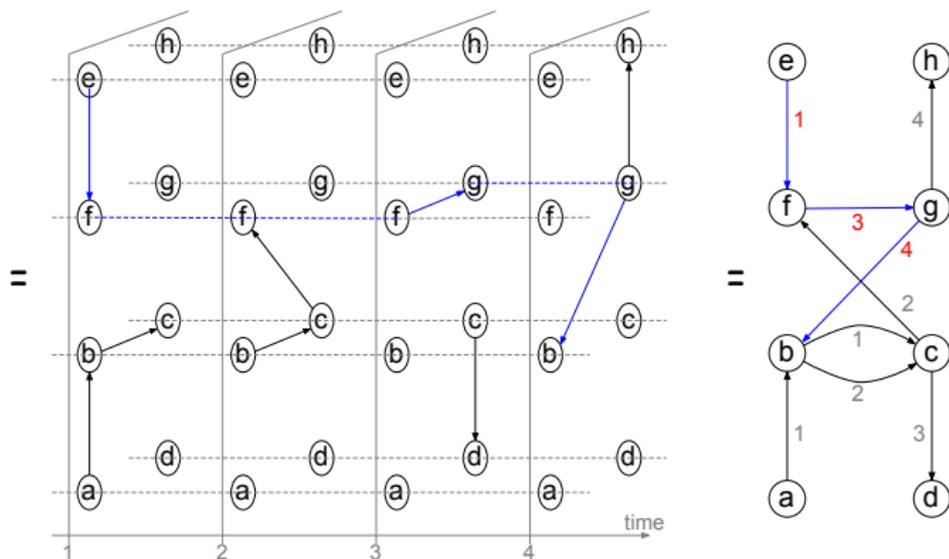
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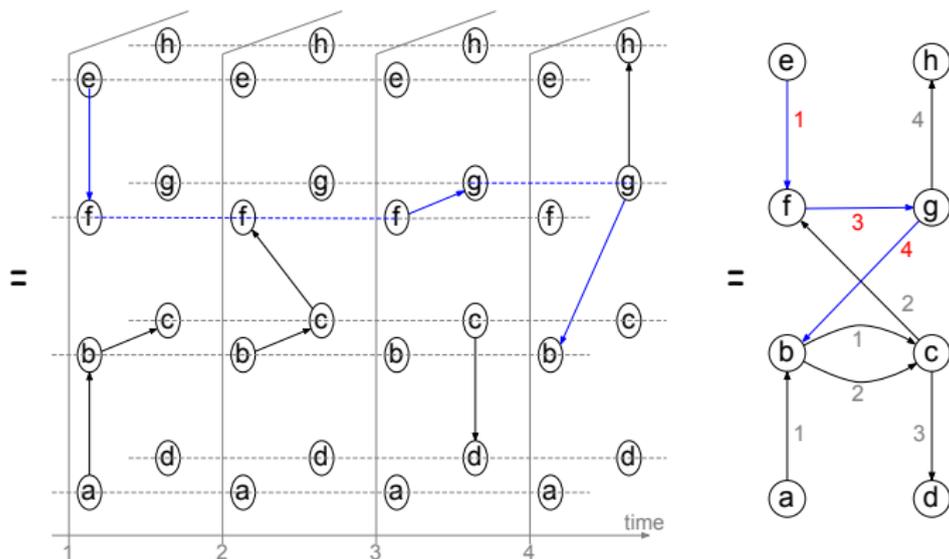
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Given a multidigraph $M = (V, A)$ assign **time labels** to edges.

Maximum reachability goal (**MRTT**): maximise total reachability $\sum_u |\mathcal{R}(u)|$.

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Undirected temporalisation and label connectivity

Label-connected graph : assign time labels to an undirected graph (V, E) such that $\mathcal{R}(u) = V$ for all $u \in V$.

Undirected setting : if uv is available at time t , both (u, v) and (v, u) are available at time t .

Known results [Göbel, Cerdeira, Veldman 1991] :

- Deciding label-connectivity is **NP-hard**.
- A graph is **label-connected** iff it contains **two edge-disjoint spanning tree-pairs** and some 4-cycle condition.

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Public transport



Temporal edge : a vehicle on a **line** going from one stop to the next one.

The **trip** of a vehicle produces a sequence of temporal edges.

