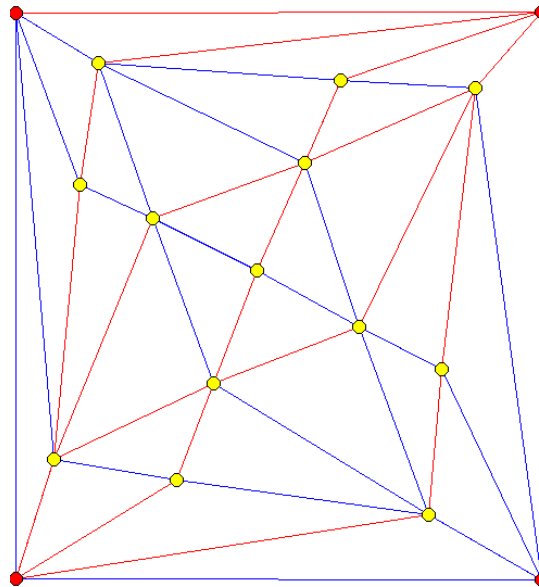


Which combinatorial triangulations can be realized as a L_∞ -Delaunay triangulation?



Nicolas Bonichon (Bordeaux U. – Inria CEPAGE-LaBRI)

Jit Bose (Carleton U.)

Sander Verdonschot (Carleton U.)

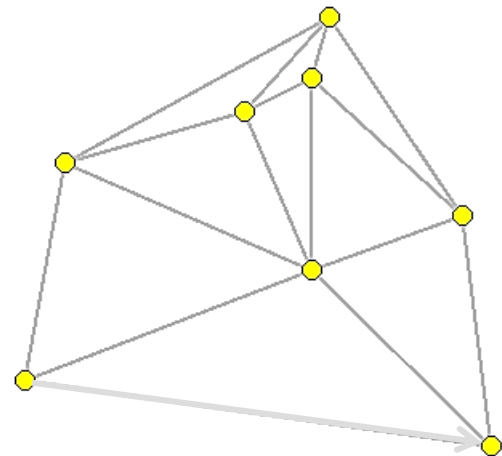
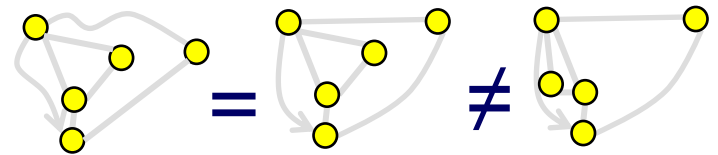
Work in progress

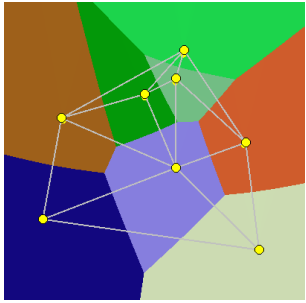


Acknowledgment: LIRCO; ANR JCJC EGOS

Combinatorial Triangulation

- **Planar map**: planar graph embedded on the sphere with no edge crossings, up to continuous deformation.
- **Rooted**: one edge is oriented. The face on its right is the **outerface**.
- **Triangulation**: each **inner** face is a triangle.





Delaunay Triangulation

[Boris Delaunay '34]

Let $S = \{s_1, s_2, \dots\}$ be a set of points in the plane.

Voronoi cell:

$$C(s_i) = \{x / d(s_i, x) \leq d(s_j, x) \forall i \neq j\}$$

Delaunay Triangulation:

s_i is a neighbour s_j i f f $C(s_i) \cap C(s_j) \neq \emptyset$

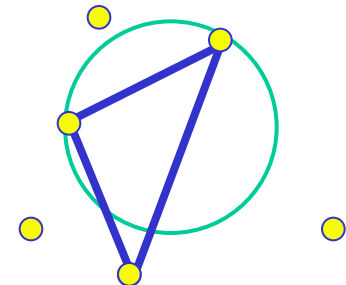
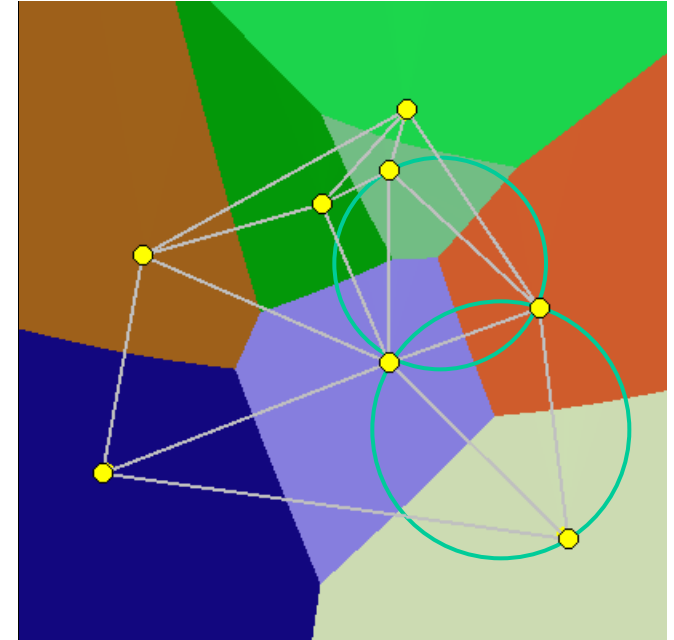
General Position:

No 3 points collinear

No 4 points co-circular.

Alternative def: There is an edge (s_i, s_j) iif there is an empty circle supporting s_i and s_j .

\Rightarrow : each face is supported by an empty circle.

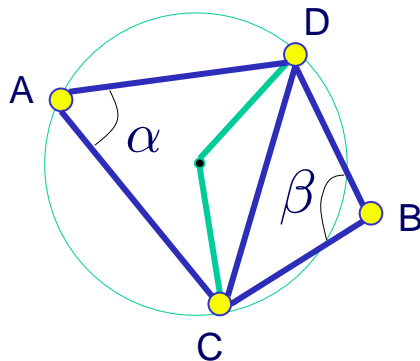


Delaunay realizability

A combinatorial triangulation T is **Delaunay realizable** if it exists S such that $T = \text{Delaunay}(S)$.

General Lemma: In a Delaunay triangulation, if 2 faces ACD and BCD share an edge then:

$$\alpha + \beta < \pi$$



$$\alpha_i + \beta_{i-1} < \pi$$

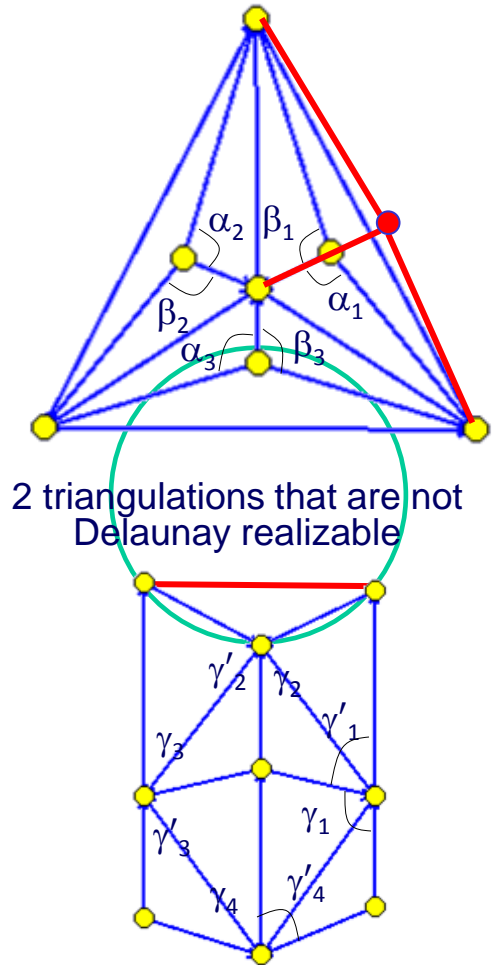
$$\sum \alpha_i + \beta_i < 3\pi$$

$$\exists i / \alpha_i + \beta_i < \pi$$

$$\gamma_i + \gamma'_{i-1} > \pi$$

$$\sum \gamma_i + \gamma'_i > 4\pi$$

$$\exists i / \gamma_i + \gamma'_i > \pi$$



Why Delaunay realizability is an interesting question?

