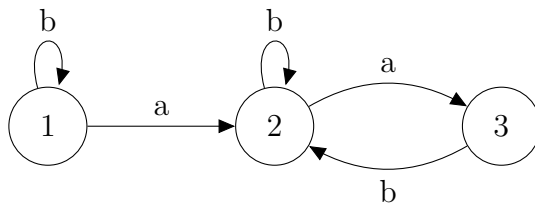


1 Automata over infinite words and trees

Exercise 1

The automaton below accepts with the parity condition (maximal priority visited infinitely often is even), with the priority of a state being equal to the state number.



1. Which language is accepted by \mathcal{A} ?
2. Give an equivalent Büchi automaton.

Exercise 2

Let $\Sigma = \{a, b, c, d\}$. For $w \in \Sigma^\omega$ we write $\text{inf}(w)$ for the set of letters occurring infinitely often in w . Let $L \subseteq \Sigma^\omega$ consist of all words w such that $\text{inf}(w) \cap \{a, b\} \neq \emptyset$ implies $c \in \text{inf}(w)$.

- Give a Büchi automaton recognizing L .
- Give a *deterministic* parity automaton recognizing L .
- Show that there is no *deterministic* Büchi automaton recognizing L .

Exercise 3

1. Given $K \subseteq \Sigma^+$ we define $\vec{K} \subseteq \Sigma^\omega$ as

$$\vec{K} = \{w_0 w_1 \cdots \mid w_1 \cdots w_i \in K \text{ for every } i \geq 0\}$$

Show that $L \subseteq \Sigma^\omega$ is recognized by a deterministic Büchi automaton iff it equals \vec{K} , for some regular language $K \subseteq \Sigma^*$.

2. A Muller automaton $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \mathcal{F} \rangle$ has as acceptance condition a set of sets of states. A run of \mathcal{A} is successful if the set of states visited infinitely often belongs to \mathcal{F} .

The acceptance condition $\mathcal{F} \subseteq \mathcal{P}(Q)$ is called *realizable* if for every $F \in \mathcal{F}$ there is some run ρ of \mathcal{A} s.t. $\text{inf}(\rho) = F$.

(Landweber) Show that $L \subseteq \Sigma^\omega$ can be recognized by a deterministic Büchi automaton iff *every* deterministic Muller automaton $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, \mathcal{F} \rangle$ that recognizes L and such that \mathcal{F} is realizable, also satisfies the following condition (upward closure):

Whenever $F \in \mathcal{F}$ and $F \subseteq G$, also $G \in \mathcal{F}$.

Exercise 4

Let L be the language of $\{a, b\}$ -labelled infinite binary trees t such that there is some path of t with infinitely many a -labelled nodes. Let K be the complement of L .

1. Give an MSO formula describing L .
2. Give a Büchi tree automaton that recognizes L .
3. Give a parity tree automaton that recognizes K .
4. Show that no Büchi tree automaton recognizes K .

Hint: Consider a tree that contains “ a ” exactly at nodes from $(1^+0)^k$, where $1 \leq k \leq n$ and n is the size of a supposed Büchi tree automaton recognizing K .

Exercise 5

In this exercise we show how to reduce the question whether a parity tree automaton recognizes at least some tree (“non-emptiness problem”), to a parity game.

1. Justify that for the non-emptiness problem we can assume wlog. that the alphabet Σ has size one.

2. Let $\mathcal{A} = \langle S, \{a\}, S_0, \xrightarrow{a}, \ell : S \rightarrow \{0, \dots, d\} \rangle$ be a parity tree automaton over alphabet $\{a\}$.

