

# Basic invariants and reciprocity theorems for the Hopf monoid of hypergraphs and its sub-monoids

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## Abstract

In arXiv:1709.07504 Aguiar and Ardila give a Hopf monoid structure on hypergraphs as well as a general construction of polynomial invariants on Hopf monoids. Using these results, we define in this paper a new polynomial invariant on hypergraphs. We give a combinatorial interpretation of this invariant on negative integers which leads to a reciprocity theorem on hypergraphs. Finally, we use this invariant to recover well-known invariants on other combinatorial objects (graphs, simplicial complexes, building sets etc) as well as the associated reciprocity theorems.

## 1 Introduction

In combinatorics, Hopf structures give an algebraic framework to deal with operations of merging (product) and splitting (coproduct) combinatorial objects. The notion of Hopf algebra is well known and used in combinatorics for over 30 years, and has proved its great strength in various questions (see for example [10]). More recently, Aguiar and Mahajan defined a notion of Hopf monoid [3],[4] akin to the notion of Hopf algebra and built on Joyal's theory of species [12]. Such as in the case of Hopf algebras, a useful application of Hopf monoids is to define and compute polynomial invariants (see [2], [7], [9] or [13] for various examples), as was put to light by the recent and extensive paper of Aguiar and Ardila [1]. In particular they give a theorem to generate various polynomial invariants and use it to recover the chromatic polynomial of graphs, the Billera-Jia-Reiner polynomial of matroids and the strict order polynomial of posets. Furthermore they also give a way to compute these polynomial invariants on negative integers hence also recovering the different reciprocity theorems associated to these combinatorial objects.

In this paper, we apply Aguiar and Ardila's theorem to the Hopf monoid of hypergraphs defined in [1]. This Hopf structure is different than the one defined and studied in [5] (the respective coproducts are different). We obtain a combinatorial description for the (basic) invariant  $\chi_I(H)(n)$  in terms of colorings of hypergraphs (Theorem 14). We then use another approach (rather technical) than the method of [1] to get a reciprocity theorem for hypergraphs (Theorem 19). Finally, we use these results to obtain polynomial invariants on sub-monoids of the Hopf monoid of hypergraphs.

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## 2 Definitions and reminders

### 2.1 Hopf monoids

We present here basic definitions on Hopf monoids. The interested reader may refer to [4] and to [1] for more information on this subject. In this paper  $\mathbb{k}$  is a field and all vector spaces are over  $\mathbb{k}$ .

**Definition 1.** A *vector species*  $P$  consists of the following data.

- For each finite set  $I$ , a vector space  $P[I]$ .
- For each bijection of finite sets  $\sigma : I \rightarrow J$ , a linear map  $P[\sigma] : P[I] \rightarrow P[J]$ . These maps should be such that  $P[\sigma \circ \tau] = P[\sigma] \circ P[\tau]$  and  $P[\text{id}] = \text{id}$ .

A *sub-species* of a vector species  $P$  is a vector species  $Q$  such that for each finite set  $I$ ,  $Q[I]$  is a sub-space of  $P[I]$  and for each bijection of finite sets  $\sigma : I \rightarrow J$ ,  $Q[\sigma] = P[\sigma]|_{Q[I]}$ .

A *morphism*  $f : P \rightarrow Q$  between vector species is a collection of linear maps  $f_I : P[I] \rightarrow Q[I]$  satisfying the naturality axiom: for each bijection  $\sigma : I \rightarrow J$ ,  $f_J \circ P[\sigma] = Q[\sigma] \circ f_I$ .

**Definition 2.** A *connected Hopf monoid in vector species* is a vector species  $M$  with  $M[\emptyset] = \mathbb{k}$  that is equipped with product and co-product linear maps

$$\mu_{S,T} : M[S] \otimes M[T] \rightarrow M[S \sqcup T] \quad \Delta_{S,T} : M[S \sqcup T] \rightarrow M[S] \otimes M[T],$$

with  $S$  and  $T$  disjoint sets, and subject to the following axioms:

- *Naturality.* For each pair of disjoint sets  $S, T$ , each bijection  $\sigma$  with domain  $S \sqcup T$ , we have  $M[\sigma] \circ \mu_{S,T} = \mu_{\sigma(S),\sigma(T)} \circ M[\sigma|_S] \otimes M[\sigma|_T]$  and  $M[\sigma|_S] \otimes M[\sigma|_T] \circ \Delta_{S,T} = \Delta_{\sigma(S),\sigma(T)} \circ M[\sigma]$ .
- *Unitality.* For each set  $I$ ,  $\mu_{I,\emptyset}$ ,  $\mu_{\emptyset,I}$ ,  $\Delta_{I,\emptyset}$  and  $\Delta_{\emptyset,I}$  are given by the canonical isomorphisms  $M[I] \otimes \mathbb{k} \cong \mathbb{k} \otimes M[I]$ .
- *Associativity.* For each triplet of pairwise disjoint sets  $R, S, T$  we have:  $\mu_{R,S \sqcup T} \circ \text{id} \otimes \mu_{S,T} = \mu_{R \sqcup S, T} \circ \mu_{R,S} \otimes \text{id}$ .
- *Co-associativity.* For each triplet of pairwise disjoint sets  $R, S, T$  we have:  $\Delta_{R,S} \otimes \text{id} \circ \Delta_{R \sqcup S, T} = \text{id} \otimes \Delta_{S,T} \circ \Delta_{R,S \sqcup T}$ .
- *Compatibility.* For each pair of disjoint sets  $A, B$ , each pair of disjoint sets  $C, D$  we have the following commutative diagram, where  $\tau$  maps  $x \otimes y$  to  $y \otimes x$ :

$$\begin{array}{ccccc} P[S] \otimes P[T] & \xrightarrow{\mu_{S,T}} & P[I] & \xrightarrow{\Delta_{S',T'}} & P[S'] \otimes P[T'] \\ \Delta_{A,B} \otimes \Delta_{C,D} \downarrow & & & & \uparrow \mu_{A,C} \otimes \mu_{B,D} \\ P[A] \otimes P[B] \otimes P[C] \otimes P[D] & \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} & P[A] \otimes P[C] \otimes P[B] \otimes P[D] & & \end{array}$$

A *sub-monoid* of a Hopf monoid  $M$  is a sub-species of  $M$  stable under the product and co-product maps.

The *co-opposite* Hopf monoid  $M^{\text{cop}}$  of  $M$  is the Hopf monoid with opposite co-product:  $\Delta_{S,T}^{M^{\text{cop}}} = \Delta_{T,S}^M$ .

A *morphism of Hopf monoids* in vector species is a morphism of vector species which preserves the products, co-products (compatibility axiom) and the unity (unitality axiom).

We will use the term Hopf monoid for connected Hopf monoid in vector species. A sub-monoid of a Hopf monoid  $M$  is itself a Hopf monoid when equipped with the product and co-product maps of  $M$ . We consider this to be always the case.

A *decomposition* of a finite set  $I$  is a sequence of pairwise disjoint subsets  $S = (S_1, \dots, S_l)$  such that  $I = \sqcup_{i=1}^l S_i$ . A *composition* of a finite set  $I$  is a decomposition of  $I$  without empty parts. We will note  $S \vdash I$  for  $S$  a decomposition of  $I$ ,  $S \models I$  if  $S$  is a composition,  $l(S) = l$  the length of a decomposition and  $|S| = |I|$  the number of elements in the decomposition.

**Definition 3.** Let  $M$  be a Hopf monoid. The *antipode* of  $M$  is the morphism of Hopf monoids  $S : M \rightarrow M$  defined by:

$$S_I = \sum_{\substack{(S_1, \dots, S_k) \models I \\ k \geq 1}} \mu_{S_1, \dots, S_k} \circ \Delta_{S_1, \dots, S_k}$$

for any finite set  $I$ .

**Definition 4.** A *character* on a Hopf monoid  $M$  is a collection of linear maps  $\zeta_I : M[I] \rightarrow \mathbb{k}$  subject to the following axioms.

- *Naturality.* For each bijection  $\sigma : I \rightarrow J$  we have  $\zeta_J \circ M[\sigma] = \zeta_I$ .
- *Multiplicativity.* For each disjoint sets  $S, T$  we have  $\zeta_{S \sqcup T} \circ \mu_{S, T} = \mu_{\mathbb{k}} \circ \zeta_S \otimes \zeta_T$ .
- *Unitality.*  $\zeta_{\emptyset}(1) = 1$ .

Let us recall from [1] the results which we will use in the sequel.