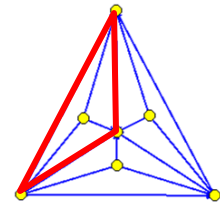
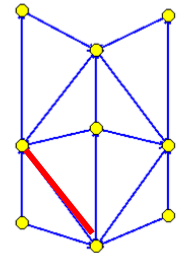
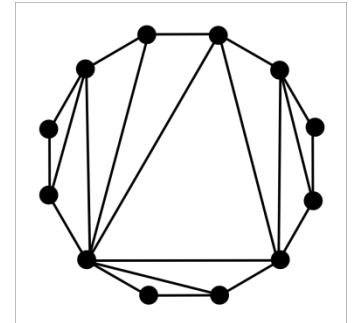
- What is the probability of a random triangulation to be Delaunay realizable?
- Is it enough to test a program on random Delaunay triangulations?
- Can I make an algorithm dedicated to Delaunay triangulations more efficient than for general triangulations?
- For instance can I encode every n -vertex Delaunay triangulation with less than $2.34n$ bits?

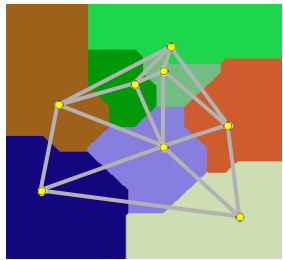
Delaunay realizability (related work)

- Sufficient condition:
 - [Dillencourt'90] outerplanar triangulation
 - [Dillencourt Smith'96] triangulation without **chords** or **nonfacial triangles**

(chord: edge between 2 nonconsecutive vertices of the outface)

- Recognizability:
 - [Hodgson Rivin Smith '92][Hiroshima Miyamoto Sugihara'00] polynomial time.





L_∞ -Delaunay Triangulation

[Chew SoCG'86]

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

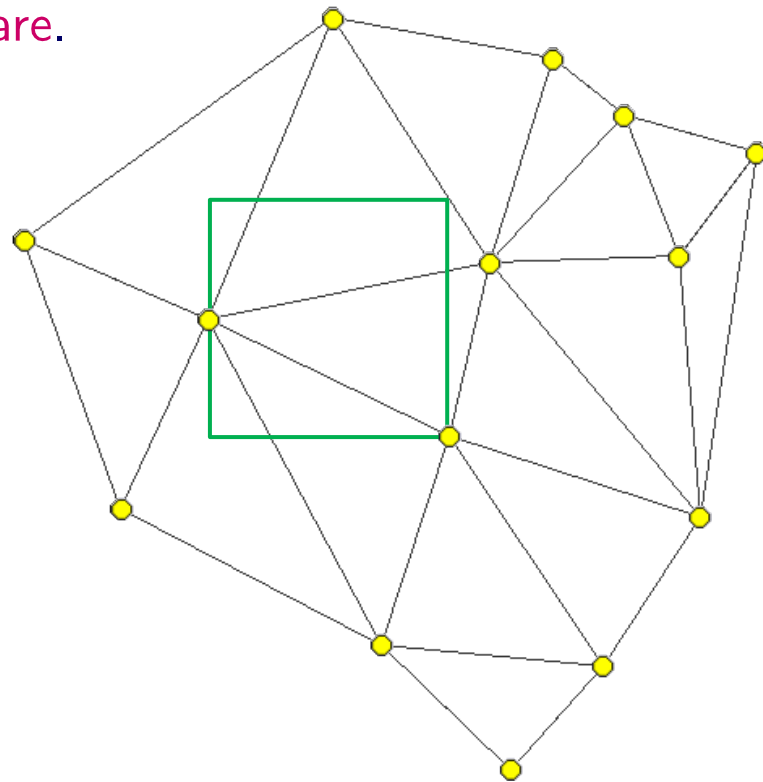
Def: There is an edge (i, j) iif there is an empty **square** supporting i and j .

\Leftrightarrow : each face is supported by an empty **square**.

General Position:

No 4 points co-”circular”.

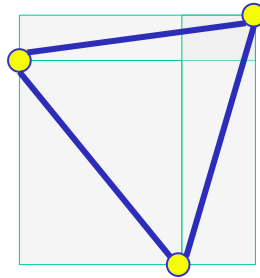
Points have \neq abscissas and ordinates.



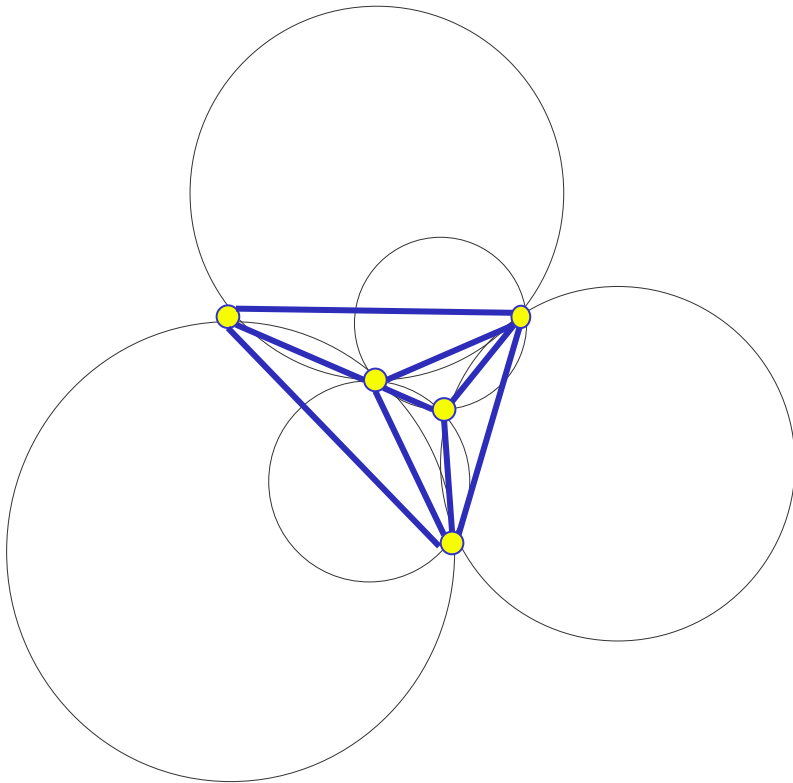
Necessary condition

Prop: If a (combinatorial) triangulation is L_∞ -Delaunay realizable then it doesn't contain nonfacial triangles.

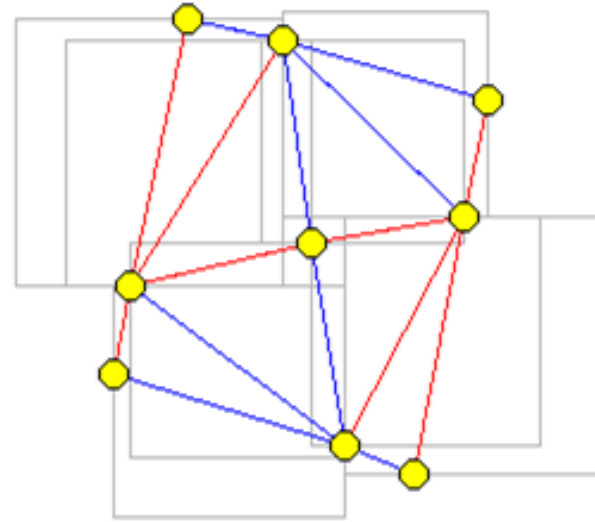
Proof: the rectangle supporting each edge is empty. The union of the rectangles supporting the edges of a triangle covers the interior of that triangle.



L_∞ -Delaunay realizable
vs
 L_2 -Delaunay realizable



L_2 -Delaunay realizable
but not L_∞ -Delaunay realizable



L_∞ -Delaunay realizable
but not L_2 -Delaunay realizable

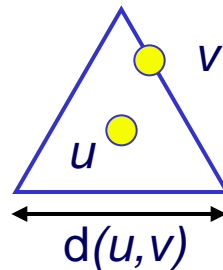
Let's make a digression
through TD-Delaunay
triangulations...

