

IF222, Introduction aux jeux Strategic games

This note is based on “A primer on Strategic Games” by Krzysztof Apt (Ch. 1 in Lectures in Game Theory for Computer Scientists, eds. K. Apt and E. Grädel), Cambridge).

1 Definitions

Compared to the settings we have seen so far, the most important difference here is that the profit function determining the gain of a player after a play will not be necessary zero-sum. In this lecture we will allow more than two players, although all examples talk about only two players. Each player will be profit maximising. The games will be played in one round: all the players choose simultaneously a strategy and then the payoff for each of the players is determined by looking it up in the payoff table.

A *strategic game* (“jeux matriciels”) for n players, $(S_1, \dots, S_n, p_1, \dots, p_n)$ consists of:

- a non-empty, possibly infinite, set S_i of *strategies*, and
- a *payoff function* $p_i : S_1 \times \dots \times S_n \rightarrow R$,

for each player $i = 1, \dots, n$.

Our general assumptions are:

- players choose strategies *simultaneously*,
- each player wants to *maximize his/her profit*,
- players have *complete knowledge* of the game and there are common assumptions about the rationality of other players.

Some classical examples of games:

		C	D
Prisoner's dilemma	C	(-2,-2)	(0,-3)
	D	(-3,0)	(-1,-1)

		F	B
Battle of the Sexes	F	(2,1)	(0,0)
	B	(0,0)	(1,2)

	H	T
Matching Pennies	(1,-1)	(-1,1)
	(-1,1)	(1,-1)

A *joint strategy* is an element $s \in S_1 \times \dots \times S_n$. We write s_{-i} for $(s_j)_{j \neq i}$. Then we can write (s_i, s_{-i}) for s .

Best response: A strategy s_i is a best response to a joint strategy s_{-i} of i -player opponents if:

$$\forall s' \in S_i. p(s_i, s_{-i}) \geq p(s'_i, s_{-i}) .$$

Nash equilibrium: A joint strategy s is a *Nash equilibrium* if each s_i is a best response to a joint strategy s_{-i} :

$$\forall_{i=1, \dots, n} \forall s'_i \in S_i. p(s_i, s_{-i}) \geq p(s'_i, s_{-i}) .$$

So in Nash equilibrium no player has an incentive to switch to some other strategy.

Pareto efficiency: A joint strategy s is *Pareto efficient* if for no joint strategy s' :

$$\forall_{i=1, \dots, n} p_i(s') \geq p_i(s) \quad \text{and} \quad \exists_{i=1, \dots, n} p_i(s') > p_i(s) .$$

Some examples

1. Prisoner's dilemma has a unique Nash equilibrium (which one?).
2. Matching Pennies has no Nash equilibrium.
3. Battle of the sexes has two Nash equilibria.
4. In Prisoner's dilemma the Nash equilibrium is the only outcome that is not Pareto efficient.

We now introduce three notions of *domination* of one strategy by another:

- s_i *strictly dominates* s'_i if for all $s_{-i} \in S_{-i}$ we have $p(s_i, s_{-i}) > p(s'_i, s_{-i})$;
- s_i *weakly dominates* s'_i if for all $s_{-i} \in S_{-i}$ we have $p(s_i, s_{-i}) \geq p(s'_i, s_{-i})$, and there is $s'_{-i} \in S_{-i}$ such that $p(s_i, s'_{-i}) > p(s'_i, s'_{-i})$;
- s_i *dominates* s'_i if for all $s_{-i} \in S_{-i}$ we have $p(s_i, s_{-i}) \geq p(s'_i, s_{-i})$.

We say that s_i is *strictly dominant* if it strictly dominates all other strategies of player i , and define similarly *weakly dominant* and *dominant* strategies.

Clearly a rational player will not choose a strictly dominated strategy (consider Prisoner's dilemma as an example).

Observation: Let G be a strategic game. Suppose s is a joint strategy such that each s_i is a dominant strategy. Then s is a Nash equilibrium of G .

Observation: If we replace dominant by strictly dominant in the above then s is a unique Nash equilibrium of G .

Example: This is not true if we replace dominant by weakly dominant. Consider

	L	R
T	(1,1)	(1,1)
B	1,1)	(0,0)

Here T is weakly dominant strategy, and L is too. The above observation says that (T, L) is a Nash equilibrium. However (T, R) and (B, L) are also Nash equilibria.

There are games with Nash equilibria and no dominant strategies. For example Battle of the sexes has two Nash equilibria and no dominant strategy.

2 Iterated elimination of strictly dominated strategies

Since every player is supposed to be rational, it should be safe to remove his strategies that are strictly dominated by some other strategy.