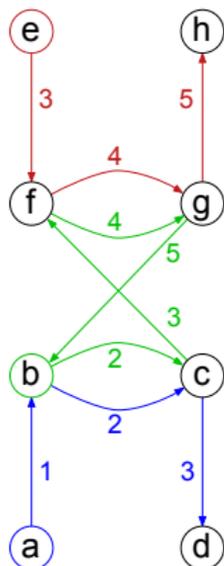
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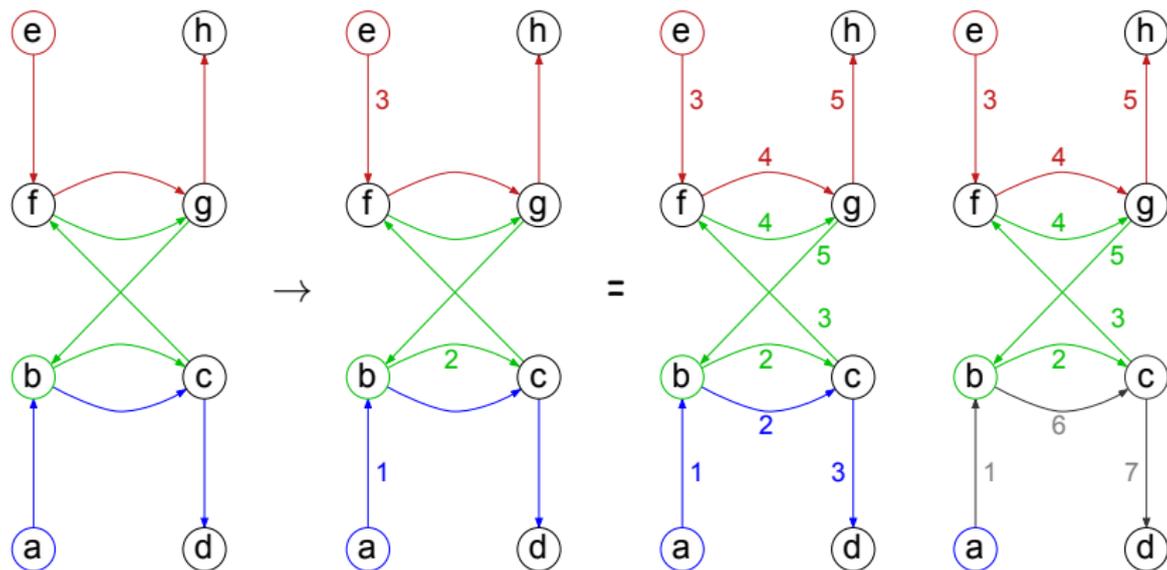
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Temporal edges are grouped into walks called **trips**.

Trip temporalisation

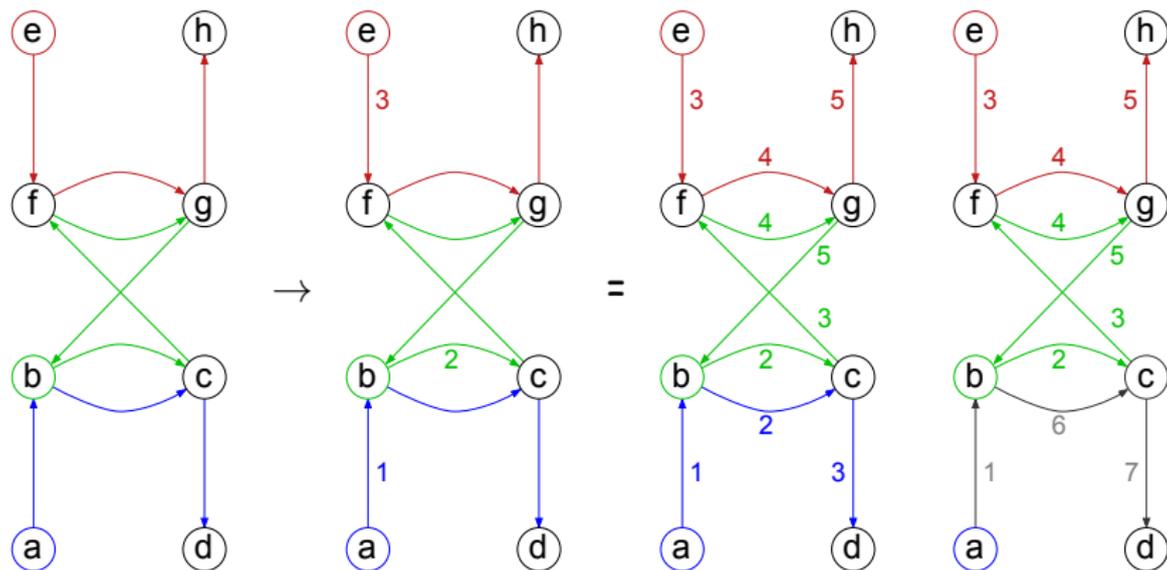
Trip temporalisation : Given a multidigraph M whose edges are partitioned into t trips, set a **starting time** for each trip (we note $n = |V(M)|$ and $m = |E(M)|$).