Triangular Distance Delaunay Triangulation

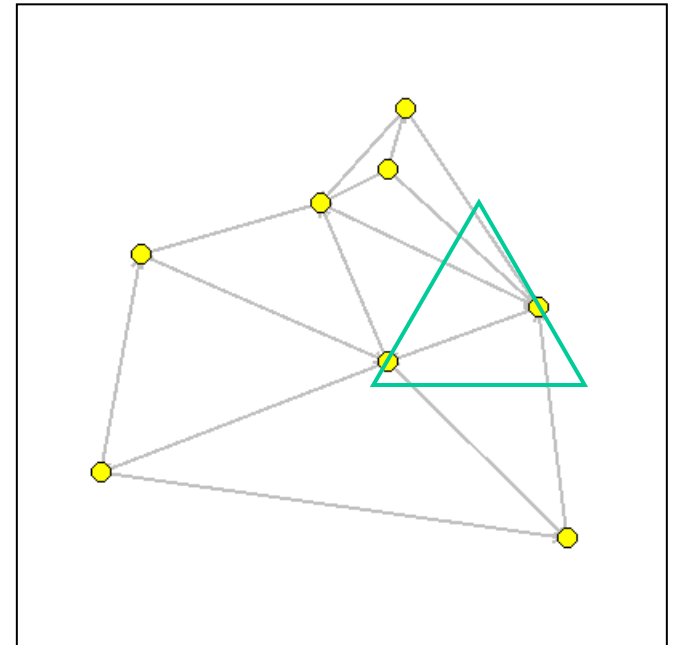
[Chew 89]

- Triangular “Distance”:

$d(u,v)$ = size of the smallest equilateral triangle centred at u touching v .



- Remark: in general $d(u,v) \neq d(v,u)$



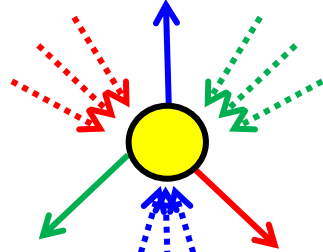
Thm [B. Gavoille Hanusse Ilcinckas'10]:

Every combinatorial triangulation is TD-Delaunay realizable.

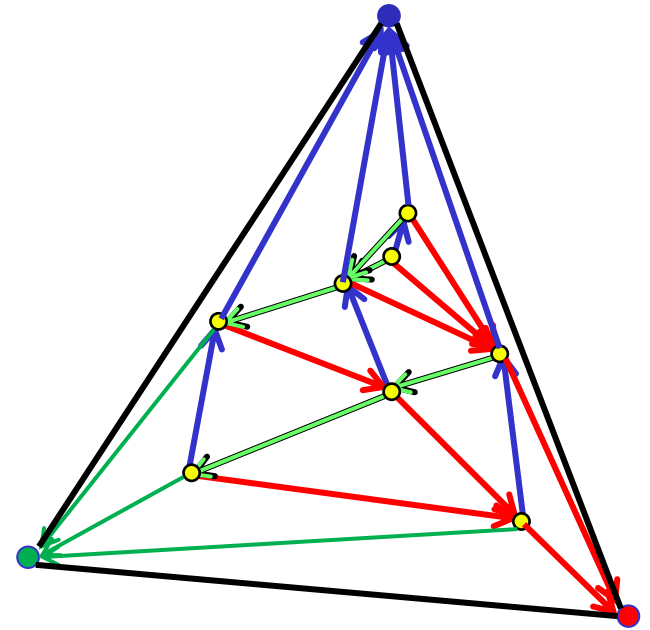
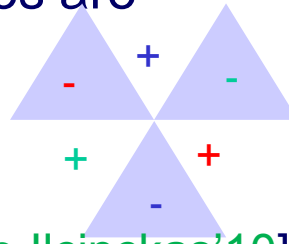
Schnyder woods

[Schnyder'89]

- **Schnyder wood**: coloration and orientation of edges such that around every inner vertex:



- We can define a SW of TD-Delaunay: color and orientation of edges are determined by the slope:



Thm [B. Gavaille Hanusse Ilcinckas'10]:
Every Schnyder wood is TD-Delaunay realizable.

Why Delaunay realizability is an interesting question?

- What is the probability of a random triangulation to be TD-Delaunay realizable? 1
- Is it enough to test a program on random TD-Delaunay triangulations? Yes
- Can I make an algorithm dedicated to TD-Delaunay triangulations more efficient than for general triangulations? No
- For instance can I encode every n -vertex TD-Delaunay triangulation with less than $2.34n$ bits? No

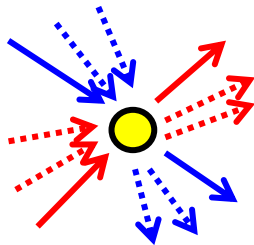
Let's come back to L_∞ - Delaunay triangulations...

Is there an equivalent to Schnyder woods for L_∞ -Delaunay triangulations?

Regular edge labelings/Transversal structures

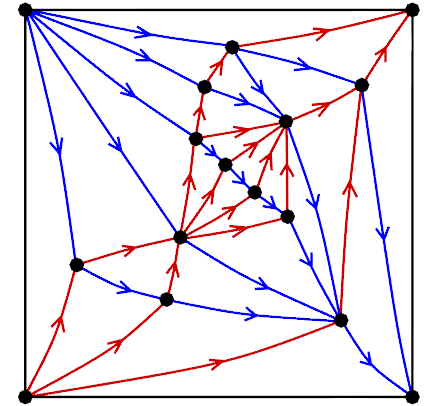
[Kant He '97][Fusy '05]

- Let T be a combinatorial triangulation without nonfacial triangles and with 4 vertices on the outerface.
- **Transversal structure**: coloration and orientation of edges such that:

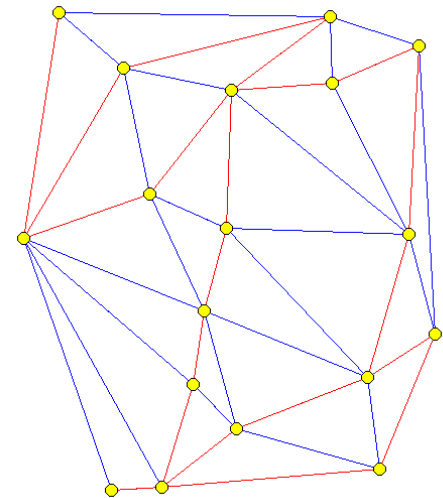


We can define transversal structures of L_∞ -Delaunay triangulation using the slope of edges.

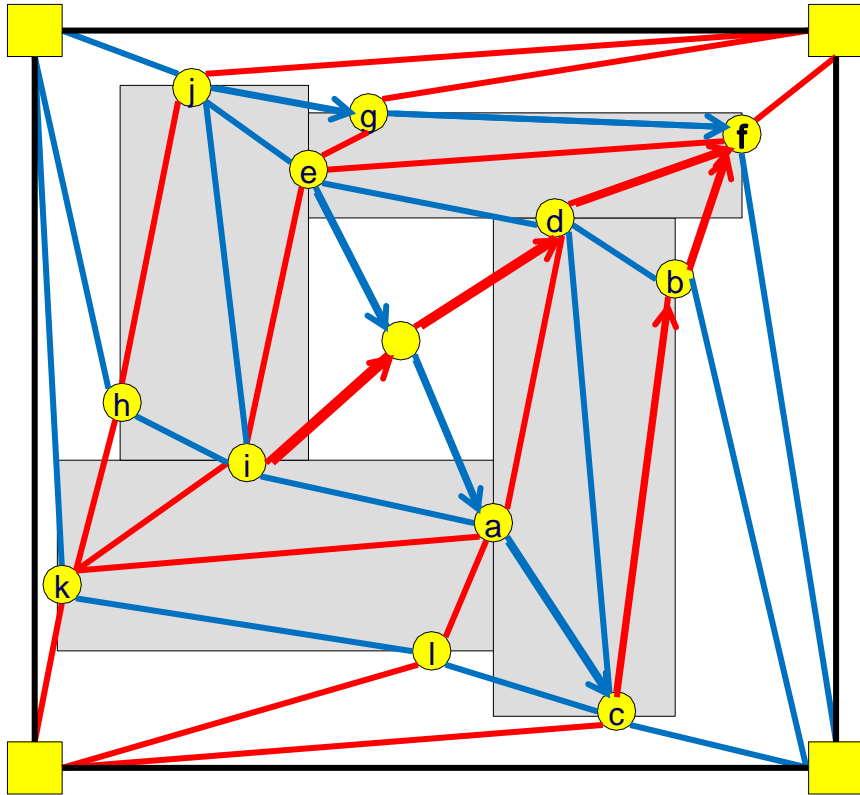
Natural question: Is every transversal structure L_∞ -Delaunay realizable?



