

Graph insertion operads

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Abstract. Using the combinatorial species setting, we propose two new operad structures on multigraphs and on pointed oriented multigraphs. The former can be considered as a canonical operad on multigraphs, directly generalizing the Kontsevich-Willwacher operad, and has many interesting suboperads. The latter is a natural extension of the pre-Lie operad in a sense developed here and related to the multigraph operad. We study some of the finitely generated suboperads of the multigraph operad and establish links between them and the associative operad and the commutative magmatic operads.

Abstract. Nous présentons ici deux nouvelles opérades: une sur les multigraphes et une sur les multigraphes pointés orientés. La première peut être considérée comme l'opérade canonique sur les multigraphes, généralisant l'opérade Kontsevich-Willwacher, et a de nombreuses sous opérades d'intérêt. La seconde est une généralisation naturelle de l'opérade pre-Lie, dans un sens développé ici et en lien avec l'opérade sur les multigraphes. Nous étudions certaines sous opérades finiment engendrées de l'opérade sur les multigraphes et établissons des liens avec l'opérade associative et l'opérade commutative magmatique.

Keywords: Graphs; Species; Operads; Pre-Lie Operad; Koszul duality.

Introduction

Operads are mathematical structures which have been intensively studied in the context of topology, algebra [11] but also of combinatorics [4] —see for example [7,13] for general references on symmetric and non-symmetric operads, set-operads through species, *etc.*). In the last decades, several interesting operads on trees have been defined. Amongst these tree operads, maybe the most studied are the pre-Lie operad **PLie** [5] and the nonassociative permutative operad **NAP** [10].

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However, it seems to us that a natural question to ask is what kind of operads can be defined on graphs and what are their properties? The need for defining appropriate graph operads comes from combinatorics, where graphs are, just like trees, natural objects to study, but also from physics, where it was recently proposed to use graph operads in order to encode the combinatorics of the renormalization of Feynman graphs in quantum field theory [9].

Other graph operads have been defined for example in [6, 8, 12–14]. In this paper, we go further in this direction and we define, using the combinatorial species [2] setting, new graph operads. Moreover, we investigate several properties of these operads: we describe an explicit link with the pre-Lie tree operad mentioned above, and we study interesting (finitely generated) suboperads.

This paper is organized as follows. Section 1 contains elementary definitions of species and operads. In Section 2 we define and study the main operad of interest of this paper. Section 3 is devoted to the study of finitely generated suboperads.

This text is an extended abstract. The long version of this work [1] contains a more general definition of graph insertion operads as well as new operad constructions and all the proofs of the results presented here.

1 Species, operads and graphs

Most definitions, results and proofs of this section can be found with more details in [13]. We refer the reader to [2] for the theory of species and to [11] for the theory of operads. In all the following, \mathbb{K} is a field of characteristic zero. For any positive integer n , $[n]$ stands for the set $\{1, \dots, n\}$.

Definition 1. A *set species* S consists of the following data. For each finite set V a set $S[V]$, and for each bijection of finite sets $\sigma : V \rightarrow V'$ a map $S[\sigma] : S[V] \rightarrow S[V']$. These maps should be such that $S[\sigma \circ \tau] = S[\sigma] \circ S[\tau]$ and $S[\mathcal{I}] = \mathcal{I}$, where \mathcal{I} is the identity map.

A *morphism of set species* $f : R \rightarrow S$ is a collection of maps $f_V : R[V] \rightarrow S[V]$ such that for each bijection $\sigma : V \rightarrow V'$, $f_{V'} \circ R[\sigma] = S[\sigma] \circ f_V$. A set species S is *positive* if $|S[\emptyset]| = 0$ and *connected* if $|S[\{v\}]| = 1$ for any singleton $\{v\}$.

Switching sets with vector spaces, maps to linear maps and cardinality to dimension in the previous definition, we obtain the definitions of *linear species*, *morphisms of linear species*, *positive linear species* and *connected linear species*. The *Hilbert series* of a vector species S is the formal series $\mathcal{H}_S(t) = \sum_{n \geq 0} \dim S[[n]] \frac{t^n}{n!}$. For S a set species, we denote by $\mathbb{K}S$ the linear species defined by $(\mathbb{K}S)[V] = \mathbb{K}S[V]$, where $\mathbb{K}S[V]$ is the free \mathbb{K} -vector space on $S[V]$. The species $\mathbb{K}S[V]$ is naturally equipped with a scalar product $(\cdot | \cdot)_{S[V]}$ by considering $S[V]$ as an orthonormal basis. For $x \in \mathbb{K}S[V]$ we denote $\text{Supp}(x) = \{y \in S[V] \mid (x|y)_{S[V]} > 0\}$ the support of x .