We define a parity game $G_{\mathcal{A}} = (V_0 \cup V_1, E, p : V \rightarrow \{0, \dots, d\})$ as

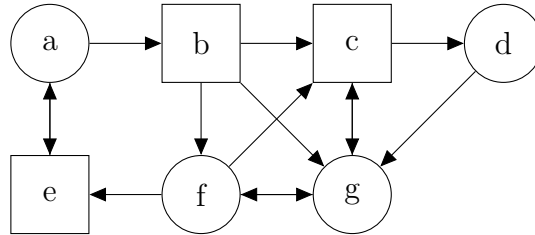
- Nodes: $V_0 = S$, $V_1 = \{(s, s_0, s_1) \mid (s, s_0, s_1) \in \xrightarrow{a}\}$,
- Priorities: $p(s) = p(s, s_0, s_1) = \ell(s)$, for $s \in V_0$, $(s, s_0, s_1) \in V_1$,
- Edges: $s \rightarrow (s, s_0, s_1)$ for every $s \in V_0$, $(s, s_0, s_1) \in V_1$, and $(s, s_0, s_1) \rightarrow s_0$, $(s, s_0, s_1) \rightarrow s_1$.

Justify that $L(\mathcal{A})$ is non-empty iff some node $s_0 \in S_0$ is winning for P_0 in the game $G_{\mathcal{A}}$.

2 Games on finite graphs

Exercise 6

Consider the following parity game, with priorities $p(a) = p(e) = 0$, $p(b) = p(f) = 1$, $p(c) = p(g) = 2$, $p(d) = 3$:



Compute the winning region W_0 of P_0 , as well as winning strategies for both players.

Exercise 7

Describe a polynomial algorithm for the following problem: given a finite arena and a positional strategy σ_0 for P_0 , check that σ_0 is winning for P_0 (starting from any vertex).

Deduce that determining if a vertex is winning for P_0 in a parity game is in the class $\text{NP} \cap \text{co-NP}$.

Exercise 8

We fix a starting vertex v_0 in a finite parity game G and we define a new game G' where all plays are finite. The game stops as soon as a vertex is visited twice. A play is thus a finite path v_0, \dots, v_n such that v_0, \dots, v_{n-1} are pairwise distinct and $v_n = v_j$ for some $j < n$. Player P_0 wins if the maximal priority of the loop, $\max\{p(v_j), \dots, \chi(v_n)\}$, is even.

Show that the parity game G starting in v_0 and G' are *equivalent* (i.e., P_0 wins in G from v_0 iff she wins in G'):

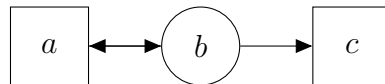
1. Show that if P_0 has a *positional* winning strategy σ in G , then σ is also winning in G' .
2. Show that if P_0 has a winning strategy in G' , then she has also one in G .

Exercise 9

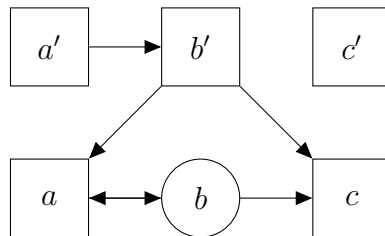
In the following variant of parity games, called handicap- k -parity, we allow player P_1 to play the k first moves of the play. Afterwards, the game continues as usual. A positional strategy for P_1 is a mapping $\sigma_1 : V \rightarrow V$ (for $v \notin V_1$, $\sigma_1(v)$ represents a move of P_1 among the first k moves that he is allowed to make at the beginning).

1. Assume first that $k = 1$. We build a new game $G' = (V = V_0 \cup V_1 \cup V', E \cup E')$ from G as follows: $V' = \{v' \mid v \in V\}$ is a copy of V and all vertices of V' belong to P_1 . For every edge $v \rightarrow w$ of G , we add to E' an edge $v' \rightarrow w'$ if $v \in V_1$, resp. an edge $v' \rightarrow w$ if $v \in V_0$.

Take as an example the game G :



The new game G' is:



Every play in the handicap-1-parity game played in G from vertex v can be simulated by a usual parity play in G' starting in v' : if $v \in V_1$, then the move $v \rightarrow w$ played by P_1 is simulated by $v' \rightarrow w'$ and the play stays in V' ; if $v \in V_0$, then the move $v \rightarrow w$ played by P_1 instead of P_0 is simulated by $v' \rightarrow w$ and the remaining play stays in G .

Priorities in G' are inherited from G : every vertex $v \in V$ keeps its priority, and v' has the same priority as v .

Show that each player has a winning strategy from v in G iff he/she has a winning strategy from v' in G' .

2. Show that handicap- k -parity is determined and that the winning strategies are positional for both players.