Source: E. Fusy



Is every transversal structure L_∞ - Delaunay realizable? **No** ☹️



Edge (d,c) is in an empty square

$$\Rightarrow y_d - y_c < x_b - x_a$$

Because of path e, O, a, c, b, f : $x_e < x_a < x_c < x_b < x_f$

$$\Rightarrow x_b - x_a < x_f - x_e$$

Edge (e,f) is in an empty square

$$\Rightarrow x_f - x_e < y_g - y_d$$

Because of path i, O, d, f, g, j : $y_i < y_d < y_f < y_g < y_j$

$$\Rightarrow y_g - y_d < y_j - y_i$$

Edge (i,j) is in an empty square

$$\Rightarrow y_j - y_i < x_e - x_h$$

Because of path k, h, j, e, O, a : $x_k < x_h < x_j < x_e < x_a$

$$\Rightarrow x_e - x_h < x_a - x_k$$

Edge (k,a) is in an empty square

$$\Rightarrow x_a - x_k < y_i - y_l$$

Because of path c, l, k, i, O, d : $y_c < y_l < y_k < y_i < y_d$

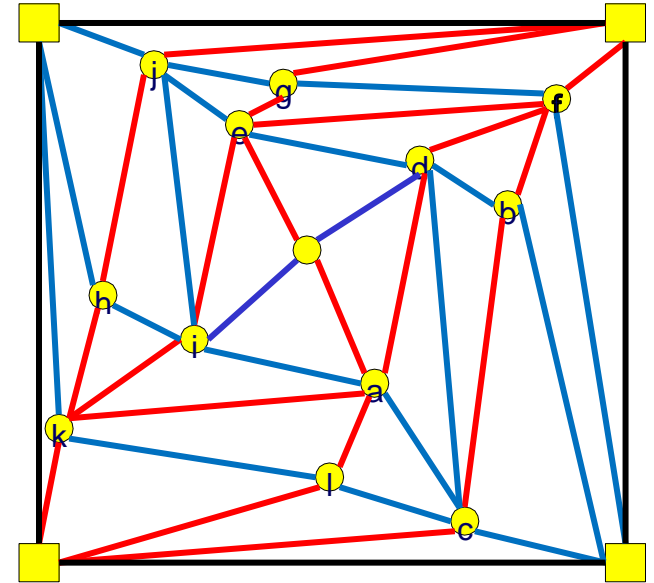
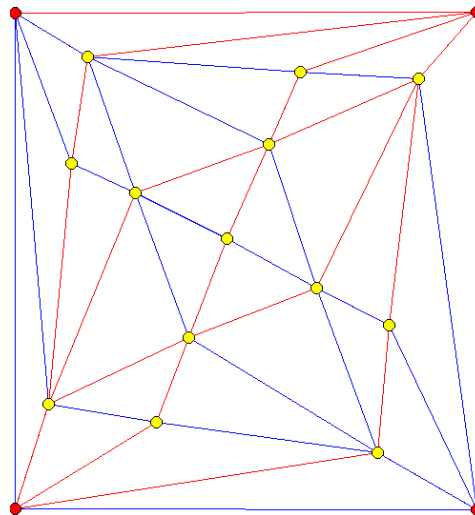
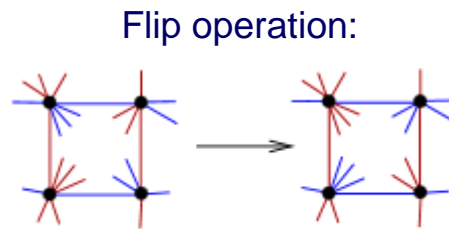
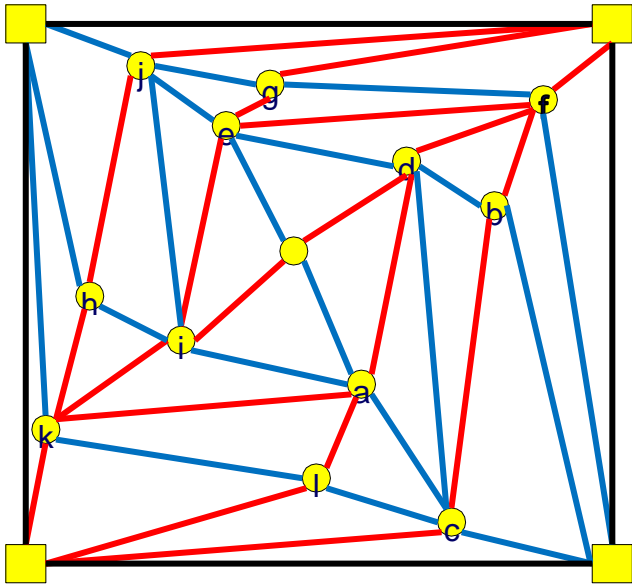
$$\Rightarrow y_i - y_l < y_d - y_c$$

Putting all together:

$$y_d - y_c < x_b - x_a < x_f - x_e < y_g - y_d < y_j - y_i < x_e - x_h < x_a - x_k < y_i - y_l < y_d - y_c$$

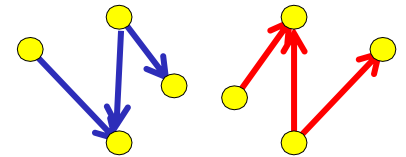
Contradiction: $y_d - y_c < y_d - y_c$

Is every transversal structure L_∞ -
Delaunay realizable? **No** ☹ **But...**



Sufficient condition

Thm: Every transversal structure without blue anti-N or red N is L_∞ -Delaunay realizable.



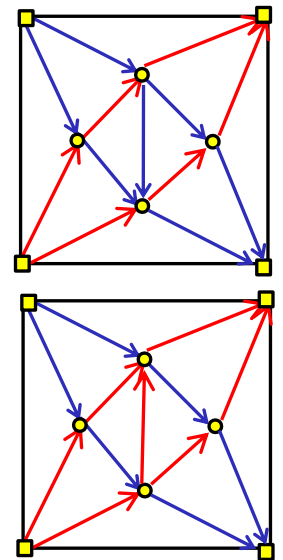
Remarks:

- Not every 4-connected triangulation admits a transversal structure without blue anti-N or red N.
- Every 4-connected triangulation admits a transversal structure without blue N or red N.

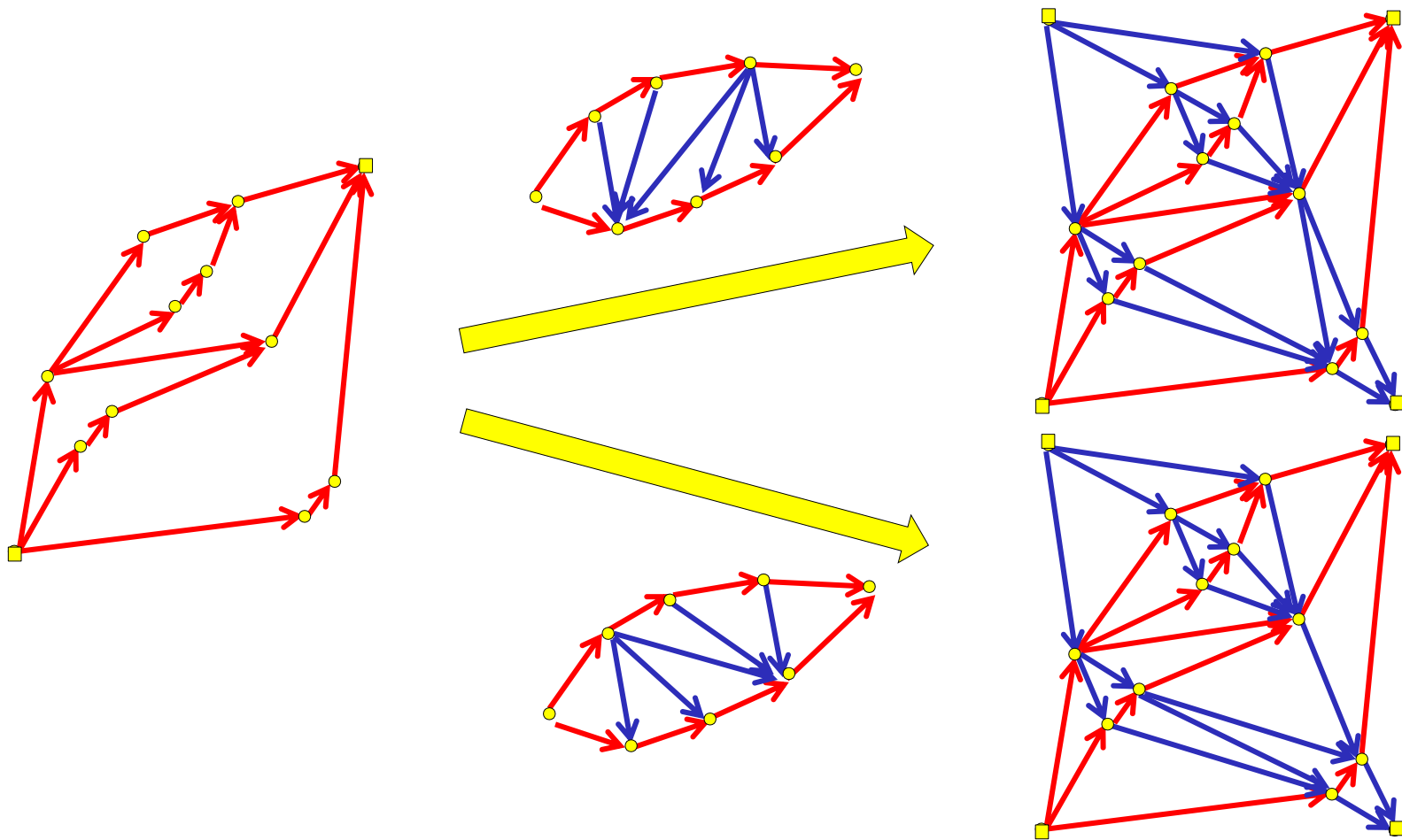
Thm [Fusy 08]: The following objects are in bijection:

- Transversal structures without blue N or red N
- Transversal structures without blue anti-N or red N
- Baxter permutations.

We will use this bijection to obtain L_∞ -Delaunay realization.



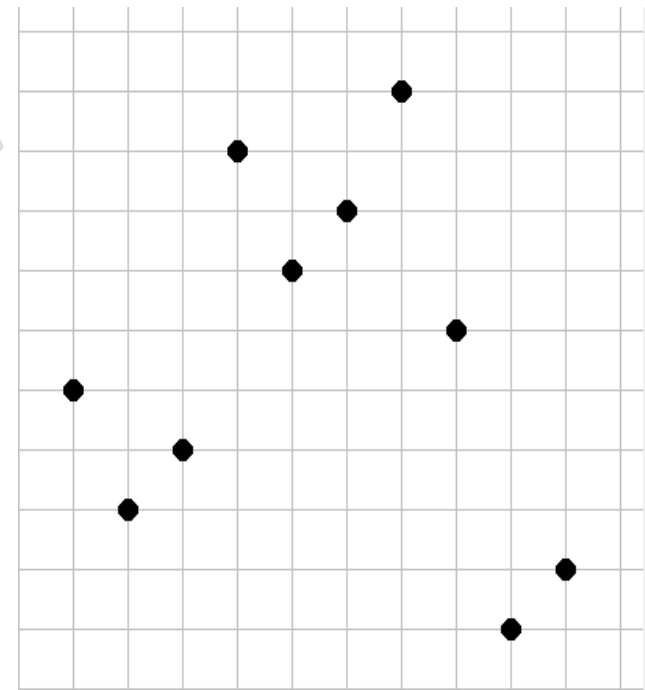
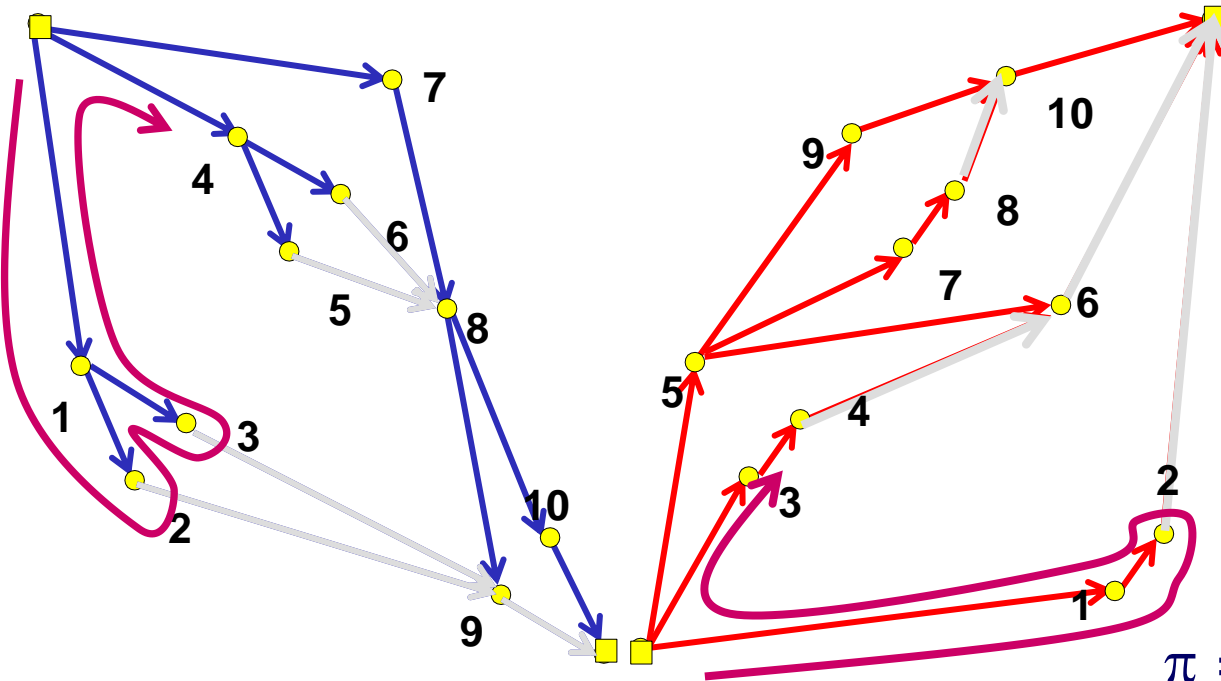
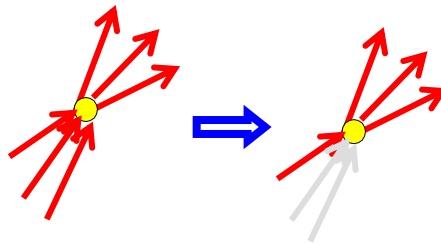
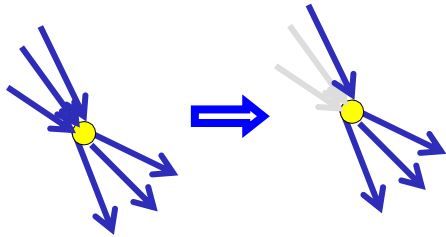
Given the **red edges**, there is a unique to add blue edges without creating a blue N (or a blue anti-N)



TS without blue N or red N



Baxter Permutations



$\pi = 5\ 3\ 4\ 9\ 7\ 8\ 10\ 6\ 1\ 2$

T_x

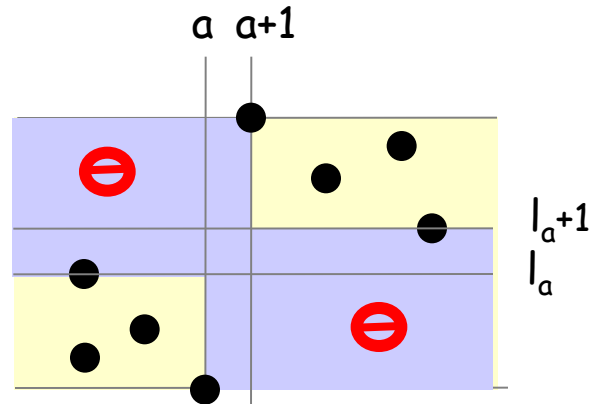
T_y

Baxter Permutation

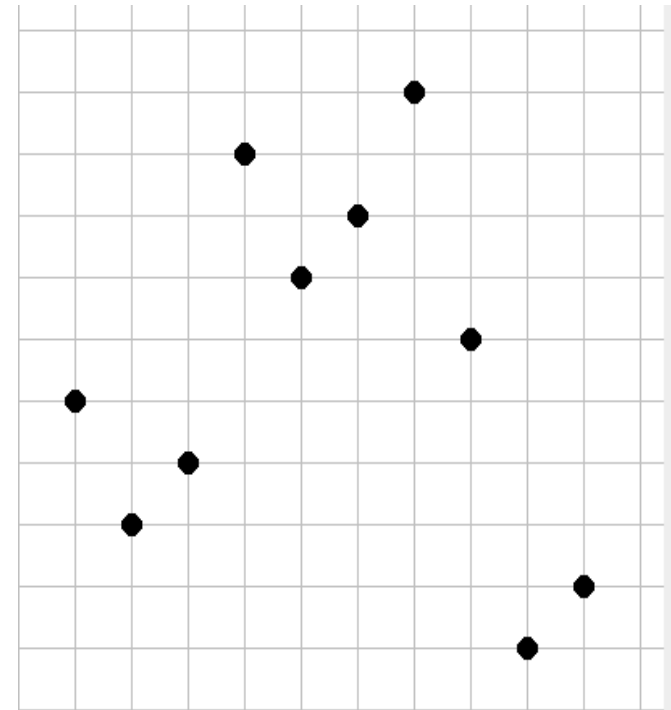
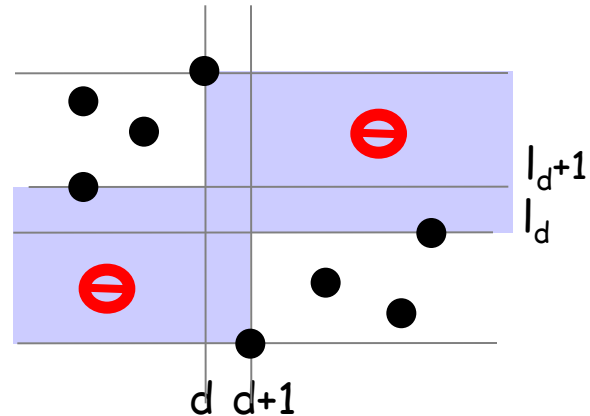
[Baxter'64]

π is a *Baxter permutation* if :

1) For each ascent a



2) For each descent d



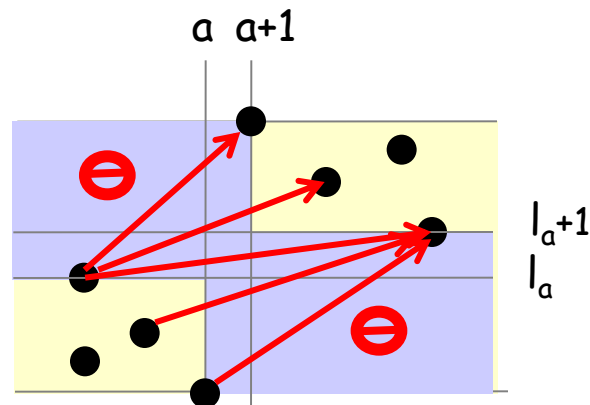
$\pi = 5\ 3\ 4\ 9\ 7\ 8\ 10\ 6\ 1\ 2$

Baxter Permutations

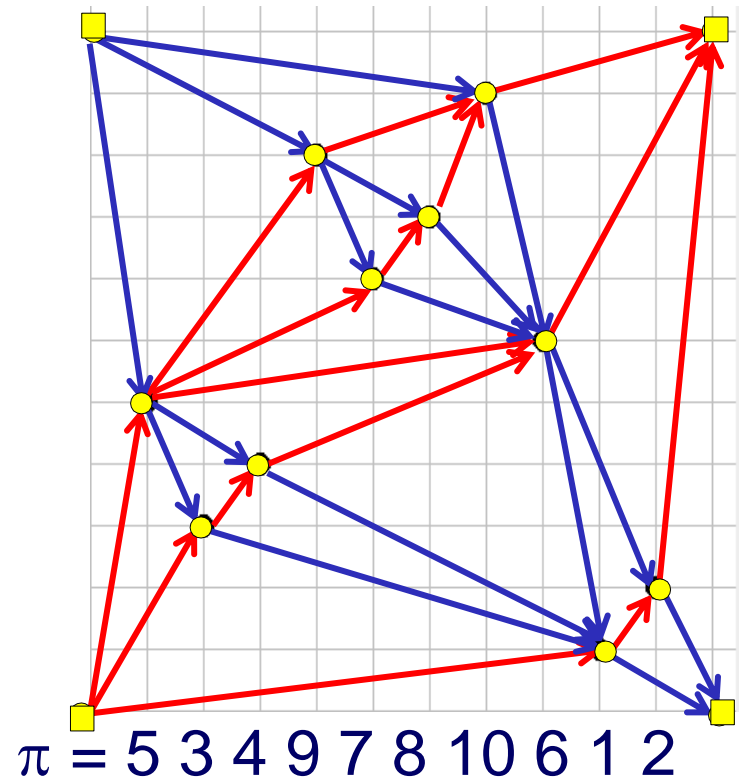
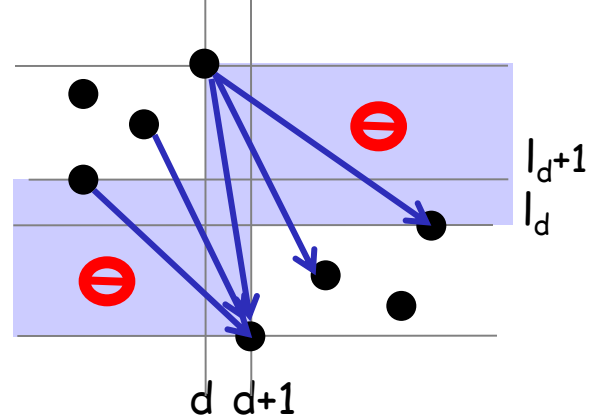