**Theorem 5** (Proposition 16.1 and Proposition 16.2 in [1]). Let  $M$  be a Hopf monoid and  $\zeta$  a character on  $M$ . For  $x \in M[I]$  and  $n$  an integer we define:

$$\chi_I(x)(n) = \sum_{(S_1, \dots, S_n) \vdash I} \zeta_{S_1} \otimes \dots \otimes \zeta_{S_n} \circ \Delta_{S_1, \dots, S_n}(x).$$

Then  $\chi_I$  is a polynomial invariant in  $n$  verifying:

- $\chi_I(x)(1) = \zeta(x)$ ,
- $\chi_{\emptyset} = 1$  and  $\chi_{S \sqcup T}(\mu(x \otimes y)) = \chi_S(x)\chi_T(y)$ ,
- $\chi_I(x)(-n) = \chi_I(S_I(x))(n)$ .

Let  $M$  be a Hopf monoid. For  $I$  a set and  $x \in M[I]$  we call  $x$  *discrete* if  $I = \{i_1, \dots, i_{|I|}\}$  and  $x = \mu_{\{i_1\}, \dots, \{i_{|I|}\}} x_1 \otimes \dots \otimes x_{|I|}$  for  $x_j \in M[\{i_j\}]$ . Then the maps that send discrete elements onto 1 and other elements onto 0 give us a character of Hopf monoid. Following the terminology introduced in Section 17 of [1], we call *basic invariant of  $M$*  the polynomial invariant obtained by applying Theorem 5 with this character. We note  $\chi^M$  this polynomial or just  $\chi$  when  $M$  is clear from the context.

**Proposition 6** (Proposition 16.3 in [1]). Let  $M$  and  $N$  be two Hopf monoids,  $\zeta^M$  and  $\zeta^N$  characters on  $M$  and  $N$  and  $f : M \rightarrow N$  a Hopf monoid morphism such that for every  $I$ :

$$\zeta_I^N \circ f_I = \zeta_I^M.$$

Denote by  $\chi(\zeta^M)$  and  $\chi(\zeta^N)$  the polynomial invariants obtained by applying Theorem 5 with  $M$  and  $\zeta^M$  and  $N$  and  $\zeta^N$ . For every  $I$ , one then has:

$$\chi(\zeta^N)_I \circ f_I = \chi(\zeta^M)_I.$$

In particular, since Hopf monoid morphisms conserve discrete elements, for  $f : M \rightarrow N$  a Hopf monoid morphism and  $I$  a set, we have  $\chi_I^N \circ f_I = \chi_I^M$ .

## 2.2 A useful combinatorial identity

We remind here a classical result of combinatorics and a direct corollary which will be useful in the following section. We shall only give a sketch of the proofs.

In all the following, given an integer  $n$  we note  $[n] = \{1, \dots, n\}$ .

**Proposition 7.** Let  $n$  and  $m$  be two integers. The number of surjections  $S_{n,m}$  from  $[m]$  to  $[n]$  is given by:

$$S_{n,m} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^m.$$

*Proof.* This formula can be obtained by the inclusion-exclusion principle.  $\square$

**Corollary 8.** For  $n$  and  $m$  two integers such that  $m < n$ , and  $P$  a polynomial of degree at most  $m$ , we have:

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} P(k) = 0.$$

*Proof.* The statement above is a direct consequence of the fact that  $S_{n,m} = 0$  for  $n < m$ .  $\square$

## 3 Basic invariant of hypergraphs

In all the following,  $I$  always denotes a finite set.

Our goal is to express the basic invariant of the Hopf monoid of hypergraphs defined in Section 20 of [1]. More specifically we intend to obtain a combinatorial interpretation of  $\chi_I(x)(n)$  and  $\chi_I(x)(-n)$ .

In this context, an *hypergraph over  $I$*  is a collection of (possibly repeated) subsets of  $I$ , which we call *edges*<sup>1</sup>, containing  $\emptyset$  exactly once. The elements of  $I$  are then called *vertices* of  $H$  and  $HG[I]$  denotes the free vector space of hypergraphs over  $I$ . Note that two hypergraphs over different sets can never be equal, e.g.  $\{\{1, 2, 3\}, \{2, 3, 4\}\} \in HG[[4]]$  is not the same as  $\{\{1, 2, 3\}, \{2, 3, 4\}\} \in HG[[4] \cup \{a, b\}]$ . This is illustrated in Figure 1

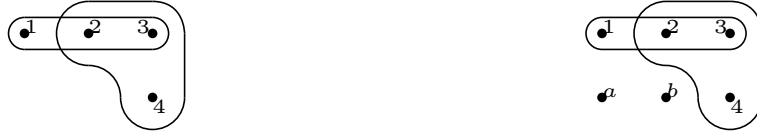


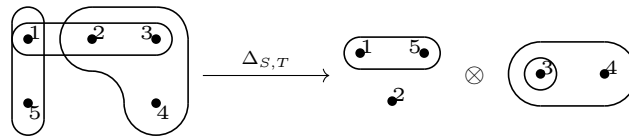
Figure 1: Two hypergraphs with the same edges but over different sets.

The product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : HG[S] \otimes HG[T] &\rightarrow HG[I] & \Delta_{S,T} : HG[I] &\rightarrow HG[S] \otimes HG[T] \\ H_1 \otimes H_2 &\mapsto H_1 \sqcup H_2 & H &\mapsto H|_S \otimes H/_S \end{aligned}$$

where  $H|_S = \{e \in H \mid e \subseteq S\}$  is the *restriction of  $H$  to  $S$*  and  $H/_S = \{e \cap T \mid e \not\subseteq S\} \cup \{\emptyset\}$  is the *contraction of  $S$  from  $H$* . The discrete hypergraphs are then the hypergraphs with edges of cardinality at most 1.

**Example 9.** For  $I = [5]$ ,  $S = \{1, 2, 5\}$  and  $T = \{3, 4\}$ , we have:



<sup>1</sup>in some references, the terms *hyperedge* or *multiedge* is used.

In [1], Ardila and Aguiar propose a method to obtain a combinatorial interpretation of any polynomial invariant given by Theorem 5 on negative integers, assuming that we have an interpretation of it on positive integers. Their method consists in using a cancellation-free grouping-free formula for the antipode and the third point of Theorem 5. We use here a different approach: we express the polynomial dependency of  $\chi_I(x)(n)$  in  $n$ , which we then use to calculate  $\chi_I(x)(-n)$  and interpret the resulting formula.

Let us begin by giving a proposition which is needed to show the polynomial dependency of  $\chi_I(x)(n)$  in  $n$ . For  $t \in \mathbb{N}^*$  and a sequence of positive integers  $p_1, p_2, \dots, p_t$ , we define  $F_{p_1, \dots, p_t}$  as a function over the integers given by, for  $n \in \mathbb{N}$ :

$$F_{p_1, \dots, p_t}(n) = \sum_{0 \leq k_1 < \dots < k_t \leq n-1} k_1^{p_1} \dots k_t^{p_t}.$$

**Proposition 10.** Let  $p_1, p_2, \dots, p_t$  be integers and note  $d_k = \sum_{i=1}^k p_i + k$  for  $1 \leq k \leq t$ . Then  $F_{p_1, \dots, p_t}$  is a polynomial of degree  $d_t$  whose constant coefficient is null and the  $(d_t - i)$ -th, (for  $i < d_t$ ) coefficient is given by

$$\sum_{j_{t-1}=0}^{\min(j_t, d_{t-1}-1)} \sum_{j_{t-2}}^{\min(j_{t-1}, d_{t-2}-1)} \dots \sum_{j_1=0}^{\min(j_2, d_1-1)} \prod_{k=1}^t \binom{d_k - j_{k-1}}{j_k - j_{k-1}} \frac{B_{j_k - j_{k-1}}}{d_k - j_{k-1}},$$

where  $j_t = i$  and  $j_0 = 0$ , and the  $B_j$  numbers are the Bernoulli numbers with the convention  $B_1 = -1/2$ .