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One-to-one reachability is hard

O2O-RTT : given $u, v \in V$ decide if there exists a **trip temporalisation** such that uRv .

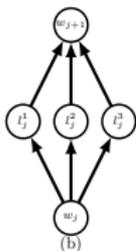
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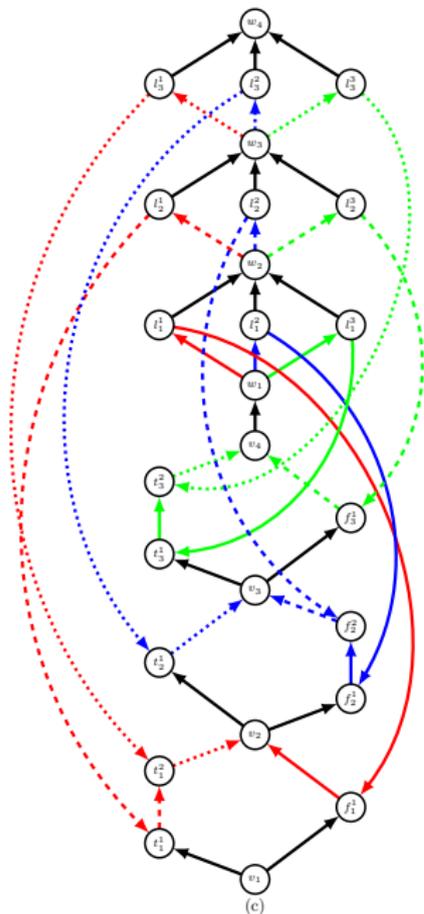
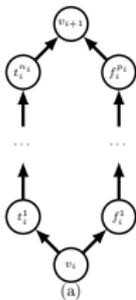
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3-SAT reduction :



Consequence and work-around

Theorem For any $\varepsilon > 0$, **MRTT** cannot be approximated within $O(n^{1-\varepsilon})$ unless $P=NP$.

Idea : connect many sources to v_1 and many sinks to w_{m+1} .

Theorem **O2O-RTT** using at most k trips can be solved in $2^{O(k)} m \log T$ where $m = |E(M)|$ and T is the number of trips.

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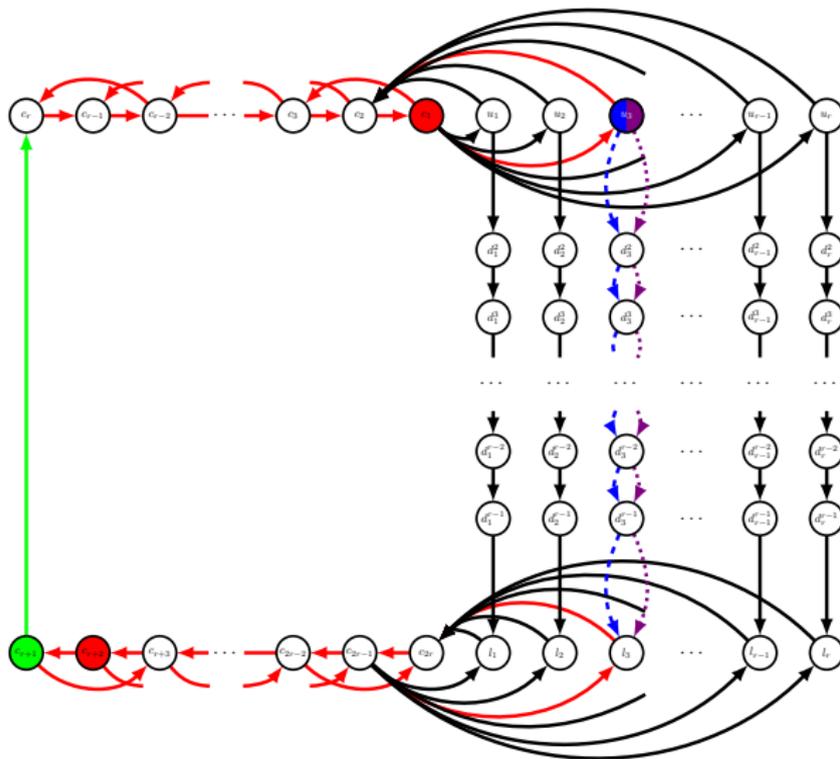
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Strong temporalisabilty

A trip network is **strongly temporisable** when for each pair $u, v \in V$, there exists a temporalisation such that uRv .