TS without blue N or red N

1) For each ascent a



2) For each descent d

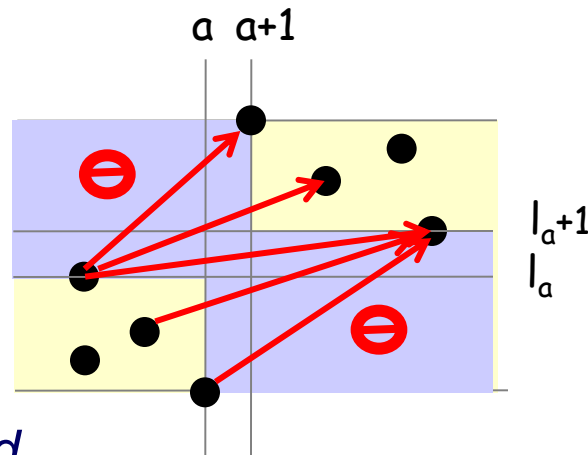


Baxter Permutations

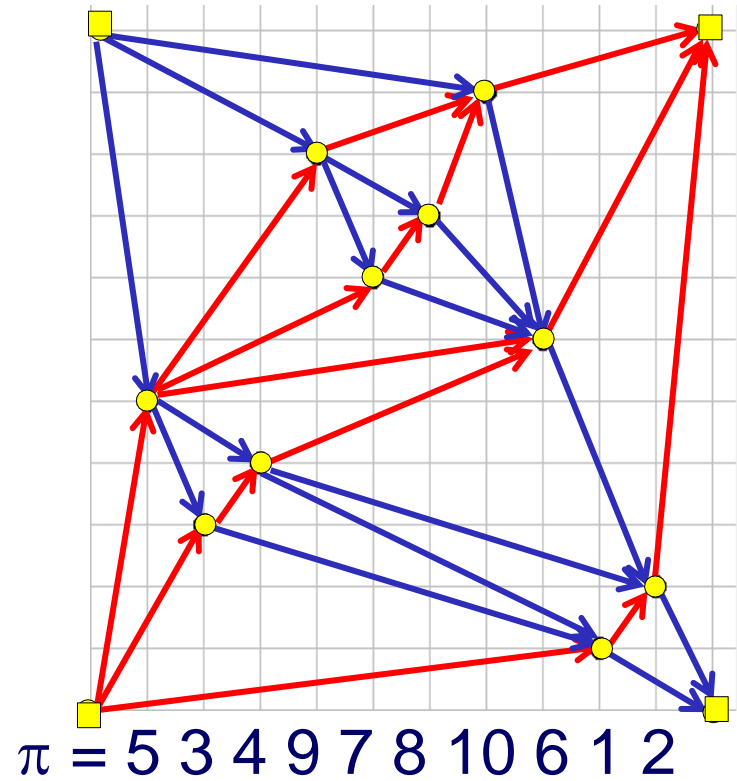
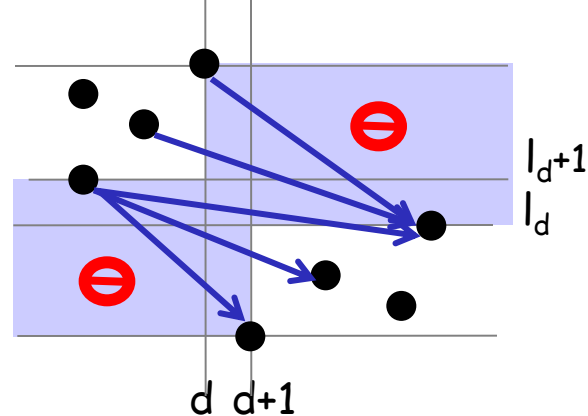