*Proof.* We show this by induction over  $t$ . For  $t = 1$  the expression of the coefficients gives us the well-known identity  $F_p(n) = \sum_{i=0}^p \binom{p+1}{i} \frac{B_i}{p+1} n^{p+1-i}$ . Hence the result is true for  $t = 1$ . Suppose now the result is true for  $t \geq 1$  and let  $p_1, p_2, \dots, p_{t+1}$  be  $t + 1$  integers. Denote by  $a_i$  the  $d_t - i$  coefficient of  $F_{p_1, \dots, p_t}(n)$ . We then have:

$$\begin{aligned} F_{p_1, \dots, p_{t+1}}(n) &= \sum_{0 \leq k_1 < \dots < k_{t+1} \leq n-1} k_1^{p_1} \dots k_{t+1}^{p_{t+1}} = \sum_{k=0}^{n-1} k^{p_{t+1}} F_{p_1, \dots, p_t}(k) = \sum_{k=0}^{n-1} k^{p_{t+1}} \sum_{j=0}^{d_t-1} a_j k^{d_t-j} \\ &= \sum_{j=0}^{d_t-1} a_j \sum_{k=0}^{n-1} k^{p_{t+1}+d_t-j} = \sum_{j=0}^{d_t-1} a_j \sum_{k=0}^{n-1} k^{d_{t+1}-1-j} \\ &= \sum_{j=0}^{d_t-1} a_j F_{d_{t+1}-1-j}(n) \\ &= \sum_{j=0}^{d_t-1} a_j \sum_{i=0}^{d_{t+1}-1-j} \binom{d_{t+1}-j}{i} \frac{B_i}{d_{t+1}-j} n^{d_{t+1}-j-i} \\ &= \sum_{j=0}^{d_t-1} \sum_{i=0}^{d_{t+1}-1-j} a_j \binom{d_{t+1}-j}{i} \frac{B_i}{d_{t+1}-j} n^{d_{t+1}-j-i} \\ &= \sum_{j=0}^{d_t-1} \sum_{i=j}^{d_{t+1}-1} a_j \binom{d_{t+1}-j}{i-j} \frac{B_{i-j}}{d_{t+1}-j} n^{d_{t+1}-i} \\ &= \sum_{i=0}^{d_{t+1}-1} \left( \sum_{j=0}^{\min(i, d_t-1)} a_j \binom{d_{t+1}-j}{i-j} \frac{B_{i-j}}{d_{t+1}-j} \right) n^{d_{t+1}-i}. \end{aligned}$$

This concludes this proof.  $\square$

Before stating our results on  $\chi_I(H)(n)$  we need to introduce some definitions. There exists a canonical bijection between decompositions and functions with co-domain of the form  $[n]$ . In

the sequel, we will want to seamlessly pass from one notion to the other. We hence give a few explanations on this bijection. Given an integer  $n$ , the canonical bijection between decompositions of  $I$  of size  $n$  and functions from  $I$  to  $[n]$  is given by:

$$b_{I,n} : \{f : I \rightarrow [n]\} \rightarrow \{P \vdash I \mid l(P) = n\}$$

$$f \mapsto (f^{-1}(1), \dots, f^{-1}(n)).$$

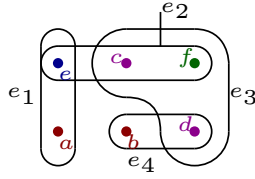
If it is clear from the context what are  $I$  and  $n$  we will write  $b$  instead of  $b_{I,n}$ . If  $P$  is a partition we will also refer to  $b^{-1}(P)$  by  $P$  so that instead of writing "i such that  $v \in P_i$ " we can just write  $P(v)$ . Similarly, if  $P$  is a function we will refer to  $b(P)$  by  $P$  so that  $P_i = P^{-1}(i)$ . Also remark that  $b_{I,n}$  induces a bijection between compositions of  $I$  of size  $n$  and surjections from  $I$  to  $[n]$ .

**Definition 11.** Let  $H$  be a hypergraph over  $I$  and  $n$  be an integer. A *coloring of  $H$  with  $[n]$*  is a function from  $I$  to  $[n]$  (or a decomposition of  $I$  of length  $n$  from what precede) and in this context the elements of  $[n]$  are called *colors*.

Let  $S \vdash I$  be a coloring of  $H$ . For  $v \in e \in H$ , we say that  $v$  is a *maximal vertex of  $e$  (for  $S$ )* if  $v$  is of maximal color in  $e$  and we call *maximal color of  $e$  (for  $S$ )* the color of a maximal vertex of  $e$ . We say that a vertex  $v$  is a *maximal vertex (for  $S$ )* if it is a maximal vertex of an edge.

If  $J \subset I$  is a subset of vertices, the *order of appearance of  $J$  (for  $S$ )* is the composition  $\text{cano}(S|_J)$  where  $S|_J = (S_1 \cap J, \dots, S_{l(S)} \cap J)$ . The map  $\text{cano}$  sends any decomposition on the composition obtained by dropping the empty parts.

**Example 12.** We represent the coloring of a hypergraph on  $I = \{a, b, c, d, e, f\}$  with  $\{1, 2, 3, 4\}$ :



The maximal vertex of  $e_1$  is  $a$  and the maximal vertices of  $e_3$  are  $c$  and  $d$ . The maximal color of  $e_2$  is  $3$ . The order of appearance of  $\{a, c, d, e\}$  is  $(\{e\}, \{c, d\}, \{a\})$ . The order of appearance of all edges is  $(\{e_2, e_3\}, \{e_1, e_4\})$ .

**Definition 13.** Let  $H$  be a hypergraph over  $I$ . An *orientation of  $H$*  is a function  $f$  from  $H$  to  $I$  such that  $f(e) \in e$  for every edge  $e$ . A *directed cycle* in an orientation  $f$  of  $H$  is a sequence of distinct edges  $e_1, \dots, e_k$  such that  $f(e_1) \in e_2 \setminus f(e_2), \dots, f(e_k) \in e_1 \setminus f(e_1)$ . An orientation is *acyclic* if it does not have any cycle. We note  $\mathcal{A}_H$  the set of acyclic orientation of  $H$ .

An orientation  $f$  of  $H$  and a coloring  $S$  of  $H$  with  $[n]$  are said to be *compatible* if  $S(f(e)) = \max(S(e))$  for every  $e \in H$ . They are said to be *strictly compatible* if  $f(e)$  is the unique maximal vertex of  $e$ .

**Theorem 14.** Let  $I$  be a set and  $H \in HG[I]$  a hypergraph over  $I$ . Then  $\chi_I(H)(n)$  is the number of colorings of  $H$  with  $[n]$  such that every edge has only one maximal vertex. This is also the number of strictly compatible pairs of acyclic orientations and colorings with  $[n]$ . Furthermore, defining  $P_{H,f} = \{P \vdash f(H) \mid v \in e \setminus f(e) \Rightarrow P(v) < P(f(e))\}$ , for every  $f \in \mathcal{A}_H$ , we have that

$$\chi_I(H)(n) = n^{|J_H|} \sum_{f \in \mathcal{A}_H} \sum_{P \in P_{H,f}} F_{p_1, \dots, p_{l(P)}}(n),$$

where  $J_H \subset I$  is the set of isolated vertices of  $H$  (i.e vertices not in an edge) and for every  $P \in P_{H,f}$ ,  $p_i = |\tilde{P}_i|$  and  $\tilde{P}_i = \left( \bigcup_{e \in f^{-1}(P_i)} e \right) \cap f(H)^c \cap \bigcap_{j < i} \tilde{P}_j^c$ .

*Proof.* For  $S$  a decomposition of  $I$  of size  $n$ , note  $H_1 \otimes \dots \otimes H_n = \Delta_{S_1, \dots, S_n}(H)$ . Let  $S$  be a decomposition of  $I$  of size  $n$ . Let  $e$  be an edge. We then have the equivalence:

$$e \in H_i \iff e \cap S_i \neq \emptyset \wedge \forall j > i, e \cap S_j = \emptyset$$

$$\iff e \cap S_i \text{ is the set of maximal vertices of } e$$

Hence, we have that

$$\begin{aligned} \zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n} \circ \Delta_{S_1, \dots, S_n}(H) = 1 &\iff \forall e \in H, e \in H_i \Rightarrow |e \cap S_i| = 1 \\ &\iff \text{each edge has only one maximal vertex.} \end{aligned}$$

The equivalence between the colorings such that every edge has only one maximal vertex and the strictly compatible pairs of acyclic orientations and colorings is given by the bijection  $S \mapsto (e \mapsto v_e, S)$ , where  $v_e$  is the unique vertex in  $e$  such that  $S(v_e) = \max(S(e))$ .

The term  $n^{|J_H|}$  in the formula is trivially obtained, in the following we hence consider that  $H$  has no isolated vertices.

Informally, the formula can be obtained by the following reasoning. To choose a coloring such that every edge has only one maximal vertex, one can proceed in the following:

1. choose the maximal vertex of each edge ( $f \in \mathcal{A}_H$ ),
2. choose in which order those vertices appear ( $P \in P_{H,f}$ ),
3. choose the color of those vertices ( $k_1 + 1, \dots, k_{l(P)} + 1$ ), (and notice that the set of such choices is empty if  $l(P) > n$ , which allow us to not add this non polynomial dependency in  $n$  at the previous choice),
4. choose the colors of the yet uncolored vertices which are in the same edge than a vertex of minimal color in  $f(H)$  ( $k_1^{|\tilde{P}_1|}$ ); then those in the same edge than a vertex of second minimal color in  $f(H)$  ( $k_2^{|\tilde{P}_2|}$ ), etc.

More formally, we show that there exists a bijection between the set of colorings such that every edge has only one maximal vertex and the set

$$\bigsqcup_{f \in \mathcal{A}_H} \bigsqcup_{P \in P_{H,f}} \bigsqcup_{0 \leq k_1 < k_2 < \dots < k_{l(P)} \leq n-1} \prod_{1 \leq i \leq l(P)} [k_i]^{\tilde{P}_i}.$$

Let  $g$  be a coloring of interest and define:

- $f : e \mapsto v \in e$  such that  $g(v) = \max(g(e))$ ,
- $P = h \circ g(f(H))$  where  $h$  is the increasing bijection from  $g(f(H))$  to  $[[f(H)]]$ ,
- $\tilde{P}_i = \left( \bigcup_{e \in f^{-1}(P_i)} e \right) \cap f(H)^c \cap \bigcap_{j < i} \tilde{P}_j^c$  for  $1 \leq i \leq l(P)$ ,
- $k_i = g(P_i) - 1$  for  $1 \leq i \leq l(P)$ .

The function  $f$  not being in  $\mathcal{A}_H$  would imply that there exists a vertex  $v$  such that  $g(v) < g(v)$ . This is not possible, hence  $f \in \mathcal{A}_H$ . We also have that  $P \in P_{H,f}$  because by definition of  $g$ ,  $v \in e \setminus f(e)$  implies  $g(v) < g(f(e))$  and  $h$  is increasing. It is also clear that  $0 \leq k_1 < \dots < k_{l(P)} \leq n - 1$ . The image of  $g$  is then  $(g_{|\tilde{P}_1}, \dots, g_{|\tilde{P}_{l(P)}})$  which is in  $\prod_{1 \leq i \leq l(P)} [k_i]^{\tilde{P}_i}$  since for every  $v \in \tilde{P}_i$  we must have  $g(v) < g(P_i)$  by definition. Let us now consider  $f \in \mathcal{A}_H$ ,  $P \in P_{H,f}$ ,  $1 \leq k_1 < \dots < k_{l(P)}$  and  $(g_1, \dots, g_{l(P)}) \in \prod_{1 \leq i \leq l(P)} [k_i]^{\tilde{P}_i}$ . Let  $h$  be the increasing bijection from  $[[f(H)]]$  to  $\{k_1 + 1, \dots, k_{l(P)} + 1\}$  and define  $g : I \rightarrow [n]$  by  $g_{|\tilde{P}_i} = g_i$  and  $g_{|f(H)} = h \circ P$  (it is sufficient since  $(\tilde{P}_1, \dots, \tilde{P}_{l(P)}, f(H))$  is a partition of  $I$ ). Let us show that  $g$  is a coloring of interest. Let be  $v \in e \setminus f(e)$ ,

- if  $v \in f(H)$  then  $P(v) < P(f(e))$  by definition and so  $g(v) < g(f(e))$  since  $h$  is increasing,
- if  $v \notin f(H)$  then  $v \in \tilde{P}_i$  with  $i \leq P(f(e))$  and so  $g(v) = g_i(v) \leq k_i < k_i + 1 \leq k_{P(f(e))} + 1 = g(f(e))$ .

We conclude the proof by remarking that the two defined transformations are inverse functions.  $\square$

**Example 15.** The coloring given in Example 12 is not counted in  $\chi_I(H)(4)$  since  $e_3$  has two maximal vertices. However by changing the color of  $d$  to 2 we do obtain a coloring where every edge has only one maximal vertex.

We are now interested in the value of  $(-1)^{|I|}\chi_I(H)(-n)$ . Let us first state two lemmas. The first lemma justifies the use of the  $F_p$  (with  $p$  a finite sequence of integers) polynomials to express the basic invariant: they have a good expression on negative integers. The second lemma is a result which can be interpreted on graphs and partitions as we do but also on posets and linear extensions. It is the crux of the proof of Theorem 19.

Given an integer  $n$ , a sequence of integers  $p = p_1, \dots, p_t$  is said to be a *decomposition* of  $n$ , and we note  $p \vdash n$ , if  $n = \sum_{i=1}^t p_i$ . If  $p$  and  $q$  are two decompositions of the same integer, we say that  $q$  *coarsens*  $p$  or  $p$  *refines*  $q$  and note  $p \prec q$  if  $p$  is the concatenation of decompositions of the terms of  $q$ .

**Lemma 16.** Let  $p$  be a sequence of integers of length  $t$ . Then

$$F_p(-n) = (-1)^{dt} \sum_{p \prec q} F_q(n+1).$$

*Proof.* Remark that  $\sum_{p \prec q} F_q(n+1)$  can also be written as  $\sum_{0 \leq k_1 \leq \dots \leq k_t \leq n} k_1^{p_1} \dots k_t^{p_t}$ . We now proceed by induction on  $t$ . For  $t = 1$ , we have

$$\begin{aligned} F_p(-n) &= \sum_{i=0}^p \binom{p+1}{i} \frac{B_i}{p+1} (-n)^{p+1-i} \\ &= \frac{(-1)^{p+1}}{p+1} n^{p+1} - \frac{1}{2} (-1)^p n^p + (-1)^{p+1} \sum_{i=2}^p \binom{p+1}{i} \frac{B_i}{p+1} n^{p+1-i} \\ &= (-1)^{p+1} \left( \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \sum_{i=2}^p \binom{p+1}{i} \frac{B_i}{p+1} n^{p+1-i} \right) \\ &= (-1)^{p+1} (F_p(n) + n^p) = (-1)^{p+1} F_p(n+1), \end{aligned}$$

where the second equality comes from the fact that  $B_i = 0$  when  $i$  is an odd number different from one. Suppose now our proposition is true up to  $t$ . In the proof of Proposition 2 we showed that  $F_{p_1, \dots, p_{t+1}}(n) = \sum_{j=0}^{d_t-1} a_j F_{d_{t+1}-1-j}(n)$  where  $a_j$  is the  $d_t - j$  coefficient of  $F_{p_1, \dots, p_t}(n)$ . This gives

$$\begin{aligned} F_{p_1, \dots, p_{t+1}}(-n) &= \sum_{j=0}^{d_t-1} a_j (-1)^{d_{t+1}-j} \sum_{k=0}^n k^{d_{t+1}-1-j} = - \sum_{j=0}^{d_t-1} a_j \sum_{k=0}^n (-k)^{p_{t+1}+d_t-j} \\ &= - \sum_{k=0}^n (-k)^{p_{t+1}} \sum_{j=0}^{d_t-1} a_j (-k)^{d_t-j} = (-1)^{p_{t+1}+1} \sum_{k=0}^n k^{p_{t+1}} F_{p_1, \dots, p_t}(-k) \\ &= (-1)^{p_{t+1}+1} \sum_{k_{t+1}=0}^n k_{t+1}^{p_{t+1}} (-1)^{d_t} \sum_{0 \leq k_1 \leq \dots \leq k_t \leq k_{t+1}} k_1^{p_1} \dots k_t^{p_t} \\ &= (-1)^{d_{t+1}} \sum_{0 \leq k_1 \leq \dots \leq k_{t+1} \leq n} k_1^{p_1} \dots k_{t+1}^{p_{t+1}} \\ &= \end{aligned}$$

where the fifth equality is our induction hypothesis.  $\square$

**Definition 17.** Let  $I$  and  $J$  be two disjoint sets and  $P = (P_1, \dots, P_l) \vDash I$  and  $Q = (Q_1, \dots, Q_k) \vDash J$  be two compositions. The *product* of  $P$  and  $Q$  is the composition  $P \cdot Q = (P_1, \dots, P_l, Q_1, \dots, Q_k)$ . The *shuffle product* of  $P$  and  $Q$  is the set  $sh(P, Q) = \{R \vDash I \sqcup J \mid P = \text{cano}(R|_I), Q = \text{cano}(R|_J)\}$ .

Let  $P' \vDash I$  be another composition of  $I$ . We say that  $P'$  *refines*  $P$  and note  $P' \prec P$  if  $P' = Q_1 \cdots Q_l$  with  $Q_i$  a composition of  $P_i$ .