In all the following, V always denotes a finite set. Let R and S be two linear species. We recall the classical constructions on species: $(R + S)[V] = R[V] \oplus S[V]$ (*sum*), $R \cdot S[V] = \bigoplus_{V_1 \sqcup V_2 = V} R[V_1] \otimes S[V_2]$ (*product*), $(R \times S)[V] = R[V] \otimes S[V]$ (*Hadamard product*), $R'[V] = R[V \sqcup \{*\}]$ (*derivative*), $R^\bullet[V] = R[V] \times V$ (*pointing*), and $E(R)[V] = \bigoplus_{\cong} \bigotimes_{W \in V/\cong} R[W]$ (*assembly*) where \cong run over the set of the equivalence relations on V . These definitions are compatible with the functor $S \mapsto \mathbb{K}S$, e.g., $\mathbb{K}(R + S) = \mathbb{K}R + \mathbb{K}S$.

Definition 2. A (*symmetric*) *set* (resp. *linear*) *operad* is a positive set (resp. linear) species \mathcal{O} together with a *unit* $e : X \rightarrow \mathcal{O}$ (resp. $e : \mathbb{K}X \rightarrow \mathcal{O}$) and a *partial composition map* $\circ_* : \mathcal{O}' \cdot \mathcal{O} \rightarrow \mathcal{O}$, such that the following diagrams commute

$$\begin{array}{ccc}
 \mathcal{O}'' \cdot \mathcal{O}^2 & \xrightarrow{\circ_{*1}} & \mathcal{O}' \cdot \mathcal{O} \\
 \downarrow \circ_{*2} \circ \mathcal{I} \cdot \tau & & \downarrow \circ_{*2} \\
 \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*1}} & \mathcal{O}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{O}' \cdot \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*1} \cdot \mathcal{I}} & \mathcal{O}' \cdot \mathcal{O} \\
 \downarrow \mathcal{I} \cdot \circ_{*2} & & \downarrow \circ_{*2} \\
 \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*1}} & \mathcal{O}
 \end{array}$$

$$\begin{array}{ccccc}
 \mathcal{O}' \cdot \mathbb{K}X & \xrightarrow{\mathcal{O}' \cdot e} & \mathcal{O}' \cdot \mathcal{O} & \xleftarrow{e' \cdot \mathcal{O}} & \mathbb{K}X' \cdot \mathcal{O} \\
 & \searrow p & \downarrow \circ_* & \cong & \swarrow \\
 & & \mathcal{O} & &
 \end{array}$$

where $\tau_V : x \otimes y \in \mathcal{O}^2[V] \mapsto y \otimes x \in \mathcal{O}^2[V]$, and $p_V : x \otimes v \mapsto \mathcal{O}[* \mapsto v](x)$ with $* \mapsto v$ the bijection that sends $*$ on v and is the identity on $V \setminus \{v\}$.

An *operad morphism* is a species morphism compatible with the units and the partial composition maps.

Remark that if (S, e, \circ_*) is a set-operad then extending e and \circ_* linearly makes $(\mathbb{K}S, e, \circ_*)$ a linear operad. In all the following, e will often be trivial and we will not mention it. From now on all the considered species will be positive and we will use species and operad for linear species and linear operad and also write X for $\mathbb{K}X$.

An *ideal* of an operad \mathcal{O} is a sub-species S such that the image of the products $\mathcal{O}' \cdot S$ and $S' \cdot \mathcal{O}$ by the partial composition maps are in S . The *quotient species* \mathcal{O}/S defined by $\mathcal{O}/S[V] = \mathcal{O}[V]/S[V]$ is then an operad with the natural partial composition and unit.

We now need to recall the notion of free operad. For this we first introduce some notations. For V a set, let $\mathcal{T}[V]$ be the set of the trees defined as follows.

- If $V = \{v\}$ is a singleton then the sole element of $\mathcal{T}[V]$ is the tree reduced to a leaf labelled with $\{v\}$.
- Else let $\pi = (\pi_1, \dots, \pi_k)$ be a partition of V and t_1, \dots, t_k respectively be elements of $\mathcal{T}[\pi_1], \dots, \mathcal{T}[\pi_k]$. Then the tree with root labelled by π and t_1, \dots, t_k as sub-trees is an element of $\mathcal{T}[V]$.

Let now G be a connected species and G_{2+} be the species such that $G = X + G_{2+}$. The *free operad* \mathcal{F}_G over G is defined as follows.

- If $V = \{v\}$ is a singleton, then $\mathcal{F}_G[V] = \mathbb{K}v$.
- Otherwise, the elements of $\mathcal{F}_G[V]$ are elements of $\mathcal{T}[V]$ with an element in $G_{2+}[\{\pi_1, \dots, \pi_k\}]$ for each internal node labelled by $\pi = \{\pi_1, \dots, \pi_k\}$.

The partial composition of \mathcal{F}_G is the grafting of trees: $t_1 \circ_* t_2$ is the tree obtained by grafting t_2 on the leaf $*$ of t_1 and by applying the necessary relabelings (*i.e.*, map every set V containing $*$ on the set $V - \{*\} + V_2$).

If R is a sub-species of \mathcal{F}_G , we denote by (R) the smallest ideal containing R and write that (R) is *generated by* R . We denote by \mathcal{F}_G^k the sub-species of \mathcal{F}_G of trees with k internal nodes.

