

Sharp Thresholds in Random Simple Temporal Graphs

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TEMPOGRAL project meeting

Of course, the graphs are temporal

First aim: connectivity

- Multiple facets in the temporal setting
- No guarantee that one source-sink is enough
- No guarantees that spanners are small
- Bad examples — exceptional or typical?

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- Directed graph strongly connected iff some source is a sink
- Connectivity always demonstrated by a linear subgraph
(incoming + outgoing tree)
Single spanning tree in undirected case
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Connectivity, temporal graph

Worst-case

- Temporal connectivity without $O(n)$ -spanners
[Kempe, Kleinberg, Kumar 2000]
- Temporal connectivity by $\Theta(n^2)$ labels without $o(n^2)$ -spanners
[Axiotis, Fotakis 2016]
- Ordered cliques have $O(n \log n)$ -spanners
[Casteigts, Peters, Schoeters 2021]
- Minimal spanner: APX-hard
[Akrida, Gasieniec, Mertzios, Spirakis 2017]

Things can be complicated!
...and hard to check

Question: are these behaviours exceptional or typical?

Typical — asymptotically almost surely ($\Pr \rightarrow 1$)
... depends on the model of random generation!

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Connectivity in random temporal settings

- Similar random temporal settings
Call-any gossiping, Rendezvous population protocols...
- Future interactions independent from past
- Connectivity happens soon
($\approx 1.5n \log n$ edges expected value)
- Spanners?
- Simple temporal graphs: no repetitions — implications?

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Random *Simple* Temporal Graphs

- Simple: each edge has one label
Labels are distinct
- Main model we study: RSTG
- Equivalent representations
 - $\mathcal{G}_{n,p}$ (Erdős–Rényi random graph) with random edge order
 - $\mathcal{G}_{n,p}$ with random edge labels from $[0; 1]$
 - $\mathcal{G}_{n,p}$ with random edge labels from $[0; p]$
 - K_n (clique) random edge labels from $[0; 1]$, edges after p removed
- Notation: $\mathcal{F}_{n,p}$
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RSTG (Call-once) vs. Call-any

- Similar behaviour
 - Few edges get actually repeated in Call-any
- But independence of past and future is destroyed (so technical tools are lost)
- There are dependencies but they are «weak»
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- $\frac{1}{n}$: footprint gets linear-size connected component
- $\frac{\log n}{n}$: footprint gets connected
- $\frac{\log n}{n}$: most pairs get connectivity
- $\frac{\log n}{n}$: $n - o(n)$ vertices get pairwise connectivity
- $\frac{2 \log n}{n}$: first temporal source
- $\frac{2 \log n}{n}$: most vertices temporal sources
- $\frac{3 \log n}{n}$: all vertices temporal sources
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Connectivity: main points

- $\frac{\log n}{n}$: $\forall u : \forall v : \mathbf{a.a.s.} u \rightsquigarrow v$
- $\frac{2 \log n}{n}$: $\forall u : \mathbf{a.a.s.} \forall v : u \rightsquigarrow v$
- $\frac{3 \log n}{n}$: $\mathbf{a.a.s.} \forall u : \forall v : u \rightsquigarrow v$

- Normally takes $\frac{\log n}{n}$ time and $O(\log n)$ hops for $u \rightsquigarrow v$
- But a vertex can «sleep» with no edges for some time
- Longest observed sleeping time: $\frac{\log n}{n}$
- Sleep determines extra $\frac{\log n}{n}$ per shift of **a.a.s.**!

Can connectivity be shown by a small subgraph?

- $\mathcal{F}_{n, (3+o(1)) \frac{\log n}{n}}$ has $(1.5 + o(1))n \log n$ edges...
- Can we go smaller?
- Need at least $2n - 4$ labels
- Optimal spanner exists in $\mathcal{F}_{n, (4+o(1)) \frac{\log n}{n}}$
Current WIP: but not earlier
- $2n + o(n)$ -spanner exists in $\mathcal{F}_{n, (3+o(1)) \frac{\log n}{n}}$

[Bumby 1970]

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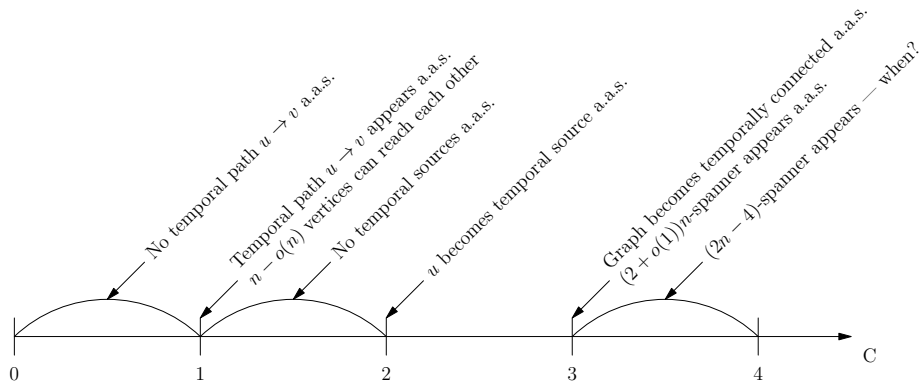
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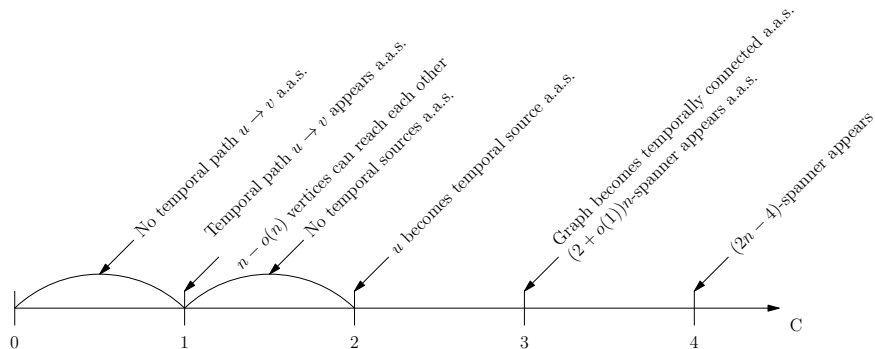
Spanners: main points

- $\frac{3 \log n}{n}$: $2n + o(n)$ -spanner
- $\frac{4 \log n}{n}$: $2n - 4$ -spanner

Timeline



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(With WIP results)

- *Routing* in a temporally connected graph:
Temporal path $u \rightsquigarrow v$ for each pair (u, v)
- *Central* vertex for a routing:
Belongs to $\Theta(n^2)$ paths
- Is there a centralised / decentralised routing?
- Are all the routings such?

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Centralisation (WIP)

- $\frac{3 \log n}{n}$: $2n + o(n)$ -spanner ... can have centralised routing
- $\frac{3 \log n}{n}$: $2n + o(n)$ -spanner ... can have decentralised routing (a different spanner)
- $\frac{4 \log n}{n}$: $2n - 4$ -spanner ... only with centralised routing
- $\frac{5 \log n}{n}$: $2n + o(n)$ -spanner ... only with decentralised routing

Building block for larger construction

Tree of temporal paths

- Start with a single vertex
 advanced version: and some starting time
- Add earliest later edge that adds a new vertex to tree
- Repeat

Same can be done in reverse

- Obstacles to small spanners and early connectivity are rare
- In random simple temporal graphs nearly optimal spanners appear nearly immediately after connectivity
- Weak dependencies can be handled
- Decentralisation overhead is low

Thanks for your attention!

Questions?

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