**Lemma 18.** Let  $I$  be a set and  $P \vDash I$  a composition of  $I$ . We have the identity:

$$\sum_{Q \prec P} (-1)^{l(Q)} = (-1)^{|P|}.$$

Let furthermore  $G$  be a directed acyclic graph on  $I$  and consider the *constrained set*  $C(G, P) = \{Q \prec P \mid \forall (v, v') \in G, Q(v) < Q(v')\}$ . We have the more general identity:

$$\sum_{Q \in C(G, P)} (-1)^{l(Q)} = \begin{cases} 0 & \text{if there exists } (v, v') \in G \text{ such that } P(v') < P(v), \\ (-1)^{|P|} & \text{else.} \end{cases}$$

*Proof.* Since  $\sum_{Q \prec P} (-1)^{l(Q)} = \prod_{i=1}^{l(P)} \sum_{Q \vDash P_i} (-1)^{l(Q)}$  we only need to show that  $\sum_{Q \vDash I} (-1)^{l(Q)} = (-1)^{|I|}$  to prove the first identity. Since the compositions of  $I$  of size  $n$  and the surjections from  $I$  to  $[n]$  are in bijection, we have that:

$$\begin{aligned} \sum_{Q \vDash I} (-1)^{l(Q)} &= \sum_{n=1}^{|I|} (-1)^n S_{|I|, n} = \sum_{n=1}^{|I|} (-1)^n \sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k^{|I|} \\ &= \sum_{k=1}^{|I|} (-1)^k \left( \sum_{n=k}^{|I|} \binom{n}{k} \right) k^{|I|} = \sum_{k=1}^{|I|} (-1)^k \binom{|I|+1}{k+1} k^{|I|} \\ &= (-1)^{|I|} \sum_{k=0}^{|I|-1} (-1)^k \binom{|I|+1}{k} (|I|-k)^{|I|} \\ &= (-1)^{|I|} \left( 1 + \sum_{k=0}^{|I|-1} (-1)^k \binom{|I|+1}{k} (|I|-k)^{|I|} \right) \\ &= (-1)^{|I|}. \end{aligned}$$

Note that the last equality is a direct consequence of Corollary 8.

To show the second identity first remark that the case where the sum is null is straightforward: if there exists  $(v, v') \in G$  such that  $P(v') < P(v)$ , then  $C(G, P) = \emptyset$  and so the sum is null. From now on we only consider non empty summation sets. In this case we have that  $\sum_{Q \in C(G, P)} (-1)^{l(Q)} = \prod_{i=1}^{l(P)} \sum_{Q \in C(G \cap P_i^2, (P_i))} (-1)^{l(Q)}$  and we only need to show that  $\sum_{P \in C(G)} (-1)^{l(P)} = (-1)^{|I|}$  where  $C(G) = C(G, (I))$ . Let us note  $S(G) = \sum_{P \in C(G)} (-1)^{l(P)}$  from now on.

If  $G$  is not connected let  $I = J \sqcup K$  and  $G = H \sqcup H'$  where  $V(H) = J$  and  $V(H') = K$ . Let  $P \in C(H)$  and  $Q \in C(H')$  and suppose without loss of generality that  $m = l(Q) < l(P) = M$ . To choose  $R$  in  $sh(P, Q)$  we can first choose its length; then which indices are going to have a part of

$Q$ ; and then which indices among them are also going to have a part of  $P$ . This leads to:

$$\begin{aligned}
\sum_{R \in sh(P, Q)} (-1)^{l(R)} &= \sum_{k=M}^{m+M} (-1)^k \binom{k}{m} \binom{m}{M - (k - m)} \\
&= \sum_{k=0}^m (-1)^{k+M} \binom{M+k}{m} \binom{m}{k-m} \\
&= (-1)^M \sum_{k=0}^m (-1)^{m-k} \binom{M+m-k}{m} \binom{m}{k} \\
&= \frac{(-1)^M}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \frac{(m+M-k)!}{(M-k)!} \\
&= \frac{(-1)^M}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} (-k)^m \\
&= \frac{(-1)^{m+M}}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^m \\
&= \frac{(-1)^{m+M}}{m!} S_{m,m} = (-1)^{m+M} = (-1)^{l(P)+l(Q)},
\end{aligned}$$

where the fifth equality follows from Corollary 8. This shows that  $S(G)$  is multiplicative (with the product being the disjoint union) and so we can restrict ourselves to showing that  $S(G) = (-1)^{|I|}$  for  $G$  a connected graph. We will do this by induction on the number of edges of  $G$ .

Suppose now that  $G$  is connected. If  $G$  has no edges then  $G$  is reduced to a single vertex and the result is trivial. Else let be  $(v, v') \in G$ . We say that  $(v, v')$  is superfluous if there exists  $v_0, v_1, \dots, v_{k+1} \in I$  such that  $v = v_0, v' = v_{k+1}$  and  $(v_i, v_{i+1}) \in G$  for all  $i \in [k]$ . If  $(v, v')$  is superfluous then  $C(G) = C(G(v, v'))$  and so  $S(G) = S(G \setminus (v, v')) = (-1)^{|I|}$  by induction. Else we have  $C(G \setminus (v, v')) = C(G) + C(t_{(v, v')}(G)) + C(G \setminus (v, v')) \cap \{P \models I \mid P(v) = P(v')\}$  where  $t_{(v, v')}$  sends  $G$  on  $G \setminus (v, v') \cup (v', v)$ . By induction, we know that  $S(G \setminus (v, v')) = (-1)^{|I|}$  and since  $C(G \setminus (v, v')) \cap \{P \models I \mid P(v) = P(v')\} = C(G \cap (I/v')^2 \cup \bigcup_{(w, v') \in G \setminus (v, v')} (w, v) \cup \bigcup_{(v', w) \in G} (v, w))$ , we also have by induction that  $\sum_{P \in C(G \setminus (v, v')) \cap \{P \models I \mid P(v) = P(v')\}} (-1)^{l(P)} = (-1)^{|I|-1}$ . Hence, we have the equivalence  $S(G) = (-1)^{|I|} \iff S(t_{(v, v')}(G)) = (-1)^{|I|}$ .

Let  $e_1, \dots, e_k$  be a sequence of edges such that for every  $i$ ,  $G_i = t_{e_i} \circ \dots \circ t_{e_1}(G)$  does not have a directed cycle. Then we have that  $S(G) = (-1)^{|I|}$  if and only if  $S(G_k) = (-1)^{|I|}$ . If  $G$  has a cycle then we can find a sequence such that  $G_k$  has a superfluous edge and hence  $S(G_k) = (-1)^{|I|}$ . If  $G$  does not have any cycle then every sequence of edges satisfies the condition " $G_i$  does not have a directed cycle" and so  $S(G) = (-1)^{|I|}$  as long as there exists a directed graph  $G'$  with the same underlying non-oriented graph than  $G$  such that  $S(G') = (-1)^{|I|}$ . Given a non-oriented connected graph  $H$  we can always find a directed graph  $G$  on it with only one vertex  $v$  such that for every  $w \in V(G)$ ,  $(w, v) \notin G$ . Then we have that  $C(G) = (\{v\}) \cdot C(G \cap (V(G) - v)^2)$  which gives us  $S(G) = -S(G \cap (V(G) - v)^2) = (-1)^{|V(G)|}$  by induction. This concludes the proof.  $\square$

We can now state the main result of this section:

**Theorem 19** (Reciprocity theorem on hypergraphs). Let  $I$  be a set and  $H \in HG[I]$  a hypergraph over  $I$ . Then  $(-1)^{|I|} \chi_I(H)(-n)$  is the number of compatible pairs of acyclic orientations and colorings with  $[n]$  of  $H$ . In particular,  $(-1)^{|I|} \chi_I(H)(-1) = |\mathcal{A}_H|$  is the number of acyclic orientations of  $H$ .

*Proof.* From Proposition 14 and Lemma 16 we have that

$$\chi_I(H)(-n) = (-n)^{|J_H|} \sum_{f \in \mathcal{A}_H} \sum_{P \in P_{H,f}} (-1)^{\sum_{i=1}^{l(P)} p_i + l(P)} \sum_{(p_1, \dots, p_{l(P)}) \prec q} F_q(n+1).$$

Remark that:

- $\sum_{i=1}^{l(P)} p_i = |I \setminus J_H| - |f(H)|$  (since  $(\tilde{P}_1, \dots, \tilde{P}_{l(P)}, f(H))$  is a partition of  $I \setminus J_H$ ),
- $\phi : \{Q \models f(H) \mid P \prec Q\} \rightarrow \{q \vdash (|I \setminus J_H| - |f(H)|) \mid (p_1, \dots, p_{l(P)}) \prec q\}$   $Q \mapsto (|\tilde{Q}_1|, \dots, |\tilde{Q}_{l(Q)}|)$  is a bijection (where  $\tilde{Q}_i$  is defined the same way than  $\tilde{P}_i$  in Theorem 14).

