

More regular games

Game arena, winning condition

$$\mathcal{A} = (V_0, V_1, E), \quad \text{Win} \subseteq V^\omega \text{ winning condition}$$

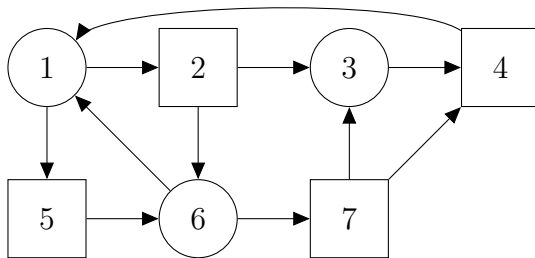
Reachability game. Given $F \subseteq V$, the winning condition Win is the set of plays that visit F at least once. The winning strategy of P_0 is the attractor strategy, and the one of P_1 is the trap strategy.

Safety game. Given $F \subseteq V$, the winning condition Win is the set of maximal plays that avoid F .

Reachability for P_0 = safety for P_1

Obligation games. Win is a boolean combination of reachability conditions. *Example:* visit either p and q and not r , or q and r and not p .
Equivalent formulation:

- Win = \mathcal{F} , where $\mathcal{F} = \{F_1, \dots, F_k\}$, $F_i \subseteq V$.
- A maximal play $\pi = v_0, v_1, \dots$ is winning (for P_0) if $\text{Occ}(\pi) \in \mathcal{F}$.
Notation: $\text{Occ}(\pi) = \{v \mid \exists i \text{ s.t. } v = v_i\}$ is the set of states visited by π .



Win: $\pi \in \text{Win}$ if and only if $\{2, 7\} \subseteq \text{Occ}(\pi)$. In order to win, P_0 needs memory in state 1: she has to visit both 2 **and** 5.

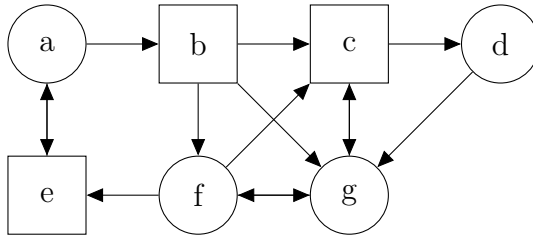
Solution for obligation games: reduction to **weak parity games**.

Weak parity games.

- The game arena \mathcal{A} is equipped with a priority function $p : V \rightarrow \{0, \dots, d\}$: $p(v)$ is the priority (color) of v .
- Win is the set of maximal plays such that the *biggest* priority is even. Formally:

$$\pi \in \text{Win} \quad \text{iff} \quad \max\{p(v) \mid v \in \text{Occ}(\pi)\} \text{ is even}$$

Example.



$$p(a) = p(e) = 0, p(b) = p(f) = 1, p(c) = p(g) = 2, p(d) = 3.$$

- $d \in W_1$ and $\text{Attr}_1(d) = \{d, c, b\} \subseteq W_1$
- $V' = V \setminus \{b, c, d\}$ is a trap for P_1 : player P_0 can keep the game within V' . In V' , P_0 can avoid priority 3, and $\text{Attr}_0(g) \upharpoonright_{V'} = \{g, f\} \subseteq W_0$
- $V'' = V' \setminus \{f, g\}$ is a trap for P_0 : P_1 can keep the game within V'' .
 $W_1 \upharpoonright_{V''} = \emptyset, W_0 \upharpoonright_{V''} = \{a, e\}$

Conclusion: $W_0 = \{a, e, f, g\}, W_1 = \{b, c, d\}$. Both players have memoryless strategies.

Polyomial-time solution for weak parity games: decompose the game arena into attractors.

Exercices

Variants of reachability. Given the arena $\mathcal{A} = (V_0, V_1, E)$ and $F \subseteq V$, propose an algorithm to compute the winning regions and winning strategies for the two players, for the winning conditions listed below. Are the games determined? are the strategies memoryless?

1. Win is the set of maximal plays that visit F at most once.
2. Win is the set of maximal plays that visit F at most k times (k is fixed).
3. Win is the set of maximal plays that visit F at least $|V| + 1$ times.

Büchi games. A recursive algorithm:

Input: Büchi game $G = (V_0, V_1, E), F \subseteq V$

Output: $\text{Büchi}(G, F) = (W_0, W_1)$

if $V = \emptyset$ **then**
| **return** (\emptyset, \emptyset)

$A := \text{Attr}_0(F);$

$B := \text{Attr}_1(V \setminus A);$

$(W'_0, W'_1) = \text{Büchi}(G \setminus B, F);$

return $(W'_0, B \cup W'_1);$

Apply the algorithm above to compute the winning regions and winning strategies for both players with $F = \{b, d, e\}$ and the game arena:

