

# Large pyramid partitions (and Aztec diamonds)

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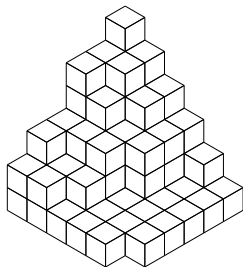
# Outline

- ▶ Motivational example: plane partitions
- ▶ Pyramid partitions
- ▶ Interlude into partitions and the Schur process
- ▶ Asymptotics of pyramid partitions
- ▶ Asymptotics of non-uniform Aztec diamonds
- ▶ Some related phenomena

Let them eat cake!

“S'ils n'ont pas de pain, qu'ils mangent de la brioche!”  
–Marie Antoinette d'Autriche (1755–1793)

## A small motivational example: plane partitions



Theorem (Major MacMahon 1916)

$$\sum_{\Lambda} q^{\text{Volume}(\Lambda)} = \prod_{n \geq 1} \frac{1}{(1 - q^n)^n}.$$

Large scale limit:  $q \rightarrow 1$

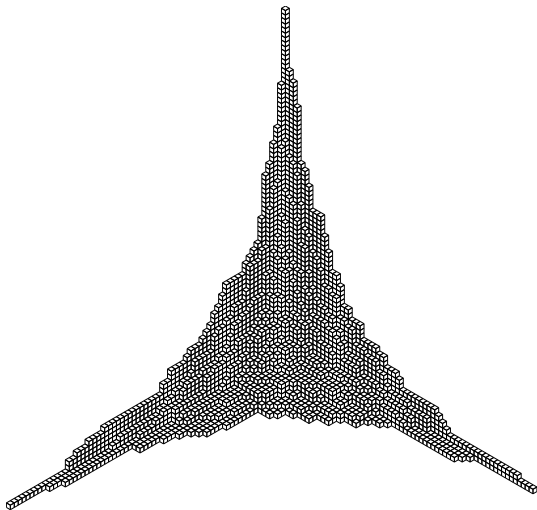
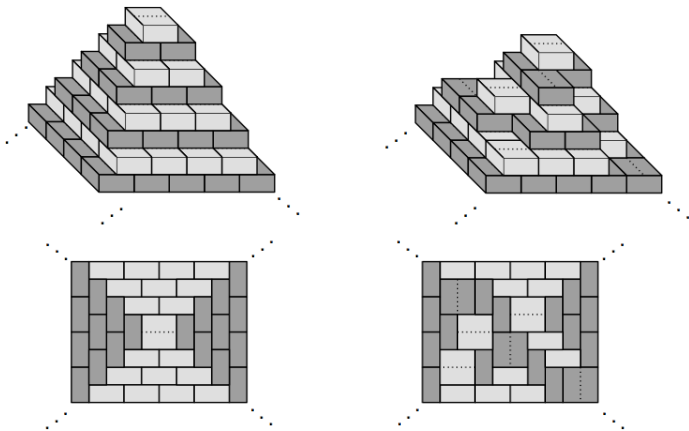


Figure: Nienhuis–Hilhorst–Blöte 1984, Cerf–Kenyon 2001, Okounkov–Reshetikhin 2003

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# Pyramid partitions



**Figure:** Piles of  $2 \times 2 \times 1$  boxes, each viewed as a pair of dominoes in the 2D projection looking downwards. On the left, the *empty* pyramid partition.

## More pictures

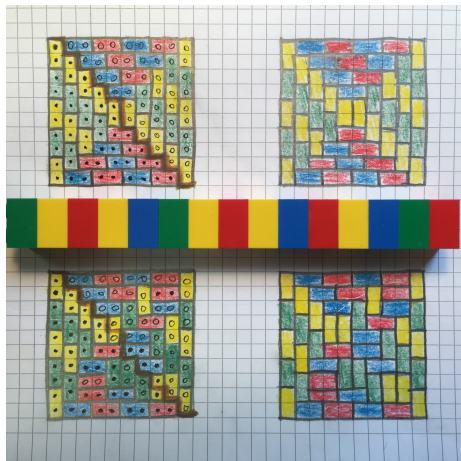


Figure: Pyramid partitions in 2D, LEGO coloring.