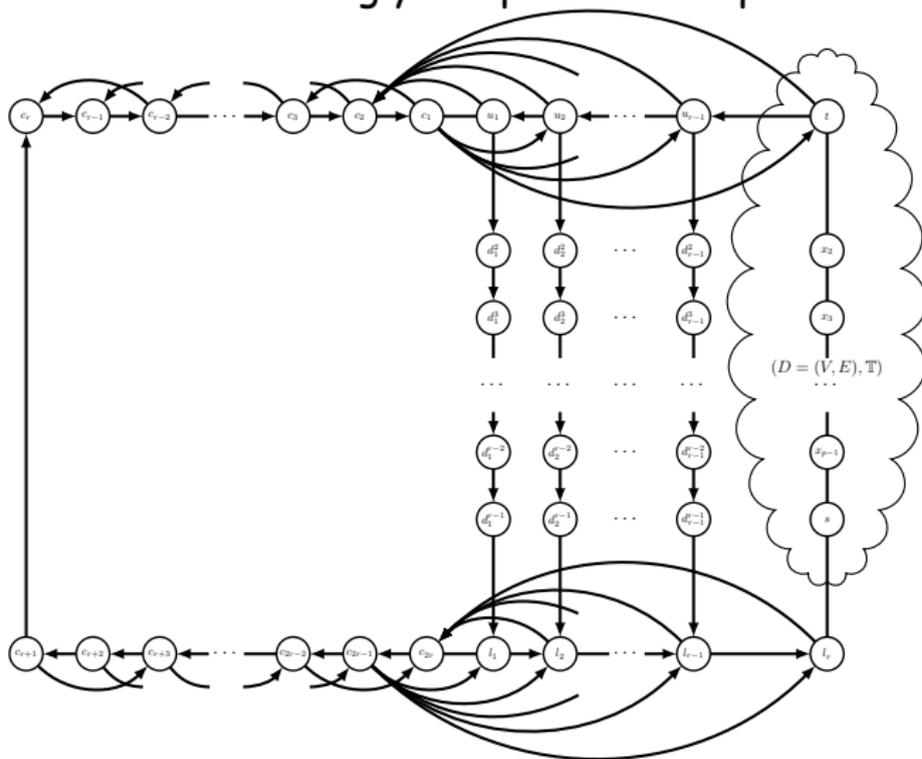
A difficult strongly temporisable network



Bounded **reachability** : $\sum_u |\mathcal{R}(u)| = O(n^{1.5})$ (\sqrt{n} gap).

Unapproximability of MRTT

Theorem MRTT cannot be approximated within $\sqrt{n}/12$ unless P=NP even in strongly temporisable trip networks.