For any linear species S we denote by S^\vee the linear species defined by $S^\vee[V] = S[V]^*$ and $S^\vee[\sigma](x) = \text{sign}(\sigma)x \circ S[\sigma^{-1}]$.

Definition 3. Let G be a connected species and R a sub-species of \mathcal{F}_G . Let $\text{Ope}(G, R) = \mathcal{F}_G / (R)$. The operad $\text{Ope}(G, R)$ is *binary* if the species of generators G is concentrated in cardinality 2. This operad is *quadratic* if R is a sub-species of \mathcal{F}_G^2 .

Definition 4. Let $\mathcal{O} = \text{Ope}(G, R)$ be a binary quadratic operad. Let us define the linear form $\langle -, - \rangle$ on $\mathcal{F}_G^2 \times \mathcal{F}_G^2$ as follows. For $V = \{a, b, c\}$, $f_1 \in G^\vee[\{a\}]$, $f_2 \in G[\{b, c\}]$, $x_1 \in G'[\{a\}]$ and $x_2 \in G[\{b, c\}]$:

$$\langle f_1 \circ_* f_2, x_1 \circ_* x_2 \rangle = f_1(x_1)f_2(x_2). \quad (1.1)$$

The *Koszul dual* of \mathcal{O} is then the operad $\mathcal{O}^\dagger = \text{Ope}(G^\vee, R^\perp)$ where R^\perp is the orthogonal of R for $\langle -, - \rangle$.

When \mathcal{O} is quadratic and its Koszul complex is acyclic [11], \mathcal{O} is a *Koszul operad*. In this case, the Hilbert series of \mathcal{O} and of its Koszul dual are related by the identity:

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^\dagger}(-t)) = t. \quad (1.2)$$

2 Graph operads

A *multigraph* on V is a multiset of pairs in V^2 which we call *edges*. In this context the elements of V are called vertices and the elements in V which are in no edge are called *isolated vertices*. A multigraph on V is *connected* if it has no isolated vertex. A *graph* on V is a multigraph on V which is a set and with no edge in $\{\{v, v\} \mid v \in V\}$. We denote by **MG** the set species of multigraphs, by **G** its sub-species of graphs and by **MG_c** and **G_c** their connected counterparts.

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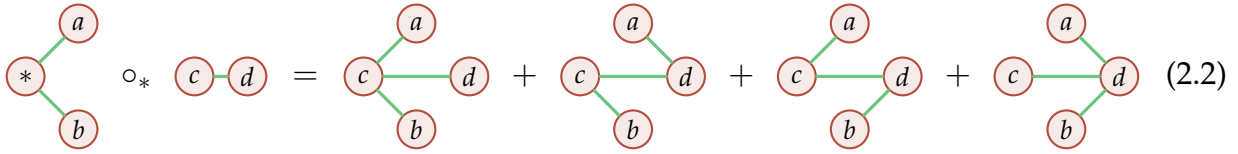
Proposition 2.1. *The species of multigraphs \mathbb{KMG} has an operad structure given by: for $g_1 \in MG[V_1 + \{*\}]$ and $g_2 \in MG[V_2]$ the partial composition $g_1 \circ_* g_2$ is the sum over all the multigraphs obtained in the following way.*

- Do the disjoint union of g_1 and g_2 .
- Remove the vertex $*$. We then have some edges with a loose end.
- Connect each loose end to any vertex in V_2 .

This makes \mathbb{KG} and \mathbb{KMG}_c suboperads of \mathbb{KMG} and \mathbb{KG}_c a suboperad of \mathbb{KG} . In particular, this structure on \mathbb{KG} is known as the Kontsevich-Willwacher operad [12]. In the case of \mathbb{KG} we can have a more formal definition of the partial composition:

$$g_1 \circ_* g_2 = \sum_{f:n(*) \rightarrow V_2} g_1 \cap V_1^2 \cup \bigcup_{v \in n(*)} \{v, f(v)\} \cup g_2, \tag{2.1}$$

where $n(*)$ is the set of neighbours of $*$. This is illustrated in Figure 2.2.

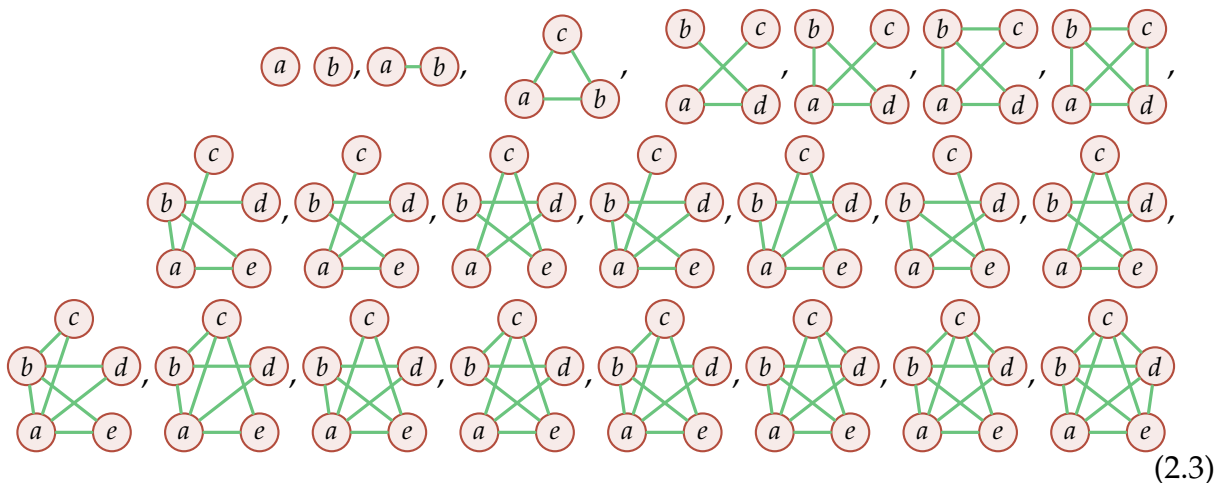


We consider this operad structure as the canonical one on \mathbb{KMG} and we call *graph insertion operads* its suboperads. While \mathbb{KMG} has an involved structure we will see that it has many interesting sub operads. Let us start by giving some basic results on \mathbb{KG} .

Let S be a species, I be a set, $\{V_i\}_{i \in I}$ be a family of finite sets, and $x_i \in S[V_i]$ for all $i \in I$. Then we denote by $\text{Spe}_S(\{x_i\}_{i \in I})$ the sub-species of S generated by $\{x_i\}_{i \in I}$ i.e., the smallest sub-species of S containing the family $\{x_i\}_{i \in I}$. If S is furthermore an operad, we denote by $\text{Ope}_S(\{x_i\}_{i \in I})$ the suboperad generated by $\{x_i\}_{i \in I}$ i.e., the smallest suboperad of S containing the family $\{x_i\}_{i \in I}$. We write that x is generated by $\{x_i\}_{i \in I}$ if $x \in \text{Ope}_S(\{x_i\}_{i \in I})$.

These definitions given, it is natural to search for a smallest family of generators of \mathbb{KG} . The search of such a family is computationally hard. We computed the graphs with less than five vertices of such a family. This is illustrated in Figure 2.3. While Figure 2.3 does not provide to us any particular insight on a possible characterisation of the generators, it does suggest that any graph with enough edges must be a generator. This is confirmed by the following lemma.

Lemma 2.2. *Let $\{V_i\}_{i \in I}$ be a family of non empty finite sets, g_i be a graph in $G[V_i]$ for all $i \in I$ and let g be a graph in $G[V]$ with at least $\binom{n-1}{2} + 1$ edges, where $n = |V|$. Then g is generated by $\{g_i\}_{i \in I}$ if and only if $g = g_i$ for some $i \in I$.*



Proof. (sketch) Remark that the number of edges of the graphs in the support of $g_1 \circ_* g_2$ is the sum of the number of edges in g_1 and the number of edges in g_2 . Hence graphs with too many edges cannot appear in the support of a partial product. \square

Proposition 2.3. *The operad $\mathbb{K}\mathbf{G}$ is not free and has an infinite number of generators.*

Proof. The fact that $\mathbb{K}\mathbf{G}$ has an infinite number of generators is a direct consequence of the previous lemma. The following relation shows that $\mathbb{K}\mathbf{G}$ is not free:

$$\begin{aligned}
& (a \text{---} * \text{---} \circ_* b \text{---} c) + (c \text{---} * \text{---} \circ_* b \text{---} a) - (b \text{---} * \text{---} \circ_* a \text{---} c) - 2(a \text{---} b \text{---} c) \\
&= (a \text{---} b \text{---} c) + (b \text{---} c \text{---} a) + (c \text{---} b \text{---} a) + (b \text{---} a \text{---} c) \\
&\quad - (b \text{---} a \text{---} c) - (a \text{---} c \text{---} b) - 2(a \text{---} b \text{---} c) \\
&= 0
\end{aligned}
\tag{2.4}$$

\square

AS a consequence of Proposition 2.3, it seems particularly involved to further investigate the structure of $\mathbb{K}\mathbf{G}$. Let us restrict further our study of graph insertion operads to the suboperad of tree $\mathbb{K}\mathbf{T}$. In particular, we show that this suboperad has a non trivial link with \mathbf{PLie} . To show this we first need to introduce a new operad.

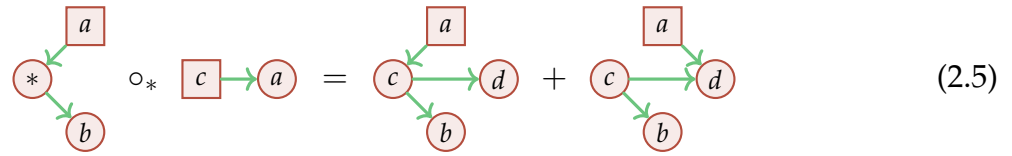
An *oriented multigraph* on V is a graph where each edge end is either unlabelled or labelled with an arrow head. Denote by \mathbf{MG}_{or} the set species of oriented graphs, \mathbf{G}_{or} the set species of oriented graphs and \mathbf{MG}_{orc} and \mathbf{G}_{orc} their connected counterparts.

Proposition 2.4. *The species $\mathbb{K}\mathbf{MG}_{orc}^\bullet$ has an operad structure given by the following. For $(g_1, v_1) \in \mathbf{G}_{orc}^\bullet[V_1 + \{*\}]$ and $(g_2, v_2) \in \mathbf{G}_{orc}^\bullet[V_2]$ the partial composition $(g_1, v_1) \circ_* (g_2, v_2)$ is*

equal to (g_3, v_3) where $v_3 = v_1$ if $* \neq v_1$ and $v_3 = v_2$ else. The vector g_3 is the sum over all the graphs obtained in the following way.

- Do the disjoint union of g_1 and g_2 .
- Remove the vertex $*$. We then have some edges with a loose end.
- Connect each non labelled loose end to v_2 .
- Connect each labelled loose end to any vertex in V_2 .

This makes $\mathbb{K}\mathbf{G}_{orc}^\bullet$ a suboperad of $\mathbb{K}\mathbf{M}\mathbf{G}_{orc}^\bullet$. An example of partial composition on oriented pointed graphs is illustrated in Figure 2.5.



$$\begin{array}{c} \boxed{*} \\ \swarrow \searrow \\ \circledast \quad \circledast \\ \downarrow \downarrow \\ \circledast \quad \circledast \\ a \quad b \end{array} \circ_* \begin{array}{c} \boxed{c} \\ \rightarrow \\ \circledast \\ a \end{array} = \begin{array}{c} \boxed{c} \\ \swarrow \searrow \\ \circledast \quad \circledast \\ \downarrow \downarrow \\ \circledast \quad \circledast \\ a \quad b \end{array} \rightarrow \begin{array}{c} \boxed{d} \\ \rightarrow \\ \circledast \\ a \end{array} + \begin{array}{c} \boxed{c} \\ \rightarrow \\ \circledast \\ a \end{array} \rightarrow \begin{array}{c} \boxed{d} \\ \rightarrow \\ \circledast \\ b \end{array} \quad (2.5)$$

In a rooted tree, each edge has a parent end and a child end. Given a rooted tree t with root r , denote by t_r the oriented tree where each parent end of t is labelled and each child end is non labelled. Then the monomorphism $\mathbf{T}^\bullet \hookrightarrow \mathbf{G}_{orc}^\bullet$ which sends each ordered pair (t, r) , where t is a tree and r is its root, on (t_r, r) induces an operad structure on the species of rooted trees which is exactly the operad \mathbf{PLie} [5].

Proposition 2.5. *The monomorphism of species ψ defined by*

$$\begin{aligned} \psi_V : \mathbb{K}\mathbf{T}[V] &\hookrightarrow \mathbb{K}\mathbf{T}^\bullet[V] \\ t &\mapsto \sum_{r \in V} (t, r), \end{aligned} \quad (2.6)$$

is a morphism of operads from $\mathbb{K}\mathbf{T}$ to \mathbf{PLie} .

A natural question to ask is how to extend this morphism to \mathbf{G}_c and $\mathbf{M}\mathbf{G}_c$. Let us introduce some notations in order to answer this question. For $g \in \mathbf{M}\mathbf{G}_c[V]$, $r \in V$, and $t \in \mathbf{T}[V]$ a spanning tree of g , let $\vec{g}^{(t,r)} \in \mathbf{M}\mathbf{G}_{orc}$ be the oriented multigraph obtained by labelling the edges of g in t in the same way as the edges of t_r , and by labelling both ends of the edges in g not in t . More formally, we have: $\vec{g}^{(t,r)} = t_r \oplus \iota_{\mathbf{G}}(g \setminus t)$, where $\iota : \mathbb{K}\mathbf{M}\mathbf{G} \rightarrow \mathbb{K}\mathbf{M}\mathbf{G}_{or}$ sends a multigraph on the oriented multigraph obtained by labelling all the edges ends.

Define $\mathbb{K}\mathcal{O}_2 \subset \mathbb{K}\mathcal{O}_1 \subset \mathbb{K}\mathbf{S}\mathbf{T}$ three sub-species of $\mathbb{K}\mathbf{M}\mathbf{G}_{orc}^\bullet$ by:

- $\mathbf{S}\mathbf{T}[V] = \{(\vec{g}^{(t,r)}, r) \mid g \in \mathbf{M}\mathbf{G}_c[V], r \in V \text{ and } t \text{ a spanning tree of } g\}$,
- $\mathcal{O}_1[V] = \{\sum_{r \in V} (\vec{g}^{(t,r)}, r) \mid g \in \mathbf{M}\mathbf{G}_c[V] \text{ and for each } r, t(r) \text{ a spanning tree of } g\}$,
- $\mathcal{O}_2[V] = \{(\vec{g}^{(t_1,r)}, r) - (\vec{g}^{(t_2,r)}, r) \mid g \in \mathbf{M}\mathbf{G}_c[V], r \in V \text{ and } t_1 \text{ and } t_2 \text{ two spanning trees of } g\}$.

Lemma 2.6. *The following holds:*

- \mathbb{KST} is a suboperad of $\mathbb{KMG}_{orc}^\bullet$ isomorphic to $\mathbb{KMG} \times \mathbf{PLie}$,
- \mathbb{KO}_1 is a suboperad of \mathbb{KST} ,
- \mathbb{KO}_2 is an ideal of \mathbb{KO}_1 .

We can see \mathbf{PLie} as a suboperad of ST by the monomorphism $(t, r) \mapsto (t_r, r)$. The image of the operad morphism ψ of Lemma 2.5 is then $\mathbb{KO}_1 \cap \mathbf{PLie}$ and we have that $\mathbb{KO}_2 \cap \mathbf{PLie} = \{0\}$ and hence $\mathbb{KO}_1 \cap \mathbf{PLie} / \mathbb{KO}_2 \cap \mathbf{PLie} = \mathbb{KO}_1 \cap \mathbf{PLie}$.

Proposition 2.7. *The operad isomorphism $\psi : \mathbb{KT} \xrightarrow{\sim} \mathbb{KO}_1 \cap \mathbf{PLie}$ extends to an operad isomorphism $\psi : \mathbb{KMG}_c \xrightarrow{\sim} \mathbb{KO}_1 / \mathbb{KO}_2$ given by*

$$\begin{aligned} \psi_V : \mathbb{KMG}_c[V] &\xrightarrow{\sim} \mathbb{KO}_1 / \mathbb{KO}_2[V] \\ g &\mapsto \sum_{r \in V} \vec{g}^{(t(r), r)}, \end{aligned} \quad (2.7)$$

where for each $r \in V$, $t(r)$ is a spanning tree of g . Furthermore, this isomorphism restricts itself to an isomorphism $\mathbb{KG}_c \xrightarrow{\sim} \mathbb{KO}_1 \cap \mathbb{KG}_c / \mathbb{KO}_2 \cap \mathbb{KG}_c$.

The last results are summarized in the following commutative diagram of operad morphisms.

$$\begin{array}{ccccccc} \mathbb{KT} & \xrightarrow{\sim} & \mathbf{PLie} \cap \mathbb{KO}_1 / \mathbb{KO}_2 & \xlongequal{\quad} & \mathbf{PLie} \cap \mathbb{KO}_1 & \hookrightarrow & \mathbf{PLie} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{KG}_c & \xrightarrow{\sim} & \mathbb{KO}_1 \cap \mathbb{KG}_c / \mathbb{KO}_2 \cap \mathbb{KG}_c & \xleftarrow{\quad} & \mathbb{KG}_{orc}^\bullet \cap \mathbb{KO}_1 & \hookrightarrow & \mathbb{KG}_{orc}^\bullet \cap \mathbb{KST} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{KMG}_c & \xrightarrow{\sim} & \mathbb{KO}_1 / \mathbb{KO}_2 & \xleftarrow{\quad} & \mathbb{KO}_1 & \hookrightarrow & \mathbb{KMG} \times \mathbf{PLie} \end{array} \quad (2.8)$$

3 Finitely generated suboperads

Let us now focus on finitely generated suboperads of \mathbb{KG} . First remark that $\text{Ope}_{\mathbb{KG}}(\uparrow, \downarrow)$ is isomorphic to the associative algebra Ass . We have:

$$\textcircled{a} \textcircled{*} \circ_* \textcircled{b} \textcircled{c} = \textcircled{a} \textcircled{b} \textcircled{c} = \textcircled{*} \textcircled{c} \circ_* \textcircled{a} \textcircled{b} \quad (3.1)$$

Recall that the operad \mathbf{ComMag} [3] is the free set-operad generated by one binary and symmetric element. More formally, $\mathbf{ComMag}[V]$ is the set of all non planar binary trees with set of leaves equal to V . Let s be the connected species defined by $|s[V]| = 1$ if $|V| = 2$, $|s[V]| = 0$ else. The action of transposition on the sole element of $s[\{a, b\}]$ is trivial. Then $\mathbf{ComMag} = \mathcal{F}_s$.

Proposition 3.1. *The operad $\langle \text{f} \rightarrow \text{g} \rangle^{\text{Ope}}$ is isomorphic to $\mathbb{K}\text{ComMag}$.*

Proof. We know from Proposition 2.5 that $\text{Ope}_{\mathbb{K}\mathbb{G}}(\text{f} \rightarrow \text{g})$ is isomorphic to the suboperad $\text{Ope}_{\text{PLie}}(\text{f} \rightarrow \text{g} + \text{f} \rightarrow \text{g})$ of PLie . Then [3] gives us that this suboperad is isomorphic to $\mathbb{K}\text{ComMag}$. This concludes the proof \square

Now the fact that we can see both *Ass* and *ComMag* as suboperad of $\mathbb{K}\mathbb{G}$ gives us a natural way to define the smallest operad containing these two as suboperads. Denote by *SP* the operad $\text{Ope}_{\mathbb{K}\mathbb{G}}(\text{f} \rightarrow \text{g}, \text{f} \rightarrow \text{g})$. Then *SP* has an interesting operad structure.

Proposition 3.2. *We have $SP = \text{Ope}(\text{Spe}_{\mathbb{K}\mathbb{G}}(\text{f} \rightarrow \text{g}, \text{f} \rightarrow \text{g}), \text{Spe}_{\mathbb{K}\mathbb{G}}(r_1, r_2))$ with*

$$r_1 = \begin{array}{c} \begin{array}{cc} \{1,2\} & \{3\} \\ \diagdown & \diagup \\ \{1\} & \{2\} \end{array} & \text{---} & \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \{2\} & \{3\} \end{array} \\ \diagdown & & \diagup \\ \text{f} & & \text{g} \end{array} \quad (3.2)$$

$$r_2 = \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \text{f} & \{2\} \end{array} & \text{---} & \begin{array}{cc} \{2\} & \{1,3\} \\ \diagdown & \diagup \\ \{2\} & \{3\} \end{array} & \text{---} & \begin{array}{cc} \{3\} & \{1,2\} \\ \diagdown & \diagup \\ \{3\} & \{1\} \end{array} \\ \diagdown & & \diagup & & \diagup \\ \text{f} & & \text{g} & & \text{f} \end{array} \quad (3.3)$$

Hence *SP* is a binary quadratic operad.

For the readers familiar with Koszulity (see [11]), remark that *SP* is a Koszul operad.

Proposition 3.3. *SP admits as Koszul dual $SP^! = \text{Ope}_{\mathbb{K}\mathbb{G}^\vee}(\text{f} \rightarrow \text{g}, \text{f} \rightarrow \text{g}), \text{Spe}_{\mathbb{K}\mathbb{G}^\vee}(r'_1, r'_2, r'_3)$ with*

$$r'_1 = \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \text{f} & \{2\} \end{array} & \text{---} & \begin{array}{cc} \{2\} & \{1,3\} \\ \diagdown & \diagup \\ \{2\} & \{3\} \end{array} \\ \diagdown & & \diagup \\ \text{f} & & \text{g} \end{array}, \quad r'_2 = \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \text{f} & \{2\} \end{array} & \text{---} & \begin{array}{cc} \{2\} & \{1,3\} \\ \diagdown & \diagup \\ \{2\} & \{3\} \end{array} & \text{---} & \begin{array}{cc} \{3\} & \{1,2\} \\ \diagdown & \diagup \\ \{1\} & \{3\} \end{array} & \text{---} & \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \text{f} & \{2\} \end{array} \\ \diagdown & & \diagup & & \diagup & & \diagdown \\ \text{f} & & \text{g} & & \text{f} & & \text{g} \end{array} \quad (3.4)$$

$$\text{and } r'_3 = \begin{array}{c} \begin{array}{cc} \{1\} & \{2,3\} \\ \diagdown & \diagup \\ \text{f} & \{2\} \end{array} & \text{---} & \begin{array}{cc} \{3\} & \{1,2\} \\ \diagdown & \diagup \\ \{3\} & \{1\} \end{array} & \text{---} & \begin{array}{cc} \{2\} & \{1,3\} \\ \diagdown & \diagup \\ \{2\} & \{3\} \end{array} \\ \diagdown & & \diagup & & \diagup \\ \text{f} & & \text{g} & & \text{f} \end{array} \quad (3.5)$$

Proof. (sketch) Denote by I the ideal generated by the relations r_1 and r_2 . Then as a vector space $I[[3]] = \langle r_1, \mathcal{F}_{\mathbb{K}\mathbb{G}}[(23)](r_1), r_2, \mathcal{F}_{\mathbb{K}\mathbb{G}}[(123)](r_2), \mathcal{F}_{\mathbb{K}\mathbb{G}}[(123)^2](r_2) \rangle$ is a subspace of dimension 5 of $\mathcal{F}_{\text{Spe}_{\mathbb{K}\mathbb{G}}(\uparrow \rightarrow \downarrow, \uparrow \rightarrow \downarrow)}[[3]]$ which is of dimension 12. Hence, since as a vector space $\mathcal{F}_{\text{Spe}_{\mathbb{K}\mathbb{G}}(\uparrow \rightarrow \downarrow, \uparrow \rightarrow \downarrow)}^\vee[[3]] \cong \mathcal{F}_{\text{Spe}_{\mathbb{K}\mathbb{G}}(\uparrow \rightarrow \downarrow, \uparrow \rightarrow \downarrow)}^*[[3]] \cong \mathcal{F}_{\text{Spe}_{\mathbb{K}\mathbb{G}}(\uparrow \rightarrow \downarrow, \uparrow \rightarrow \downarrow)}[[3]]$, $I^\perp[[3]]$ must be of dimension 7.