# LEGO

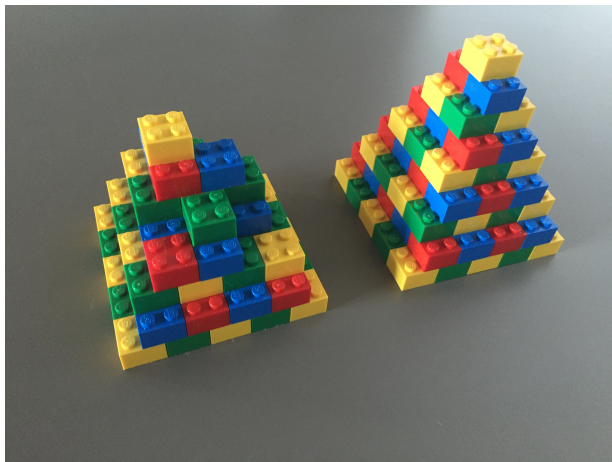


Figure: Pyramid partitions in 3D, natural light coloring.

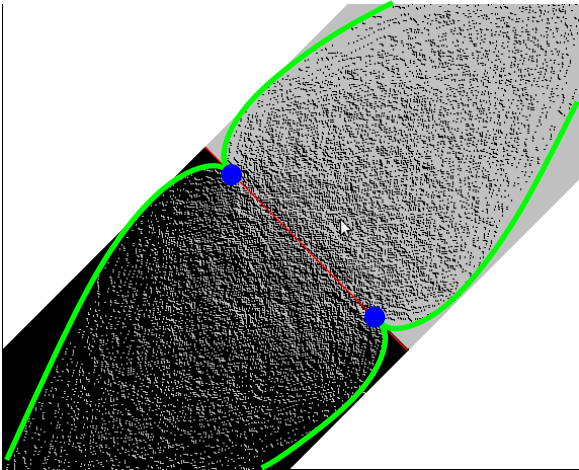
## Flips and the volume

- ▶ pyramid partition = what's left after a finite number of box removals from the empty configuration (introduced by Kenyon and Szendrői)
- ▶ removal = flip (adjacent vertical dominoes  $\leftrightarrow$  adjacent horizontal dominoes)
- ▶ Volume = Number of flips

Theorem (Young 2010)

$$\sum_{\Lambda} q^{\text{Volume}(\Lambda)} = \prod_{n \geq 1} \frac{(1 + q^{2n-1})^{2n-1}}{(1 - q^{2n})^{2n}}.$$

How do large pyramid partitions look like?



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# Partitions

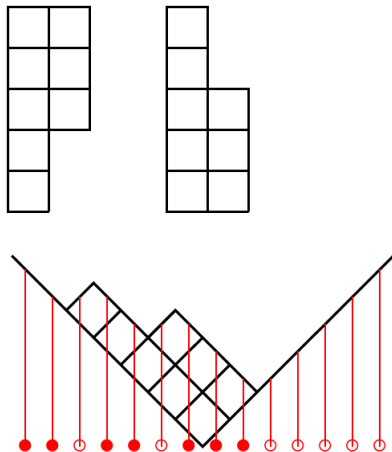


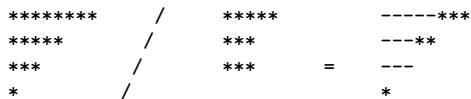
Figure: Partition  $(2, 2, 2, 1, 1)$  in English, French and Russian notation, with associated Maya diagram (particle-hole representation).

## Horizontal and vertical strips

Given partitions  $\mu \subseteq \lambda$ , we can form skew diagram  $\lambda/\mu$ , which we call a

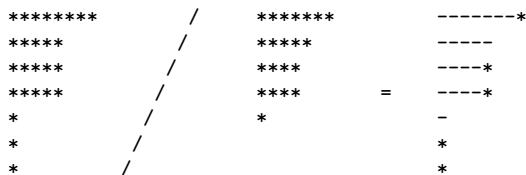
- ▶ horizontal strip, and write  $\mu \prec \lambda$  if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \dots$$



- ▶ vertical strip, and write  $\mu \prec' \lambda$ , if  $\lambda' \prec \mu'$  ( $' = \text{conjugate}$ ) or

$$\lambda_i - \mu_i \in \{0, 1\}$$



# The Schur process

Let  $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \{\prec, \succ, \prec', \succ'\}^n$  be a word. We say a sequence of partitions  $\Lambda = (\emptyset = \lambda(0), \lambda(1), \dots, \lambda(n) = \emptyset)$  is  $\omega$ -interlaced if  $\lambda(i-1) \omega_i \lambda(i)$ , for  $i = 1, \dots, n$ . The *Schur process* of word  $\omega$  with parameters  $Z = (z_1, \dots, z_n)$  is the measure on the set of  $\omega$ -interlaced sequences of partitions

$$\Lambda = (\emptyset = \lambda(0), \lambda(1), \dots, \lambda(n) = \emptyset)$$

given by

$$\text{Prob}(\Lambda) \propto \prod_{i=1}^n z_i^{|\lambda(i)| - |\lambda(i-1)|}.$$

## Remark

For a more general definition, see the original work of Okounkov–Reshetikhin 2003, or Borodin–Rains 2006.



# The Schur process is a determinantal point process

Theorem (OR 2003; BR 2006)

$$\text{Prob}(\lambda(i_s) \text{ contains a particle at position } k_s, 1 \leq s \leq n) = \det_{1 \leq u, v \leq n} K(i_u, k_u; i_v, k_v)$$

where

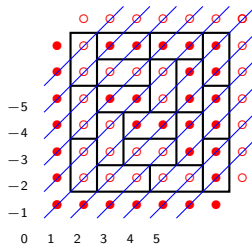
$$K(i, k; i', k') = \begin{cases} \left[ \frac{z^k}{w^{k'}} \right] \frac{\Phi(z; Z, \omega; i)}{\Phi(w; Z, \omega; i')} \frac{\sqrt{zw}}{z-w}, & i \leq i', \\ - \left[ \frac{z^k}{w^{k'}} \right] \frac{\Phi(z; Z, \omega; i')}{\Phi(w; Z, \omega; i)} \frac{\sqrt{zw}}{w-z}, & i > i' \end{cases}$$

with

$$\Phi(z; Z, \omega; i) = \prod_{\substack{j: j \leq i, \omega_j \in \{\gamma, \gamma'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \gamma', \\ -1, & \omega_j = \gamma. \end{cases}}} (1 + \epsilon_j z_j z)^{\epsilon_j} \prod_{\substack{j: j > i, \omega_j \in \{\gamma, \gamma'\} \\ \epsilon_j = \begin{cases} 1, & \omega_j = \gamma', \\ -1, & \omega_j = \gamma. \end{cases}}} \left(1 + \epsilon_j \frac{z_j}{z}\right)^{-\epsilon_j}$$

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## Pyramid partitions as Schur processes, pictorially



**Figure:** A pyramid partition of width 5 corresponding to the sequence

$\emptyset \prec (1) \prec' (2) \prec (2, 2) \prec' (3, 3) \prec (3, 3, 2) \prec' (2, 2, 1) \prec (2, 1) \prec' (1, 1) \prec (1) \prec' \emptyset$ .

## Pyramid partitions as Schur processes II

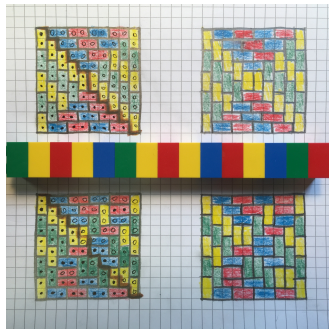
Let  $n = 2n_0$  be an even integer. A pyramid partition is (bijectively) a sequence of  $2n + 1$  partitions

$$\Lambda = (\emptyset = \lambda(-n) \prec \lambda(-n+1) \prec' \lambda(-n+2) \prec \dots \prec' \lambda(0) \succ \lambda(1) \succ' \lambda(2) \succ \dots \succ' \lambda(n) = \emptyset).$$

It is this a Schur process for the word  $\omega_{pyr} = (\prec, \prec')^{n_0} (\succ, \succ')^{n_0}$  and parameters  $Z = (z_{-n}, \dots, z_{-1}, z_1, \dots, z_n)$ .

### Remark

For volume weighting,  $z_{-i} = z_i = q^{i-\frac{1}{2}}$ ,  $1 \leq i \leq n$ .



**Figure:** Pyramid partitions in 2D (LEGO coloring), with partitions as Maya diagrams on left: empty process top and  
 $\dots \gamma' 0 \gamma (3) \gamma' (4, 1, 1) \gamma (2) \gamma' (1) \gamma (1) \gamma' (1) \gamma (1) \gamma' 0 \gamma \dots$  process bottom.

## A simple word on asymptotics

Everything we'd like to know about asymptotics of large pyramid partitions can be translated into asymptotics of large particle-hole systems associated to the corresponding Schur process.

## How to compute the limit shape

Let  $t = 2t_0 < n$ ,  $k \in \mathbb{Z} + \frac{1}{2}$ . A weak Wick lemma shows that:

Lemma (db–Boutillier–Vuletić 2016)

$$\begin{aligned} \text{Prob}(\lambda(-t) \text{ contains a particle at position } k) &= \\ &= \left[ \frac{z^k}{w^k} \right] \frac{J(z; t_0)}{J(w; t_0)} \frac{\sqrt{zw}}{z-w} \\ &= \int \int \frac{J(z; t_0)}{J(w; t_0)} \frac{1}{z^{k-\frac{1}{2}} w^{-k-\frac{1}{2}}} \frac{1}{z-w} \frac{dz}{2\pi iz} \frac{dw}{2\pi iw} \end{aligned}$$

where (with  $(u; q)_m = \prod_{i=0}^{m-1} (1 - q^i u)$ )

$$J(z; t_0) = \frac{(-q^{2t_0 + \frac{1}{2}} z; q^2)_{n_0 - t_0} \left(\frac{q^{\frac{1}{2}}}{z}; q^2\right)_{n_0}}{(q^{2t_0 + \frac{3}{2}} z; q^2)_{n_0 - t_0} \left(-\frac{q^{\frac{3}{2}}}{z}; q^2\right)_{n_0}}.$$

# Asymptotics regime

We let the size of the partition grow with  $q \rightarrow 1$  as  $\epsilon \rightarrow 0$  like so:

$$q(\epsilon) = \exp(-\gamma\epsilon),$$

$$n_0(\epsilon) = a_0/\epsilon,$$

$$t_0(\epsilon) = x_0/\epsilon,$$

$$k(\epsilon) = y/\epsilon.$$



## A few limit formulas

If  $q = \exp(-r)$  and  $r \rightarrow 0+$ , we have

$$\log(z; q)_\infty \sim -\frac{Li_2(z)}{r}$$

and furthermore,

$$\log(z; q)_{\frac{A}{r}} \sim \frac{1}{r}(Li_2(e^{-A}z) - Li_2(z))$$

where

$$Li_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2}, \quad |z| < 1$$

with analytic continuation given by

$$Li_2(z) = -\int_0^z \frac{\log(1-u)}{u} du, \quad z \in \mathbb{C} \setminus [1, \infty).$$

# Asymptotics of the kernel

## Lemma (db–Boutillier–Vuletić 2016)

In the limit ( $x = 2x_0$  is rescaled  $t$ ,  $y$  is rescaled  $k$ ),

$$\text{Prob}(\lambda(-t) \text{ contains a particle at position } k) \sim \int \int e^{S(z;x,y) - S(w;x,y)} \frac{d\mathbb{T}}{z - w}$$

where

$$S(z; x, y) = \frac{1}{2\gamma} \left( Li_2(-Az) - Li_2(-Xz) + Li_2\left(\frac{A}{z}\right) - Li_2\left(\frac{1}{z}\right) + \right. \\ \left. + Li_2(Xz) - Li_2(Az) + Li_2\left(-\frac{1}{z}\right) - Li_2\left(-\frac{A}{z}\right) \right) - y \log z$$

and  $X = \exp(-\gamma x)$ ,  $A = \exp(-2\gamma a_0)$ .

## The arctic curve

To compute the arctic curve, one solves for  $(x, y)$  (or  $X = \exp(-\gamma x)$ ,  $Y = \exp(2\gamma y)$ ) corresponding to double critical points of  $S(z; x, y)$ . That is,

### Theorem (db–Boutillier–Vuletić 2016)

*The arctic curve is the locus  $(x, y)$  satisfying:*

$$\begin{aligned}f(z; X) &= Y, \\f'(z; X) &= 0\end{aligned}$$

where  $f(z; X) = \frac{(z+1)(z-A)(z-1/A)(z+1/X)}{(z-1)(z+A)(z+1/A)(z-1/X)}$ .

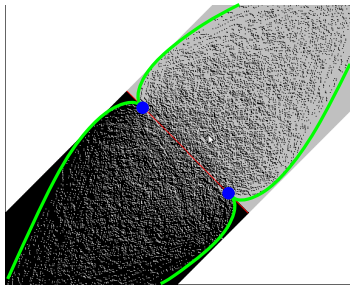
### Remark

Alternatively, it can be seen as given by the *algebraic* equation

$$\Delta [(z+1)(z-A)(z-1/A)(z+1/X) - Y(z-1)(z+A)(z+1/A)(z-1/X)] = 0$$

where  $\Delta$  represents taking the discriminant.

## The arctic curve, pictorially



Notice the cusps (which correspond to the *double* critical point of  $S$  at  $z = 0$ ).

## Arctic curve in the infinite regime

What happens when  $a_0 \rightarrow \infty$ , or equivalently,  $A \rightarrow 0$ ?

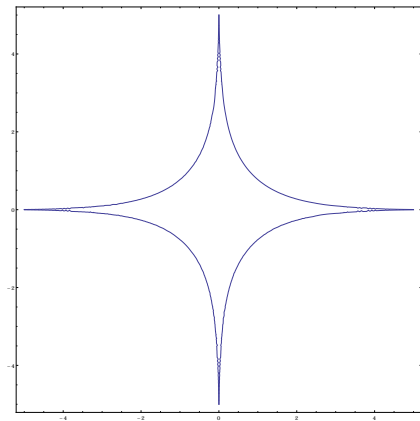
The cusps move to  $\infty$  and the arctic curve becomes

$$(1 + Z + W - ZW)(1 + Z - W + ZW)(1 - Z + W + ZW)(1 - Z - W - ZW) = 0$$

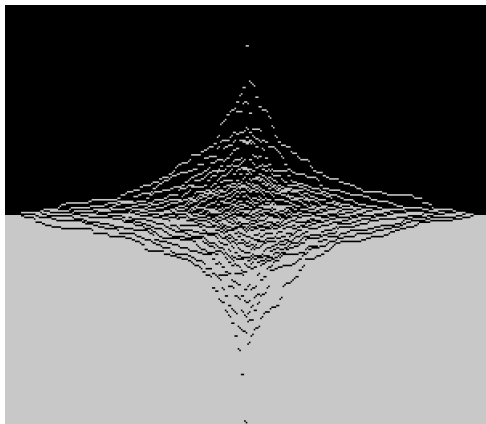
where  $(Z, W) = (\sqrt{X}, \sqrt{Y})$  which is the boundary of the amoeba of the (square lattice determined) polynomial

$$P(Z, W) = 1 + Z + W - ZW.$$

## Arctic curve in the infinite regime, pictorially



A large sample in the infinite regime, up to affine transformations



## A word on what happens on the arctic curve

Everywhere but at the cusps and tangency points, fluctuations are of Airy type (cf., for example, Okounkov–Reshetikhin 2006). At the turning points, one has two correlated GUE minors processes. At the cusps, one (conjecturally as of yet) gets the cusp Airy process of Duse–Johansson–Metcalf (2015).



## A slide on details: fermionic vertex operators cf. the Kyoto School

$$\Gamma_+(x)\Gamma_-(y) = \frac{1}{1-xy}\Gamma_-(y)\Gamma_+(x),$$

$$\tilde{\Gamma}_+(x)\tilde{\Gamma}_-(y) = \frac{1}{1-xy}\tilde{\Gamma}_-(y)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_+(x)\Gamma_-(y) = (1+xy)\Gamma_-(y)\tilde{\Gamma}_+(x),$$

$$\Gamma_+(x)\tilde{\Gamma}_-(y) = (1+xy)\tilde{\Gamma}_-(y)\Gamma_+(x),$$

$$\Gamma_+(x)\psi(z) = \frac{1}{1-xz}\psi(z)\Gamma_+(x),$$

$$\Gamma_+(x)\psi^*(w) = (1-xw)\psi^*(w)\Gamma_+(x),$$

$$\Gamma_-(y)\psi(z) = \frac{1}{1-\frac{y}{z}}\psi(z)\Gamma_-(y),$$

$$\Gamma_-(y)\psi^*(w) = (1-\frac{y}{w})\psi^*(w)\Gamma_-(y),$$

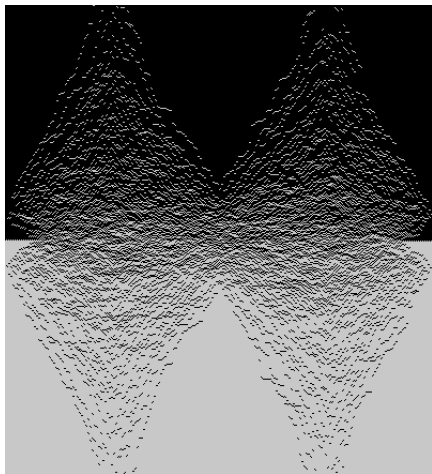
$$\tilde{\Gamma}_+(x)\psi(z) = (1+xz)\psi(z)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_+(x)\psi^*(w) = \frac{1}{1+xw}\psi^*(w)\tilde{\Gamma}_+(x),$$

$$\tilde{\Gamma}_-(y)\psi(z) = (1+\frac{y}{z})\psi(z)\tilde{\Gamma}_-(y),$$

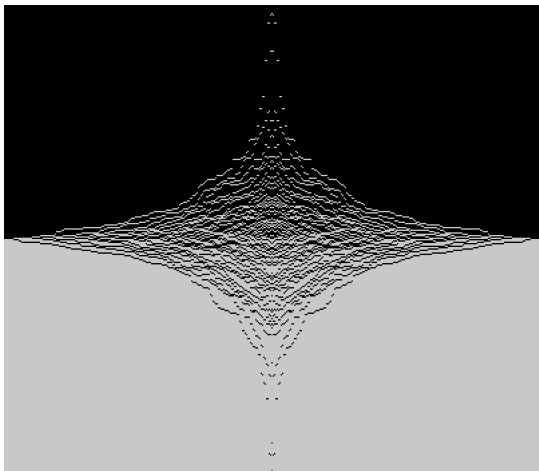
$$\tilde{\Gamma}_-(y)\psi^*(w) = \frac{1}{1+\frac{y}{w}}\psi^*(w)\tilde{\Gamma}_-(y).$$

## Other stuff: “skew pyramid partitions”

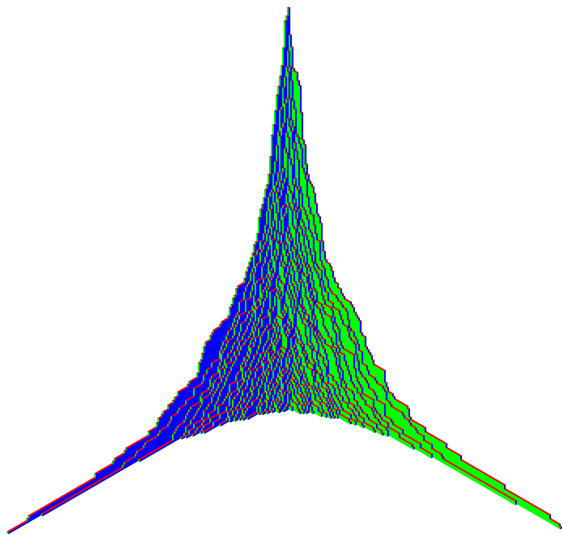


**Figure:** Skew pyramid partitions: word  $(\prec, \prec')^{50}(\succ, \succ')^{50}(\prec, \prec')^{50}(\succ, \succ')^{50}$ ,  $q = 0.99$ . The analogue in pyramid partition land of OR 2006's skew plane partitions. Vertical cusps should have Pearcey fluctuations generically, cusp Airy if symmetric.

Other stuff: symmetric “pyramid partitions”



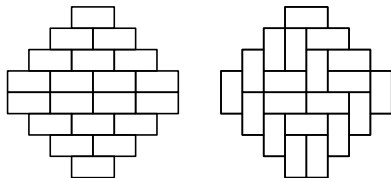
## Symmetric “pyramid partitions” as plane overpartitions



This limit shape seems to be the same that Vuletić 2009 analyzed in the context of strict plane partitions and Pfaffian processes. Work in progress with M. Vuletić and J. Bouttier.

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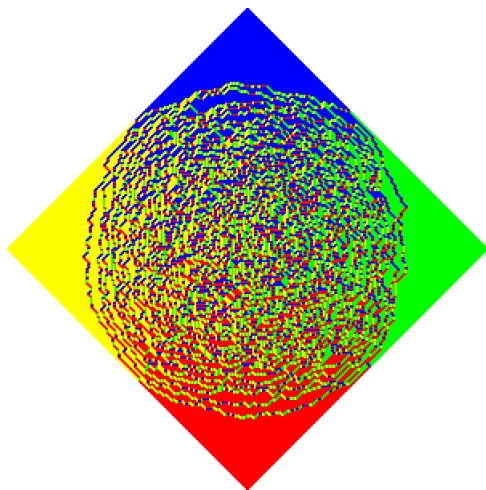
# The Aztec diamond



**Figure:** Two tilings of the size  $n = 4$  Aztec diamond. One can define the volume of a tiling as the number of flips needed to reach it from the all horizontal (zero volume) tiling.

There are  $2^{\binom{n+1}{2}}$  tilings of the  $n \times n$  Aztec diamond (Elkies–Kuperberg–Larsen–Propp 1992).

# The original arctic circle theorem (Jockusch–Propp–Shor 1998)



## The Aztec diamond as a Schur process

The order  $n$  Aztec diamond is (bijectively equivalent to) the sequence of  $2n + 1$  partitions

$$\Lambda = (\emptyset = \lambda(0) \prec \lambda(1) \succ' \lambda(2) \prec \cdots \succ' \lambda(2n - 1) \prec \lambda(n) \succ' \lambda(2n) = \emptyset).$$

It is a Schur process of word  $(\prec, \succ')^n$  and parameters  $(z_1, \dots, z_n)$ .

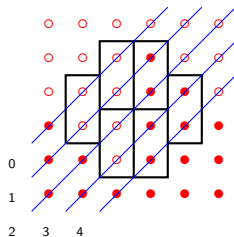


Figure: A  $2 \times 2$  Aztec diamond corresponding to the sequence  $\emptyset \prec (2) \succ' (1) \prec (1, 1) \succ' \emptyset$ .

### Remark

If  $z_{2i-1} = q^{-2i+1}$ ,  $z_{2i} = q^{2i}$ , one obtains a  $q^{\text{Volume}}$  weighting on the Aztec diamond where volume = number of flips from the all horizontal tiling.



## Periodic weights, arbitrary parameters

For (say)  $k < l$ , pick  $z$  parameters as follows:  $z_1 = a_1, z_2 = b_1, z_3 = a_2, z_4 = b_2, \dots, z_{2k-1} = a_k, z_{2k} = b_k, z_{2k+1} = a_1, z_{2k+2} = b_{k+1}, \dots, z_{2l} = b_l$ , repeat

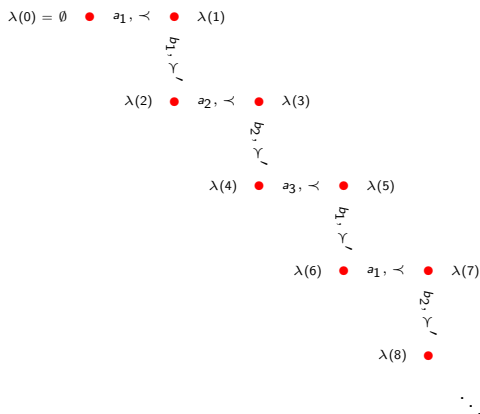


Figure:  $k = 3, l = 2$   $a, b$  parameters.

Similar weights have been considered before (Mkrtychyan 2013, case of plane partitions), but note here there is no need for any of the parameters to be  $< 1$ .

How do such large Aztec diamonds look?

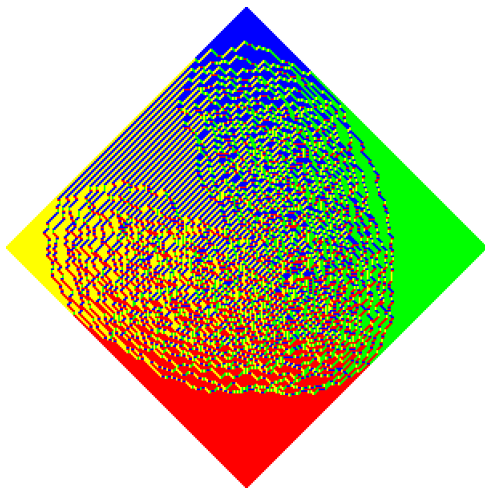
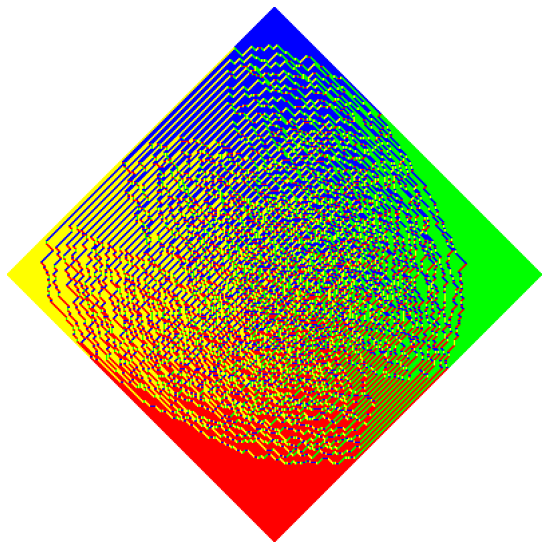


Figure: A random  $150 \times 150$  Aztec diamond with  $a, b$  parameters  $a_1 = 4, a_2 = 1/4, b_1 = 1$ .

## More fingers



**Figure:** A random  $200 \times 200$  Aztec diamond with  $a, b$  parameters  $(a_1, a_2, a_3, b_1, b_2) = (8, 1, 1/8, 3, 1/3)$ .

# Snake

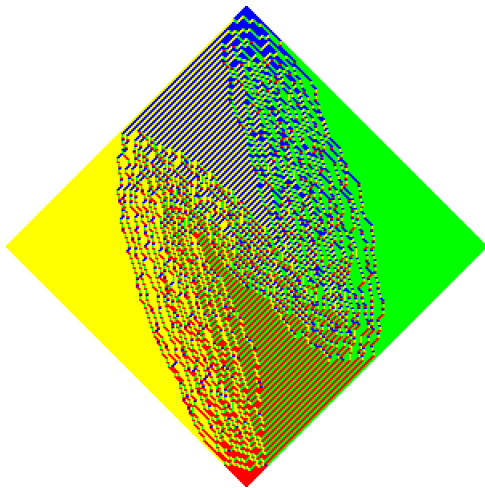
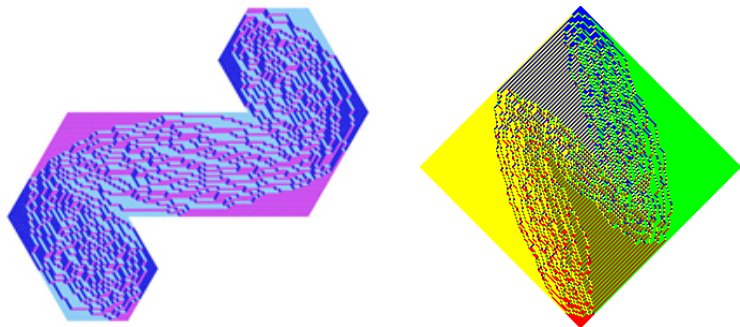