TS without blue anti-N or red N

1) For each ascent a



2) For each descent d

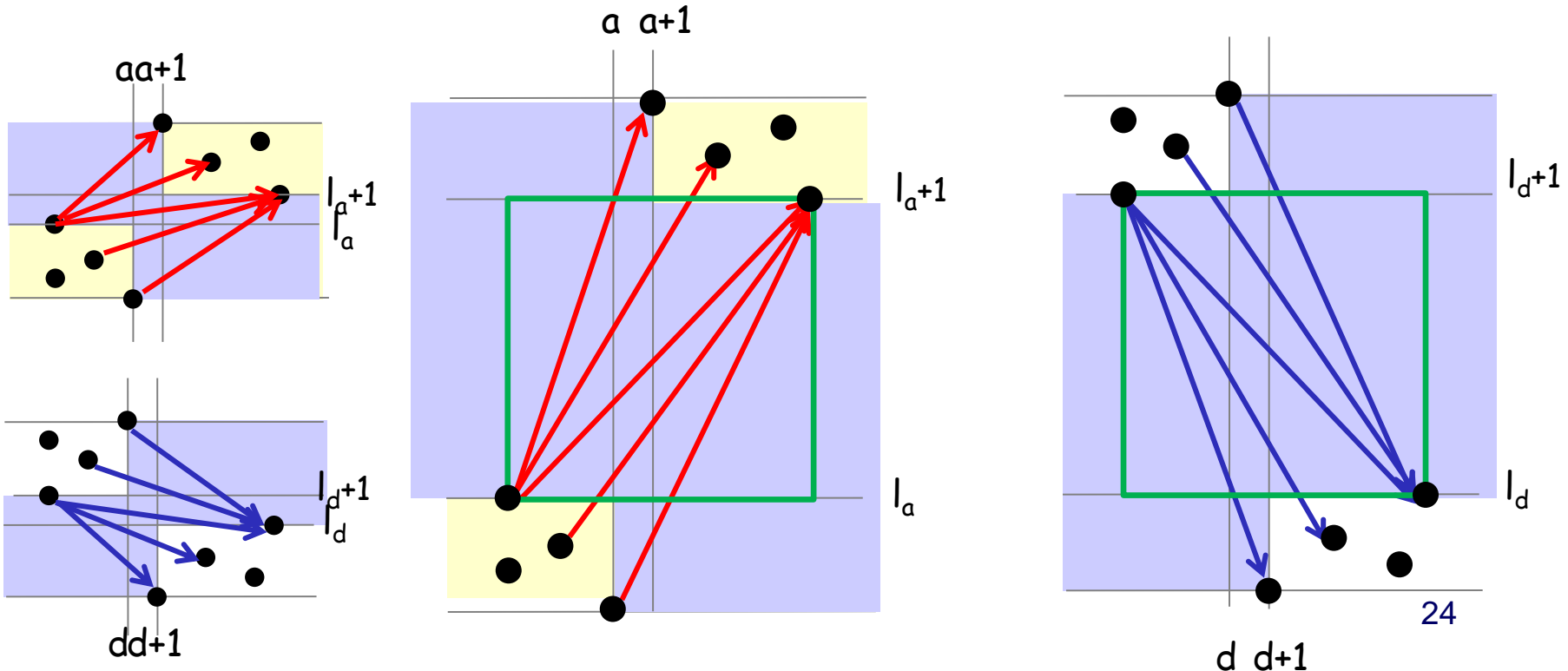


TS without blue anti-N or red N



L_∞ -Delaunay triangulations

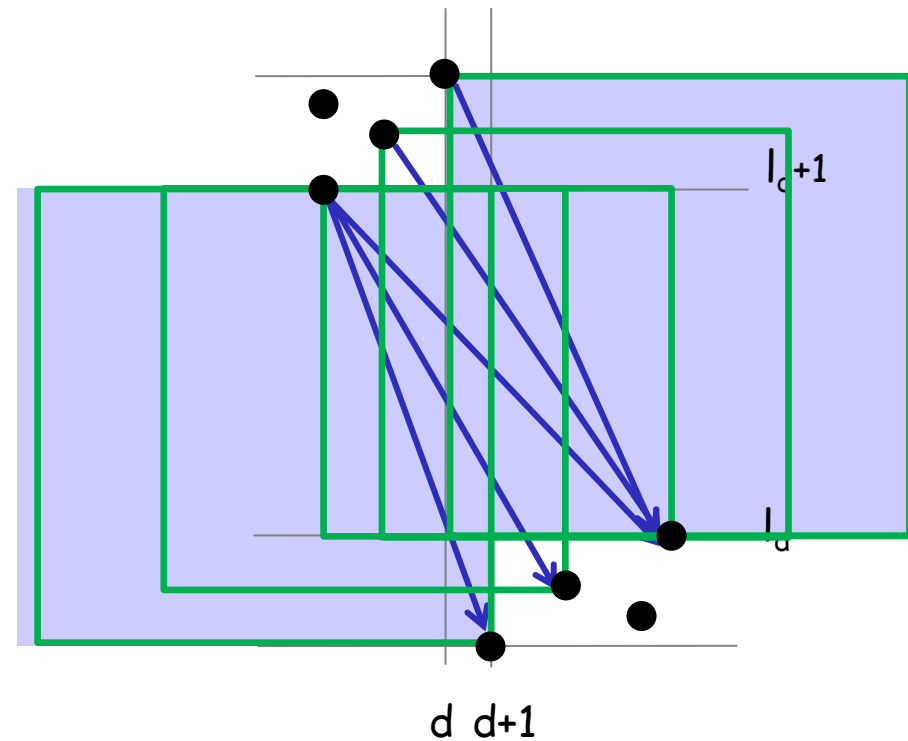
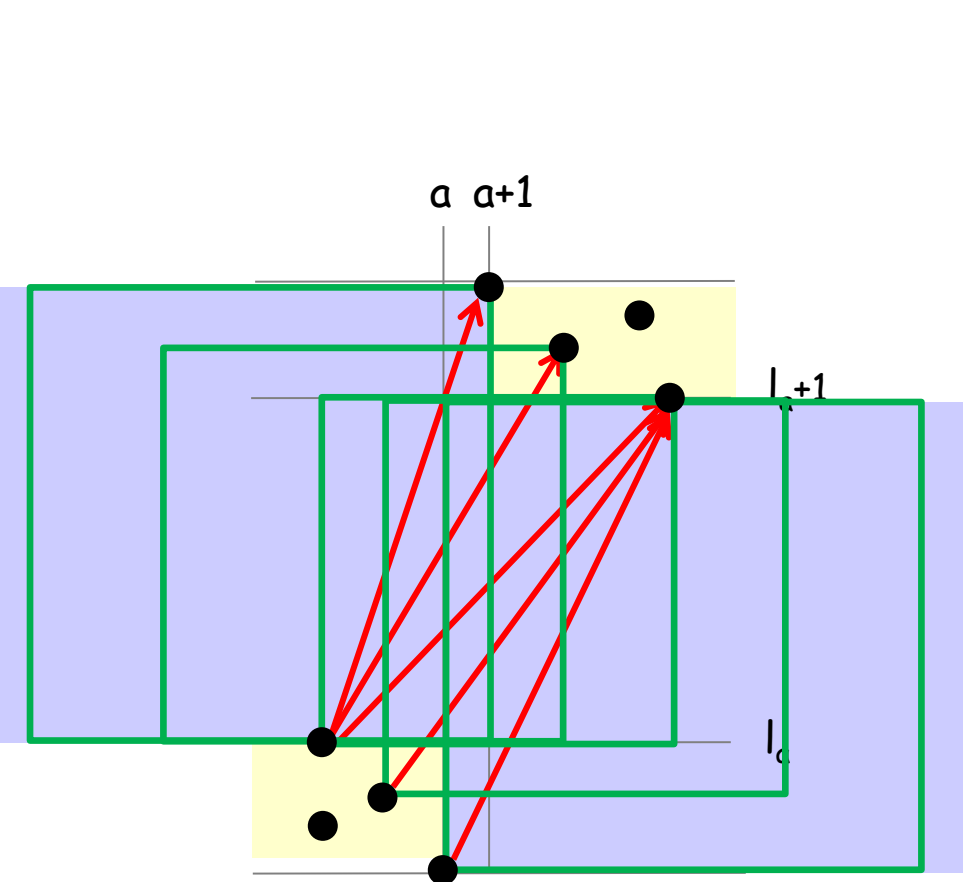
For each ascent a (resp. descent d), inflate the corresponding row l_a (resp. l_d) such that the diagonal edge has slope $\pm 45^\circ$.



TS without blue anti-N or red N



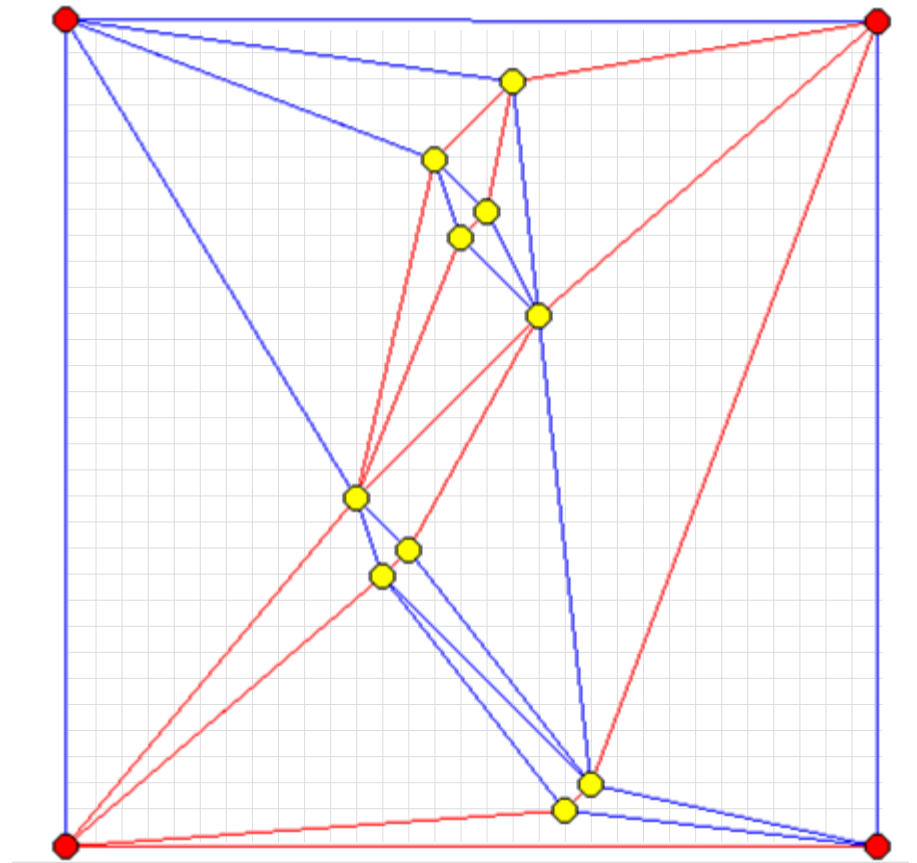
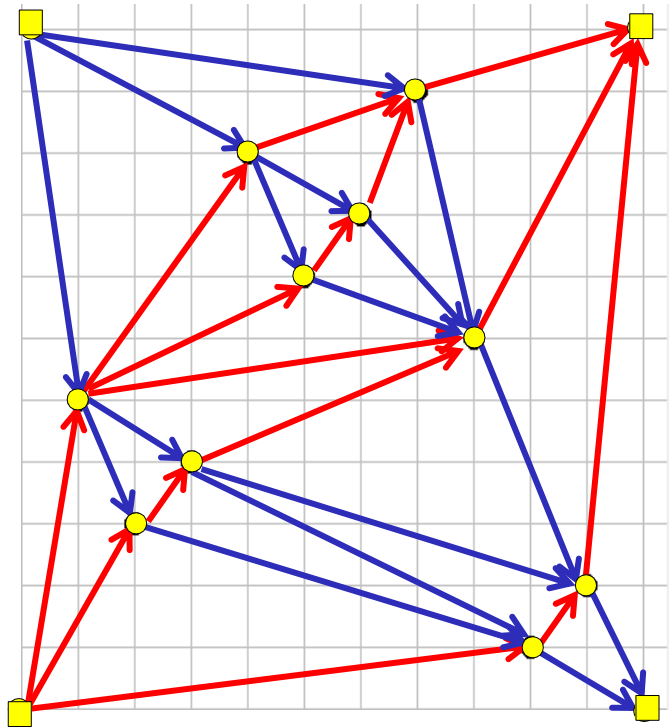
L_∞ -Delaunay triangulations



TS without blue anti-N or red N



L_∞ -Delaunay triangulations



Conclusion

- Is every triangulation without non-facial triangle L_∞ -Delaunay realizable?
 - Not so many progress on this question ☹
- If yes, can we compute efficiently a corresponding L_∞ -Delaunay triangulation?
- If not can we decide in polynomial time if a triangulation is L_∞ -Delaunay realizable?
- If not can we decide in polynomial time if a transversal structure is L_∞ -Delaunay realizable?