Denote by J the ideal generated by r'_1, r'_2 and r'_3 . Then as a vector space $J = \langle r'_1, -\mathcal{F}_{\mathbb{K}\mathbb{G}^\vee}[(12)](r'_1), -\mathcal{F}_{\mathbb{K}\mathbb{G}^\vee}[(13)](r'_1), r'_2, -\mathcal{F}_{\mathbb{K}\mathbb{G}^\vee}[(123)](r'_2), -\mathcal{F}_{\mathbb{K}\mathbb{G}^\vee}[(123)^2](r'_2), r'_3 \rangle$ is of dimension 7. To conclude we need to show that for any $x \in I$ and $y \in J$, $\langle x, y \rangle = 0$. For example, denoting $p_{1,2} = \uparrow \rightarrow \downarrow$, $s_{1,2} = \uparrow \rightarrow \downarrow$ and $p_{1,2}^\vee$ and $s_{1,2}^\vee$ their duals in $\mathbb{K}\mathbb{G}[[2]]^*$ we have:

$$\begin{aligned} \langle r'_1, r_1 \rangle &= \langle s_{1,*}^\vee \circ_* s_{2,3}^\vee, p_{*,3} \circ_* p_{1,2} - p_{1,*} \circ_* p_{2,3} \rangle \\ &= \langle s_{1,*}^\vee \circ_* s_{2,3}^\vee, p_{*,3} \circ_* p_{1,2} \rangle - \langle s_{1,*}^\vee \circ_* s_{2,3}^\vee, p_{1,*} \circ_* p_{2,3} \rangle \\ &= s_{1,*}^\vee(p_{*,3})s_{2,3}^\vee(p_{1,2}) - s_{1,*}^\vee(p_{1,*})s_{2,3}^\vee(p_{2,3}) \\ &= 0. \end{aligned}$$

□

Proposition 3.4. *The Hilbert series of $SP^!$ is $\mathcal{H}_{SP^!}(x) = \frac{(1 - \log(1 - x))^2 - 1}{2}$.*

Before ending this section let us mention the operad $\text{Ope}_{\mathbb{K}\mathbb{M}\mathbb{G}}(\uparrow, \uparrow \rightarrow \downarrow)$. This operad presents a clear interest. This comes from the fact that the elements \uparrow and $\uparrow \rightarrow \downarrow$ can be considered as minimal elements in the sense that a partial composition with $\uparrow \rightarrow \downarrow$ adds exactly one vertex and no edges and a partial composition with \uparrow adds exactly one edge and no vertex. A natural question to ask at this point is what is the description of $\text{Ope}_{\mathbb{K}\mathbb{M}\mathbb{G}}(\uparrow, \uparrow \rightarrow \downarrow)$, i.e the description of the multigraphs generated by the minimal elements.

Proposition 3.5. *The following holds:*

- the operad SP is a suboperad of $\text{Ope}_{\mathbb{K}\mathbb{M}\mathbb{G}}(\uparrow, \uparrow \rightarrow \downarrow)$,
- the operad $\text{Ope}_{\mathbb{K}\mathbb{G}}(\uparrow, \uparrow \rightarrow \downarrow)$ is a strict suboperad of $\mathbb{K}\mathbb{M}\mathbb{G}$. In particular $\uparrow \rightarrow \downarrow \circ \uparrow$ is not in $\text{Ope}_{\mathbb{K}\mathbb{G}}(\uparrow, \uparrow \rightarrow \downarrow)[[3]]$.

Proof. • The first item is a direct consequence of the following identity:

$$\begin{array}{c} \circlearrowleft \\ * \end{array} \circ_* \begin{array}{c} \bullet \\ 1 \end{array} \begin{array}{c} \bullet \\ 2 \end{array} - \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \begin{array}{c} \bullet \\ 1 \end{array} - \begin{array}{c} \circlearrowleft \\ \bullet \end{array} \begin{array}{c} \bullet \\ 2 \end{array} = 2 \begin{array}{c} \bullet \\ 1 \end{array} \begin{array}{c} \bullet \\ 2 \end{array}. \quad (3.6)$$

- **Samuele:** si tu veux developper une justification avec les calculs de generateurs...

□

Concluding remarks We defined in this extended abstract a notion of graph insertion operad. In the long version [1] of this paper, we give an even more general definition of graph insertion operads which also naturally extends to hypergraphs.

There are two main questions, with reciprocal goals, raised by this paper: the description of the multigraphs generated by \mathfrak{P} and \mathfrak{P}_2 , and the description of the generators of the various operads defined here ($\mathbb{K}\mathbf{G}_{orc}^\bullet$, $\mathbb{K}\mathbf{G}_c$, $\mathbb{K}\mathbf{T}$ etc). Another question interesting perspective for future work is to study appropriate examples of algebras on SP and $SP^!$.

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