For a game $G \equiv (S_1, \dots, S_n, p_1, \dots, p_n)$ and, possibly empty, sets of strategies R_1, \dots, R_n such that $R_i \subseteq S_i$, we say that $(R_1, \dots, R_n, p_1, \dots, p_n)$ is a *restriction of G* . We write $R \rightarrow_s R'$ if $R \neq R'$, $R'_i \subseteq R_i$ for $i = 1, \dots, n$ and $(R_i \setminus R'_i)$ contains only strictly dominated strategies.

Example:

	L	M	R
T	(3,0)	(2,1)	(1,0)
C	(2,1)	(1,1)	(1,0)
B	(0,1)	(0,1)	(0,0)

Observe that B is strictly dominated by T , and R is strictly dominated by M . By eliminating the two we get

	L	M
T	(3,0)	(2,1)
C	(2,1)	(1,1)

Now C is strictly dominated by T , so we get:

	L	M
T	(3,0)	(2,1)

Since in this game L is strictly dominated by M we get

	M
T	(2,1)

We say that R is an *outcome of IESDS* from G , iterative elimination of strictly dominated strategies, if R cannot be reduced further by the process as above. If the process leaves each player with exactly one strategy in R then we say that G is *solved by IESDS*.

Theorem 1 *Suppose G' is an outcome of IESDS from G .*

1. *If s is a Nash equilibrium of G then s is a Nash equilibrium of G' .*
2. *If G is finite and s is a Nash equilibrium of G' , then it is a Nash equilibrium of G .*
3. *If G is finite and solved by IESDS then the resulting joint strategy is a unique Nash equilibrium.*

Example: Location game of Hotelling.

Theorem 2 (Order Independence, Gilboa et al., Stegeman) *For a given finite strategic game, all iterated eliminations of strictly dominated strategies yield the same outcome.*

3 Elimination of weakly dominated strategies

We can define the same process of elimination, but now for weakly dominated strategies. We will denote it by \rightarrow_W , and refer to it by IEWDS.

Example: Consider an extension of Matching pennies game.

	H	T	E
H	(1,-1)	(-1,1)	(-1,-1)
T	(-1,1)	(1,-1)	(-1,-1)
E	(-1,-1)	(-1,-1)	(-1,-1)

Observe that (E, E) is its only Nash equilibrium. On the other hand E is the only strategy that is weakly dominated, so it will be removed by IEWDS.

	L	M	R
Example: T	(0,1)	(1,0)	(0,0)
B	(0,0)	(0,0)	(1,0)

The game has Nash equilibria (T, L) , (B, L) , and (B, R) . This game can be solved by IEWDS, but only if the first round we keep at least one weakly dominated strategy, i.e., either M or R .

In conclusion, elimination of weakly dominated strategies:

- can lead to a deletion of Nash equilibria,
- does not always yield a unique outcome,
- is too restrictive if we stipulate that we eliminate all weakly dominated strategies in every round.

4 Never best responses

	X	Y
Consider the game	A (2,1)	(0,0)
	B (0,1)	(2,0)
	C (1,1)	(1,2)

No strategy in this game is strictly or weakly dominated. However C is never a *best response*: to every strategy of the opponent there is a better response than C . So this game is solved by iterated elimination of never best responses, but it is not solved by IESDS, nor IEWDS. We will write $R \rightarrow_N R'$ for one step of elimination of never best responses.

Theorem 3 *Suppose G' is an outcome of IENBR from a game G .*

1. *If s is a Nash equilibrium of G then it is a Nash equilibrium of G' .*
2. *If G is finite and s is a Nash equilibrium of G' then it is a Nash equilibrium of G .*
3. *If G is finite and solved by IENBR then the resulting strategy is a unique Nash equilibrium.*

Theorem 4 (Order independence) *Given a finite strategic game all iterated eliminations of never best responses yield the same outcome.*

5 Mixed strategies

A *probability distribution* over a finite set A is a function

$$\pi : A \rightarrow [0, 1] \quad \text{such that} \quad \sum_{a \in A} \pi(a) = 1 .$$

Consider a strategic game $G \equiv (S_1, \dots, S_n, p_1, \dots, p_n)$. A *mixed strategy* for player i is a probability distribution over S_i . The *support* of a mixed strategy m_i is the set of pure strategies that contribute to m_i with non-zero coefficient.

A tuple (m_1, \dots, m_n) gives a probability distribution over S_1, \dots, S_n that is the joint mixed strategy.

$$m(s) = m_1(s_1) \cdot \dots \cdot m_n(s_n)$$

The payoff of a mixed strategy is

$$p_i(m) = \sum_{s \in S} m(s) \cdot p_i(s) .$$

Example: Matching pennies.