This gives us:

$$\begin{aligned} (-1)^{|I|} \chi_I(H)(-n) &= n^{|J_H|} \sum_{f \in \mathcal{A}_H} (-1)^{|f(H)|} \sum_{P \in P_{H,f}} (-1)^{l(P)} \sum_{P \prec Q} F_{\phi(Q)}(n+1) \\ &= n^{|J_H|} \sum_{f \in \mathcal{A}_H} (-1)^{|f(H)|} \sum_{Q \models f(H)} \left( \sum_{\substack{P \prec Q \\ P \in P_{H,f}}} (-1)^{l(P)} \right) F_{\phi(Q)}(n+1). \end{aligned}$$

By definition of  $\mathcal{A}_H$ ,  $G = \{(v, f(e)) \mid v \in e \setminus f(e)\}$  is a directed acyclic graph on  $f(H)$ . Hence, remarking that  $\{P \prec Q \mid P \in P_{H,f}\} = C(G, Q)$ , Lemma 18 leads to:

$$\begin{aligned} (-1)^{|I|} \chi_I(H)(-n) &= n^{|J_H|} \sum_{f \in \mathcal{A}_H} (-1)^{|f(H)|} \sum_{\substack{P \models f(H) \\ P(v) \leq P(v') \forall (v, v') \in G}} (-1)^{|f(H)|} F_{\phi(P)}(n+1) \\ &= n^{|J_H|} \sum_{f \in \mathcal{A}_H} \sum_{P \in P'_{H,f}} F_{\phi(P)}(n+1) \\ &= n^{|J_H|} \sum_{f \in \mathcal{A}_H} \sum_{P \in P'_{H,f}} F_{p_1, \dots, p_{l(P)}}(n+1), \end{aligned}$$

where  $P'_{H,f} = \{P \models f(H) \mid P(v) \in e \setminus f(e) \leq P(f(e))\}$ . To conclude we now need to show that the set of compatible pairs (acyclic orientation, coloring with  $n$ ) is in bijection with

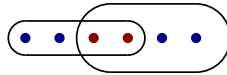
$$\bigsqcup_{f \in \mathcal{A}_H} \bigsqcup_{P \in P'_{H,f}} \bigsqcup_{0 \leq k_1 < \dots < k_{l(P)} \leq n} [k_1]^{Q_1} \times \dots \times [k_{l(P)}]^{Q_{l(P)}}.$$

This can be done in a way analogous to the one used in the proof of Theorem 14, the only difference being that we choose (with the same terms than in the proof)  $g(P_i) = k_i$  instead of  $g(P_i) = k_i + 1$ .  $\square$

**Example 20.** For any  $I$  and any  $H \in HG[I]$ , we have  $\chi_I(H)(n) \leq (-1)^{|I|} \chi_I(H)(-n)$ . This comes from the fact that any strictly compatible pair is compatible.

The coloring given in Example 12 has two compatible acyclic orientations: both send  $e_1$  on  $a$ ,  $e_2$  on  $c$  and  $e_4$  on  $b$  but one sends  $e_3$  on  $c$  and the other  $e_3$  on  $d$ .

For the color set  $\{1, 2\}$ , the following coloring has 4 compatible orientations but only two are acyclic.



## 4 Application to other Hopf monoids

In this section we use Theorem 14 and Theorem 19 to obtain a combinatorial interpretation of the basic invariants for the Hopf monoids presented in Sections 20 to 25 of [1].

The general method to do this will be to use the fact that these Hopf monoids can be seen as sub-monoids of the Hopf monoid of simple hypergraphs, and then present an interpretation of what is an acyclic orientation on these particular Hopf monoids.

The result from subsection 4.1 is new, while the results of subsections 4.2 to 4.7 appear, at least implicitly, in previous works (details are provided at the beginning of each subsections).

In all the following, we denote by  $\chi$  the basic invariant of the Hopf monoid of hypergraphs.

## 4.1 Simple hypergraphs

A hypergraph is *simple* if it has no repeated edges. The vector species  $SHG$  of simple hypergraphs is not stable by the contraction defined on hypergraphs but it still admits a Hopf monoid structure. The product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : SHG[S] \otimes SHG[T] &\rightarrow SHG[I] & \Delta_{S,T} : SHG[I] &\rightarrow SHG[S] \otimes SHG[T] \\ H_1 \otimes H_2 &\mapsto H_1 \sqcup H_2 & H &\mapsto H_{|S} \otimes H_{|T}, \end{aligned}$$

where  $H_{|S} = \{e \in H \mid e \subseteq S\}$  and  $H_{|T} = \{e \cap T \mid e \not\subseteq S\} \cup \{\emptyset\}$  but this time without repetition, i.e  $H_{|S}$  can also be defined as  $\{B \subseteq I \mid \exists A \subseteq S, A \sqcup B \in H\}$ . A discrete simple hypergraph is then a simple hypergraph with edges of cardinality at most one.

**Proposition 21.**  $\chi^{SHG}$  is the restriction of  $\chi$  to the vector species of simple hypergraphs.

*Proof.* Let  $s : HG \rightarrow SHG$  be the Hopf monoid morphism which removes any repetition of edges and let  $H$  be a simple hypergraph over  $I$ . Considering  $SHG$  as a sub-species of  $HG$  and  $s$  as a morphism of vector species we have:  $\chi_I^{HG}(H) = \chi_I^{HG}(s(H))$ . Then using the fact that  $s$  is a Hopf monoid morphism stable on the sub-species of discrete elements we have that:  $\chi_I^{HG}(s(H)) = \chi_I^{SHG}(H)$ . This concludes the proof.  $\square$

## 4.2 Graphs

The result of this subsection has already been given in Section 18 of [1], but we give it here as a consequence of our result in the previous section.

A *graph* can be seen as a hypergraph whose edges are all of cardinality 2. As for the vector species of simple hypergraphs, the vector species  $G$  of graphs is not stable by the contraction defined on hypergraphs, but it still admits a Hopf monoid structure. The product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : G[S] \otimes G[T] &\rightarrow G[I] & \Delta_{S,T} : G[I] &\rightarrow G[S] \otimes G[T] \\ g_1 \otimes g_2 &\mapsto g_1 \sqcup g_2 & g &\mapsto g_{|S} \otimes g_{|T}, \end{aligned}$$

where  $g_{|S}$  is the sub-graph of  $g$  induced by  $S$  and  $g_{|T} = g_{|I}$ . A discrete graph is then a graph with no edges.

A *proper coloring* of a graph is a coloring such that no edge has its two vertices of the same color. The chromatic polynomial of a graph is the polynomial  $T$  such that  $T(n)$  is the number of proper colorings with  $n$  colors.

**Corollary 22.** The basic invariant of  $G$  is the chromatic polynomial.

*Proof.* Let  $s : HG \rightarrow G$  be the Hopf monoid morphism which removes edges of cardinality 1. Using the same reasoning than in the proof of Proposition 21, we get that  $\chi^G$  is the restriction of  $\chi$  to  $G$ . Furthermore, for  $g$  a graph and a  $S$  coloring of  $g$ , we have the equivalence between "each edge has a unique maximal vertex" and " $S$  is a proper coloring". The result follows.  $\square$

In particular, by evaluating  $\chi$  on negative integers for a graph, we recover the classical reciprocity theorem of Stanley [15].

## 4.3 Simplicial complexes

In [6] Benedetti, Hallam, and Machacek constructed a combinatorial Hopf algebra of simplicial complexes and in particular they obtained results which generalise those given here.

An *abstract simplicial complex*, or simplicial complex, on  $I$  is a collection  $C$  of subsets of  $I$ , called *faces*, such that any subset of a face is a face i.e  $J \in C$  and  $K \subset J$  implies  $K \in C$ . By Proposition 21.1 of [1], the vector species  $SC$  of simplicial complexes is a sub-monoid of the Hopf monoid of simple hypergraphs.

The *1-skeleton* of a simplicial complex is the graph formed by its faces of cardinality 2.

**Corollary 23.** Let  $I$  be a set,  $C \in SC[I]$  and  $g$  its 1-skeleton. Then  $\chi_I^{SC}(C)$  is the chromatic polynomial of  $g$ .

*Proof.* It is clear that any coloring of  $C$  such that each edge has a unique maximal vertex induces a proper coloring of  $g$ . On the other hand if  $J$  is a face of  $C$  then it also is a clique of  $g$  and so any proper coloring of  $g$  must color all the vertices in  $J$  in different colors, in particular there must be a unique maximal vertex in  $J$ .  $\square$

#### 4.4 Building sets

Building sets and graphical building sets have been studied in a Hopf algebraic context by Grujić in [11] where he gave similar results to the ones obtained here and the next subsection.

Building sets were independently introduced by De Concini and Procesi in [8] and by Schmitt in [14]. A *building set* on  $I$  is collection  $B$  of subsets of  $I$ , called *connected sets*, such that if  $J, K \in B$  and  $J \cap K \neq \emptyset$  then  $J \cup K \in B$  and for all  $i \in I$   $\{i\} \in B$ . By Proposition 22.3 of [1] the vector species  $BS$  of building sets is a sub-monoid of the Hopf monoid of simple hypergraphs.

The maximal sets of a building set are called *connected components*.

**Lemma 24.** Let  $B$  be a building set on  $I$ . Consider a graph  $g$  given by the following inductive definition:

- Initial state: a discrete graph where the vertices are couples of a connected component and an element of this connected component.
- Induction: choose a vertex with no outgoing edges  $(J, v)$  such that  $J$  is not a singleton. For each maximal connected set  $K$  in  $J$  not containing  $v$ , choose a vertex  $v_K$  in it and add  $(K, v_K)$  as a vertex and an edge going from  $(K, v_K)$  to  $(J, v)$ .