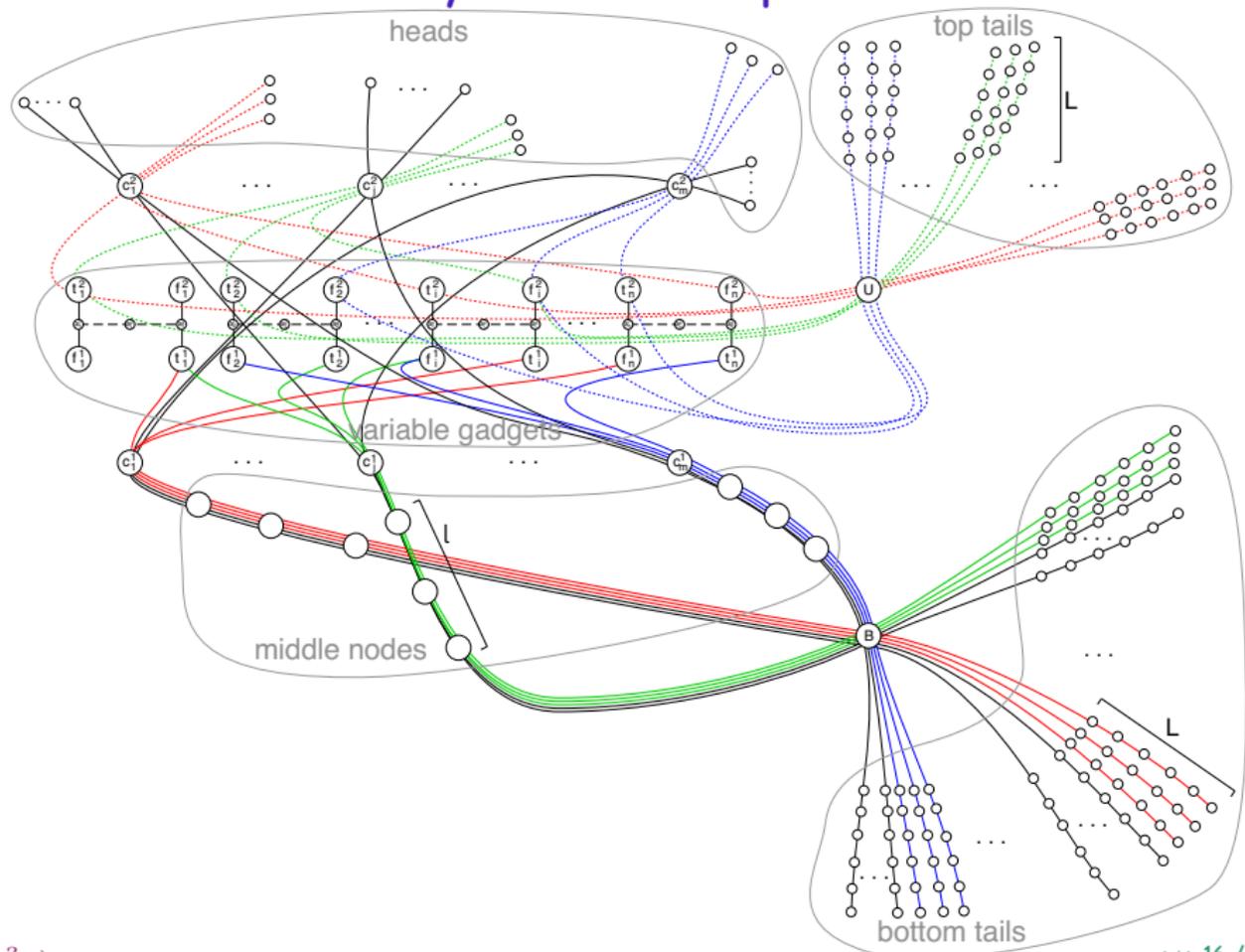
Reduction from O2O-MRTT.

Symmetry



For every trip $\langle v_1, \dots, v_k \rangle$, there is a **reverse** trip $\langle v_k, \dots, v_1 \rangle$.

MRTT is hard in symmetric trip networks



Approximability of symmetric trip networks

Fact A symmetric trip network is **strongly temporisable** iff M is (strongly) **connected**.

Theorem MRTT can be approximated within $9/2$ in strongly temporisable symmetric trip networks.

Idea :

- group trips and reverse trips into pairs,
- compute a **centroid pair** C , and a partition $A, \{C\}, B$ of trip pairs,
- schedule trips pairs in $A, \{C\}, B$ respectively so that :
 - each node in $V(A \cup \{C\})$ can reach $V(\{C\} \cup B)$.

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Going further

All results hold with arbitrary positive **delays** on edges,

and with fixed **waiting times** at stops.

Open : **varying waiting times** can be chosen in the temporalisation.

Open : each trip is **duplicated K times**.

Open : **directed graph** temporalisation (more tomorrow).

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Define the **temporal reachability** of a temporal graph as the **number of node pairs** that are **temporally connected**.

Problem : given a **strongly connected** multi-digraph, assign one time label per edge so that **temporal reachability** is maximal.

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Given an **undirected graph** G , deciding whether there exists an assignment of one time label per edge such that **all pairs** are temporally connected is **NP-complete**. [Göbel, Cerdeira, Veldman 1991]

Approximation is obvious : take a spanning tree and assign time labels that connect $(n/2)^2$ **pairs**.

Related (Gossip/telephone problem [Bumby 1981]) : The minimum number of time labels allowing to temporally connect all pairs is $2n - 2$, or $2n - 4$ if G has a C_4 (one or two time labels per edge).

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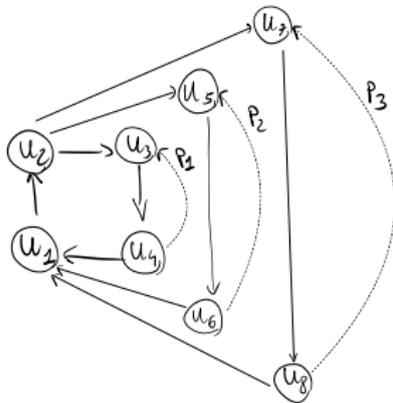
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It is **NP-hard** to decide if a strong digraph has such a **pair**.
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Conjecture : There exist c such that any **c -edge-connected** digraph has such a **pair**. [Thomassen 1989]

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