Theorem 5 (Nash) *Every finite strategic game has a mixed Nash equilibrium.*

Lemma 6 *Let $(S_1, \dots, S_n, p_1, \dots, p_n)$ be a finite strategic game. The following are equivalent.*

1. m is a Nash equilibrium in mixed strategies, i.e.

$$p_i(m) \geq p_i(m'_i, m_{-i})$$

for all $i = 1, \dots, n$, and all m'_i mixed strategies of player i .

2. for all $i = 1, \dots, n$ and all $s_i \in S_i$

$$p_i(m) \geq p_i(s_i, m_{-i})$$

3. for all $i = 1, \dots, n$ and all $s_i \in \text{support}(m_i)$

$$p_i(m) = p_i(s_i, m_{-i})$$

and for all $i = 1, \dots, n$ and all $s_i \notin \text{support}(m_i)$

$$p_i(m) \geq p_i(s_i, m_{-i}) .$$

Example: Battle of the sexes.

	F	B
F	(2,1)	(0,0)
B	(0,0)	(1,2)

Let us look at the general form of mixed strategies in this game

$$m_1 = r_1 \cdot F + (1 - r_1) \cdot B$$

$$m_2 = r_2 \cdot F + (1 - r_2) \cdot B$$

By definition of the payoff we have

$$p_1(m_1, m_2) = 2 \cdot r_1 \cdot r_2 + (1 - r_1) \cdot (1 - r_2) ,$$

$$p_2(m_1, m_2) = r_1 \cdot r_2 + 2 \cdot (1 - r_1) \cdot (1 - r_2) .$$

We can now use item 3 of the above lemma and examine what happens if m is a Nash equilibrium. We get $p_1(F, m_2) = p_1(B, m_2)$ that gives $r_2 = 1/3$. Similarly we get $r_1 = 2/3$. So the expected payoff in the Nash equilibrium in this game is $2/3$.

6 Relating the notions

Observation: If a strategy is strictly dominated then it is never a best response.

Observation: A strategy may be a never best response and at the same time neither strictly nor weakly dominated

Indeed in the game

	X	Y
A	(2,1)	(0,0)
B	(0,1)	(2,0)
C	(1,1)	(1,2)

C is a never best response, yet it

is neither strictly nor weakly dominated.

In the example

	L	M	R
T	(0,1)	(1,0)	(0,0)
B	(0,0)	(0,0)	(1,0)

strategy M is weakly dominated, and is also a best response to B .

The situation is clearer in mixed extensions. Below *totally mixed strategy* means mixed strategy with the full support.

Theorem 7 (Pearce) *Consider a finite two-player strategic games.*

1. *A pure strategy is strictly dominated by a mixed strategy iff it is not a best response to a mixed strategy.*

2. A pure strategy is weakly dominated by a mixed strategy iff it is not a best response to a totally mixed strategy.

7 Mechanism design

In 2007 the Nobel Prize in Economics was awarded to L. Hurwicz, E. Maskin, and R. Myerson for “having laid the foundations of mechanism design theory”. The theory deals with the problem

How to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like.

Assume a set $\{1, \dots, n\}$ of players, and a non-empty set D of *decisions*. Each player i has

- a non-empty set of *types* Θ_i , and
- an *initial utility function* $v_i : D \times \Theta_i \rightarrow \mathbb{R}$.

For example, in an auction, a type θ_i can be the value player i attaches to the auctioned item. The utility function $v_i(d, \theta_i)$ tells how much player i values the outcome d given that she has type θ_i . So in an auction if the outcome is that she gets the item for price p then the utility is $(\theta_i - p)$. The utility is positive if she gets the item for less than she values it.

A *decision rule* is a function $f : \theta \rightarrow D$.

A *decision problem* is a tuple

$$(D, \theta_1, \dots, \theta_n, v_1, \dots, v_n, f)$$

The mechanics of decision making is as follows.

- each player receives her type θ_i .
- each player announces a type θ'_i , this gives a joint type $\theta' = (\theta'_1, \dots, \theta'_n)$,
- the decision rule determines the outcome $d = f(\theta')$
- the resulting utility for player i is $v_i(d, \theta_i)$.

The objective of the players is to maximise their utility. The difficulty in the above scheme is to make the players announce their true types.

Example: Public project problem The objective is to decide if it is worth to construct a public good. Each player announces its utility for the good. If the sum of utilities is at least the cost of the good, the project takes place, and every player pays the same amount (the fraction of the cost of the good). Otherwise the project is cancelled.

This problem can be formalised as follows

- $D = \{0, 1\}$, the cancel/construct decision,
- $\Theta_i = \mathbb{R}^+$,
- $v_i(d, \theta_i) = d \cdot (\theta_i - c/n)$
- $f(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^n \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$

In this case a player who values the good more than c/n should announce c , and the other players should announce 0. To prevent this kind of solution, tax payments are introduced.

Example: Second price auction Suppose that there are three players A , B , and C bidding for an object. Let their true appreciations of the value of the object be 18, 21 and 24. If all the players announce their true appreciations then player C wins and the utility for each player is 0. But player C can announce 23 and his utility will be 1.

In second price auction the player who wins the auction pays second price. This can be shown to prevent the incentive to cheat (in theory). In practice Ebay uses second price auctions but humans are not rational and usually exhibit an anchoring effect.

Example: Back to public project problem The mechanism for public project problem can be improved by introducing taxes. The idea is that each player should pay taxes that are determined by the type he announces. For example one of the most known mechanisms, called pivotal mechanism, imposes the following taxes. (We use r_i for $\sum_{j \neq i} \theta'_j$, and recall that $d = 1$ if the project is to be constructed, that is $\sum_{j=1}^n \theta'_j \geq c$)

$$t_i(\theta') = \begin{cases} 0 & \text{if } r_i \geq \frac{n-1}{n}c \text{ and } d = 1 \\ r_i - \frac{n-1}{n}c & \text{if } r_i < \frac{n-1}{n}c \text{ and } d = 1 \\ \frac{n-1}{n}c - r_i & \text{if } r_i \geq \frac{n-1}{n}c \text{ and } d = 0 \\ 0 & \text{if } r_i < \frac{n-1}{n}c \text{ and } d = 0 \end{cases}$$

To illustrate the pivotal mechanism suppose that $c = 30$ and that the three players, A, B and C have true types 6, 7, and 25, respectively. If they announce these types then the project takes place and the resulting utility is as follows

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

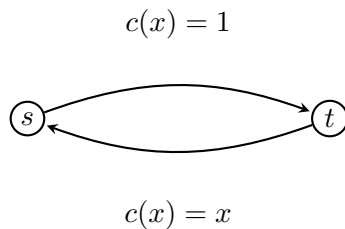
Suppose now that the true types of the players are respectively 4, 3, and 22. When they announce these types the project is cancelled. Still, some players need to pay a tax as illustrated in by the following table

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	25	0	0

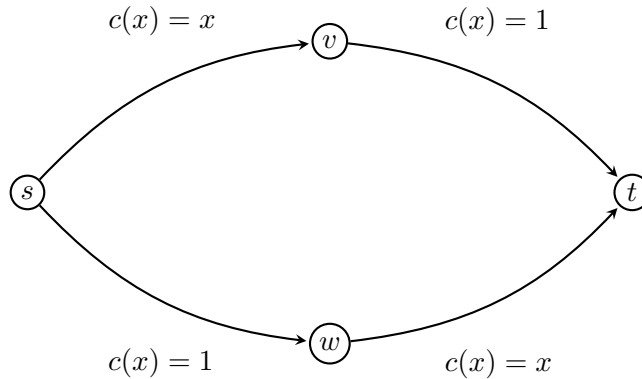
8 Selfish routing problems

Selfish routing problems are very general class of equilibria problems. We just give two examples.

Example: Pigou's example Consider a very simple network consisting of two places, s and t , connected by two roads. The upper road has a constant cost 1, while the cost of the lower one varies with the traffic. Suppose that the traffic of total cost 1 is accumulated in s and wants to get to t . Since the traffic is selfish, all of the traffic will take the lower route that has the cost not bigger than the upper one. In the result the total cost of the traffic will be 1. An optimal flow splits the traffic equally between two roads, resulting in the total cost $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = 3/4$.

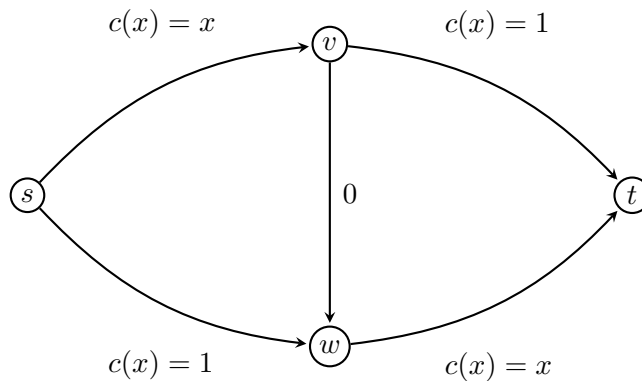


Example: Braess's Paradox This is a famous example showing that intuitively helpful actions may actually increase the cost. Consider the network presented below



Assuming that there is one unit traffic, the equilibrium is achieved when it is split evenly between the two routes. This gives $3/2$ cost.

Now suppose that in effort to decrease the cost encountered by traffic, a zero-cost road is build from v to w . The resulting network becomes:



In this network the cost of the route $s \rightarrow v \rightarrow w \rightarrow t$ is never worse than that along the two original paths, and is strictly less when some traffic does not use it. As a consequence the unique equilibrium flow routes all the traffic through this new route. In consequence all the traffic now experiences 2 units cost.