Then the oriented graph  $\pi_2(g)$  is a rooted forest on  $I$ , where  $\pi_2(g)$  is the graph obtained by applying the projection on the second coordinate on the vertices of  $g$  and combining the possible multiple occurrences of an element.

*Proof.* First remark that for any vertex  $(J, v)$  with children  $(K_1, v_1), \dots, (K_l, v_l)$ ,  $(\{v\}, K_1, \dots, K_l)$  is a partition of  $J$ . Indeed, one has:

- by definition  $\{v\} \cap K_i = \emptyset$  for all  $i$ ,
- if  $K_i \cap K_j \neq \emptyset$  then  $K_i \cup K_j$  would be in  $B$  and in contradiction with the maximality of  $K_i$  and  $K_j$ ,
- all the elements of  $J$  appear in  $(\{v\}, K_1, \dots, K_l)$  since all the singletons are in  $B$ .

In particular, if  $k$  is a descendant of  $j$  then  $\pi_1(k) \subsetneq \pi_1(j)$ .

It is clear that the initial state is a rooted forest where any second coordinate appears only once. Suppose now we are in the middle of the construction and the graph  $g$  is a rooted forest with no repeated second coordinate. Let  $(J, v)$  be a leaf such that  $J$  is not a singleton and  $(K_1, v_1), \dots, (K_l, v_l)$  its children obtained by the induction step. Then all the  $v_i$ s are distinct since  $(\{v\}, K_1, \dots, K_l)$  is a partition and for all  $i$ ,  $v_i \in K_i$ . Suppose there exists an  $i$  such that  $v_i$  did already appear in a vertex of  $g$  before this induction step and let  $k$  and  $k'$  be two vertices of  $g$  with second component  $v_i$ . If  $k$  and  $k'$  do not have a common ancestor they would be in different connected components of  $g$  and so  $v_i$  would be in two different connected components of  $B$ , which is not possible. If  $k$  and  $k'$  have a common ancestor, note  $k''$  their minimal common ancestor. Then, there exist two different children  $m$  and  $m'$  of  $k''$  such that  $\pi_1(k) \subseteq \pi_1(m)$  and  $\pi_1(k') \subseteq \pi_1(m')$  but then  $\pi_1(m)$  and  $\pi_1(m')$  would have a non empty intersection. This contradicts the fact that they must be part of a partition. Hence the  $v_i$ s did not appear before this induction step and  $g$  is still a rooted forest with no repeated second component.

We now show that all the elements of  $I$  appear as a vertex of a skeleton. To do this, we show by induction on the height that all the elements of  $J$  appear in a sub-tree of root  $(J, v)$ . It is

true for the leaf since they are of the form  $(\{v\}, v)$  (else we could still apply the induction step). The induction is straightforward using our remark about partitioning at the beginning of this proof.  $\square$

This lemma proven, we can give the following definition:

**Definition 25.** Let  $B$  be a building set on  $I$ . A *skeleton* of  $B$  is a rooted forest on  $I$  defined by:

- there is exactly one root in each connected component of  $B$ ,
- given a node  $v$ , let  $J$  be the maximal connected set containing  $v$  but not its parent. Then for each maximal connected set  $K$  in  $J$  not containing  $v$ ,  $v$  has exactly one child in  $K$ .

**Remark 26.** Since we know that skeletons are rooted forests on  $|I|$ , it easy to see that they are exactly the *B-forest* where all the vertices are singletons as defined in Definition 22.6 of [1].

Given a skeleton  $\pi_2(g)$  there is a unique way to recover  $g$ : for each  $v$  vertex of  $\pi_2(g)$  define  $J(v)$  the maximal connected set containing  $v$  but not its parent. Then  $g$  is obtained by applying  $v \mapsto (J(v), v)$  on the vertices of  $\pi_2(g)$ .

A rooted forest can be seen as a forest with an orientation which sends each edge on the parent vertex. Hence, one can define compatible and strictly compatible colorings of a rooted forest. Moreover, these notions correspond to the notions of *natural-T-partition* and *strict-T-partition* of [11].

**Corollary 27.** Let  $I$  be a set and  $B \in BS[I]$ . Then  $\chi_I^{BS}(B)(n)$  is the number of strictly compatible pairs of skeletons and colorings with  $[n]$  and  $\chi_I^{BS}(B)(-n)$  is the number of compatible pairs of skeletons and colorings with  $[n]$ . In particular,  $\chi_I^{BS}(B)(-1)$  is the number of skeletons of  $B$ .

*Proof.* Since  $BS$  is a sub-monoid of  $SHG$  we know that  $\chi^{BS}$  is the restriction of  $\chi$  to  $BS$ . Hence, we only need to show that there exists a bijection which preserves compatibility between the acyclic orientations of  $B$  seen as a hypergraph and its skeletons.

If  $f$  is an acyclic orientation of  $B$ , define the rooted forest  $g$  as follows:

- the roots of  $g$  are the  $f(J)$  for  $J$  a connected components of  $B$ ,
- given a node  $v$ , its children are the  $f(K)$  for  $K$  maximal connected set in  $J(v)$  not containing  $v$ .

It is clear by definition that  $g$  is a skeleton. Let  $g$  be a skeleton of  $B$ , given a connected set  $J$ , let us search for the minimal  $v$  such that  $J \subseteq J(v)$  and define the orientation  $f$  by  $f(J) = v$ . It is clear that  $f$  is acyclic because it would else induce a cycle in  $g$  which is not possible. It is also clear that the two previous constructions are inverse functions.

We now need to show that these transformations preserve the compatible colorings. Let  $f$  be an acyclic orientation and  $g$  its associated skeleton. Then, by construction, for any connected set  $J$ ,  $J = J(f(J))$  and all the elements of  $J$  are in the sub-tree of  $g$  with root  $f(J)$ . Hence  $S$  is compatible (resp. strictly compatible) with  $f$  if and only if it is compatible (resp. strictly compatible) with  $g$ .  $\square$

## 4.5 Simple graphs, ripping and sewing

A *simple* graph is a graph that is both a simple hypergraph and a graph. The vector species  $W$  of simple graphs admits a Hopf monoid structure, the product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : W[S] \otimes W[T] &\rightarrow W[I] & \Delta_{S,T} : W[I] &\rightarrow W[S] \otimes W[T] \\ w_1 \otimes w_2 &\mapsto w_1 \sqcup w_2 & w &\mapsto w|_S \otimes w|_T \end{aligned}$$

where  $w_{|_S}$  is the sub-graph of  $w$  induced by  $S$  and  $w_{/S}$  is the simple graph on  $T$  with an edge between  $u$  and  $v$  if there is a path from  $u$  to  $v$  in which all the vertices which are not ends are in  $S$ . These two operations are respectively called *ripping out  $T$*  and *sewing through  $S$* . A discrete simple graph is then a simple graph with no edges.

**Definition 28** (Definition 23.1 in [1]). Let be  $w \in W[I]$ . A *tube* is a subset  $J \subset I$  such that  $w_{|_J}$  is connected. The set of tubes of  $w$  is a building set called *graphical building set of  $w$*  and which we note  $\text{tubes}(w)$ .

By Proposition 23.3 of [1] we know that  $w \mapsto \text{tubes}(w)$  is a Hopf monoid morphism between  $W$  and  $BS$ .

Given a rooted tree we call its *direct sub-tree* the sub-trees with roots the children of the root.

**Definition 29.** Let be  $w \in W[I]$  a connected simple graph. We define the set of *partitioning trees of  $w$*  inductively by the following:

- if  $I = \{v\}$ , then the unique partitioning of  $w$  is the graph with  $\{v\}$  as only vertex,
- else choose  $v \in I$  and a partitioning tree for each connected component of  $w_{|_{I \setminus \{v\}}}$ . The tree with root  $v$  and direct sub-trees these partitioning trees is then a partitioning tree of  $w$ .

If  $w$  is not connected anymore, a *partitioning forest of  $w$*  is the disjoint union of partitioning trees of each connected component of  $w$ .

**Corollary 30.** Let  $I$  be a set and  $w \in W[I]$ . Then  $\chi_I^W(w)(n)$  is the number of colorings of  $w$  with  $[n]$  such that every path with ends of the same color has a vertex of color strictly greater to the color of the ends. It is also the number of strictly compatible pairs of partitioning forests and colorings with  $[n]$ .  $\chi_I^W(w)(-n)$  is the number of compatible pairs of partitioning forests and colorings with  $[n]$ . In particular,  $\chi_I^W(w)(-1)$  is the number of partitioning trees of  $w$ .

*Proof.* Since  $\chi_I^W(w) = \chi_I^{BS}(\text{tubes}(w)) = \chi_I(\text{tubes}(w))$ , to prove the corollary we only need to prove these two points:

- A coloring  $I \rightarrow [n]$  is a coloring of  $\text{tubes}(w)$  such that all edges have a unique maximal vertex if and only if it is a coloring of  $w$  such that every path with ends of the same color has a vertex of color strictly greater to the color of the ends.
- The partitioning forests of  $w$  are exactly the skeletons of  $\text{tubes}(w)$ .

We begin by the first assertion. Let  $S$  be a coloring of  $\text{tubes}(w)$  of interest and  $v_1, \dots, v_k$  a path of  $w$  such that  $S(v_1) = S(v_k)$ . Then  $w_{|_{\{v_1, \dots, v_k\}}}$  is connected and so  $\{v_1, \dots, v_k\}$  is an edge of  $\text{tubes}(w)$ . Since  $v_1$  and  $v_k$  are of the same color, their color can not be the maximal color. Hence there exists an  $i$  such that  $S(v_i) > S(v_1) = S(v_k)$  and  $S$  is a coloring of  $w$  of interest. Let now  $S$  be a coloring of  $w$  of interest and  $e$  an edge of  $\text{tubes}(w)$  with two vertices  $v_1$  and  $v_2$  of the same color. Then, since  $w_{|_e}$  is connected by definition, there exists a path in  $e$  from  $v_1$  to  $v_2$  and hence  $v_3$  such that  $S(v_3) > S(v_1) = S(v_2)$ . Thus there can only be one vertex of maximal color in  $e$  and  $S$  is a coloring of  $\text{tubes}(w)$  of interest.

To show that partitioning forests and skeletons are the same objects, just remark that given a vertex  $v \in J$ , where  $w_{|_J}$  is a connected component of  $w$ , the connected components of  $w_{|_{J \setminus \{v\}}}$  are exactly the maximal connected sets of  $\text{tubes}(w)$  included in  $J$  but not containing  $v$ .  $\square$

## 4.6 Set partitions

Proposition 24.4 of [1] states that there exists an isomorphism between the Hopf monoid of permutahedra and the Hopf monoid of set partitions. Furthermore, Propositions 17.3 and 17.4 of [1] give a combinatorial interpretation of the basic invariant of the Hopf monoid of generalized permutahedra  $GP$ . The Hopf monoid of permutahedra being a sub-monoid of a quotient of  $GP$ , it should be possible to deduce the result presented here from the aforementioned propositions.

A *partition* of  $I$  is a subset of  $\mathcal{P}(I) \setminus \{\emptyset\}$  such that all elements, which are called *parts*, are disjoint and their union equals  $I$ . The vector species  $\Pi$  of partitions admits a Hopf monoid structure, the product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : \Pi[S] \otimes \Pi[T] &\rightarrow \Pi[I] & \Delta_{S,T} : \Pi[I] &\rightarrow \Pi[S] \otimes \Pi[T] \\ \pi_1 \otimes \pi_2 &\mapsto \pi_1 \sqcup \pi_2 & \pi &\mapsto \pi|_S \otimes \pi|_T, \end{aligned}$$

where for  $\pi = \{\pi^1, \dots, \pi^l\}$ ,  $\pi|_S$  is the partition of  $S$  obtained by taking the intersection with  $S$  of each part  $\pi^i$  and forgetting the empty parts. A discrete partition is then a partition where all parts are singletons.

A *cliquey graph* is a disjoint union of cliques. By Proposition 24.2 of [1] we know that  $\pi \mapsto c(\pi)$  is a Hopf monoid from  $\Pi$  to  $W$ , where  $c(\pi)$  is the cliquey graph with a clique on each part of  $\pi$ .

**Corollary 31.** Let  $I$  be a set and  $\pi = \{\pi^1, \dots, \pi^l\} \in \Pi[I]$ . Then  $\chi_I^\Pi(\pi)(n) = \prod_{i=1}^l p_i! \binom{n}{p_i}$  where  $p_i = |\pi^i|$ .

*Proof.* Since  $\chi^\Pi$  is multiplicative and  $\pi \mapsto c(\pi)$  is a Hopf monoid morphism, we only need to show that  $\chi_I^W(w)(n) = |I|! \binom{n}{|I|}$  where  $w$  is the clique on  $I$ . A coloring  $S$  of  $w$  is such that every path with ends of the same color has a vertex of color strictly superior to the color of the ends if and only if all vertices are of different colors (because for each pair  $v_1, v_2$  of vertices  $v_1, v_2$  is a path in  $w$ ). Hence the number of such colorings is the number of injections from  $I$  to  $[n]$ . This concludes the proof.  $\square$

## 4.7 Paths

As for the previous subsection, Proposition 25.7 of [1] states that the Hopf monoid of set of paths is isomorphic to the Hopf monoid of associahedra which is a sub-monoid of a quotient of  $GP$ . Hence, it should also be possible to deduce the result of this subsection from [1].

A *word* on  $I$  is a total ordering of  $I$ . The *paths* on  $I$  are the words on  $I$  quotiented by the relation  $w_1 \dots w_{|I|} \sim w_{|I|} \dots w_1$ . A *set of paths*  $\alpha$  of  $I$  is a partition  $(I_1, \dots, I_l)$  of  $I$  with a path  $s_i$  on each part  $I_i$  and we will note  $\alpha = s_1 | \dots | s_l$ . The vector species  $F$  of sets of paths admits a Hopf monoid structure, the product and co-product are given by, for  $I = S \sqcup T$ :

$$\begin{aligned} \mu_{S,T} : F[S] \otimes F[T] &\rightarrow F[I] & \Delta_{S,T} : F[I] &\rightarrow F[S] \otimes F[T] \\ \alpha_1 \otimes \alpha_2 &\mapsto \alpha_1 \sqcup \alpha_2 & \alpha &\mapsto \alpha|_S \otimes \alpha|_T \end{aligned}$$

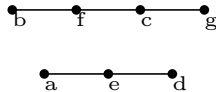
where if  $\alpha = s_1 | \dots | s_l$ ,  $\alpha|_S = s_1 \cap S | \dots | s_l \cap S$  forgetting the empty parts and  $\alpha|_S$  is the set of paths obtained by replacing each occurrence of an element of  $S$  in  $\alpha$  by the separation symbol  $|$ . A discrete set of paths is then a set of paths where all paths have only one element.

**Example 32.** For  $I = \{a, b, c, d, e, f, g\}$  and  $S = \{b, c, e\}$  and  $T = \{a, d, f, g\}$  we have:

$$\Delta_{S,T}(bfcg|aed) = bc|e \otimes f|g|a|d$$

By Proposition 25.1 of [1] we know that  $\alpha \mapsto l(\alpha)$  is a morphism of Hopf monoids from  $F$  to  $W^{cop}$ ; where  $l(s_1 | \dots | s_l)$  is the simple graph whose connected components are the paths induced by  $s_1, \dots, s_l$ .

**Example 33.** For  $I = \{a, b, c, d, e, f, g\}$  and  $\alpha = bfcg|aed$ ,  $l(\alpha)$  is the following graph:



**Corollary 34.** Let  $I$  be a set and  $\alpha$  be a path on  $I$ . Then  $\chi_I^F(\alpha)(n)$  is the number of strictly compatible pairs of binary trees with  $|I|$  vertices and colorings with  $[n]$  and  $\chi_I^F(\alpha)(-n)$  is the number of compatible pairs of binary trees with  $|I|$  vertices and colorings with  $[n]$ . In particular  $\chi_I^F(\alpha)(-1) = C_{|I|}$  where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n$ -th Catalan number.



*Proof.* First remark that by definition of the polynomial invariant of Theorem 5  $\chi^{W^{cop}} = \chi^W$  and so  $\chi_I^F(\alpha)(n) = \chi_I^W(l(\alpha))(n)$ . Fix one of the two total orderings of  $I$  induced by  $\alpha$  so that we can consider the left and the right of a vertex  $v$  of  $l(\alpha)$ . Then each vertex of  $l(\alpha)$  is totally characterised by the number of vertices on its left (and on its right) and hence the partitioning trees of  $l(\alpha)$  are exactly the binary trees with  $|I|$  vertices.  $\square$

### Concluding remark

Let us end this paper by presenting some perspectives for future work. We plan to generalize the results of this paper to all characters on the Hopf monoid of hypergraphs. While Theorem 14 does generalize easily for characters with value in  $\{0, 1\}$ , the conditions on the characters are slightly more involved for Theorem 19 to hold.

Finally, an open question that appears interesting to us is to recover Theorem 19 using the antipode formula given in [1]. We refer the reader to [5] where this has been done for a different Hopf structure on hypergraphs.

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