Figure: A random  $150 \times 150$  Aztec diamond with  $a, b$  parameters  $a_1, a_2, b_1, b_2 = (48, 1, 16, 1/8)$ .

## Compare with Kenyon–Okounkov 2003 (another snake)



Simple (uniform) measure and (moderately) complicated boundary conditions vs. (moderately) complicated measure and simple boundary conditions (limit shapes look similar, but there are equally important differences).

## Something like the tacnode process

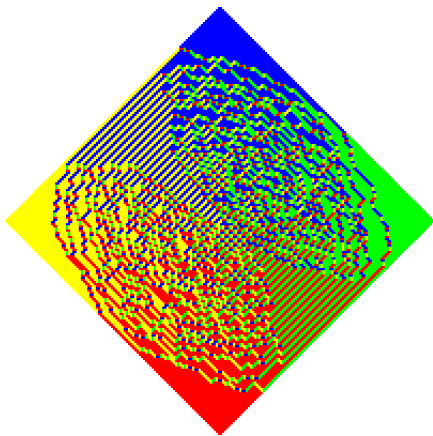
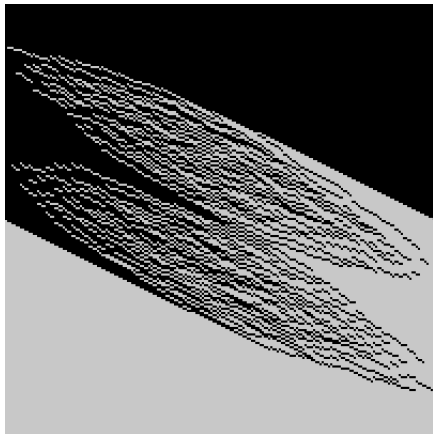


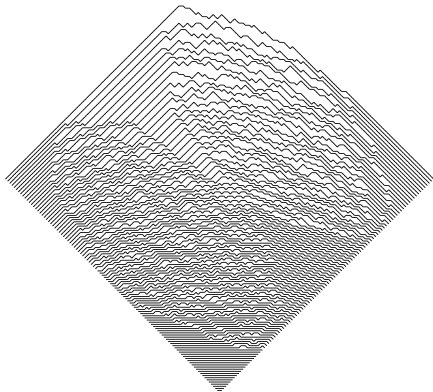
Figure:  $100 \times 100$  Aztec diamond with  $a, b$  parameters  $(a_1, a_2) = (b_1, b_2) = (\alpha, 1/\alpha)$ ,  $\alpha = 30$ .

## Particle-hole view, up to affine transformations



Compare this to the work of Borodin–Duits 2011 (maybe!).

## Non-intersecting paths picture





## The $S$ function, and what can we say 'bout the model

$$S(z; x, y) = \frac{x}{k} \log \left( \prod_{i=1}^k (1 + a_i z) \right) + \left(1 - \frac{x}{l}\right) \log \left( \prod_{i=1}^l \left(1 - \frac{b_i}{z}\right) \right) - y \log z$$

Analyzing  $S$ , we can obtain:

- ▶ formula for the arctic curve, as before
- ▶ location of the points of tangency to the boundary
- ▶ angle made by the cusps
- ▶ fluctuations

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Some pictures of things we can't yet do using our methods. Some are harder than others.

## Partial Aztec diamonds, uniformly weighted

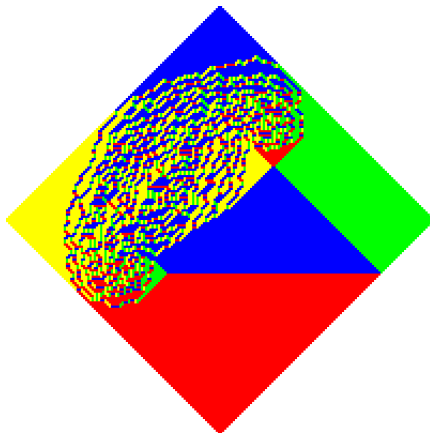
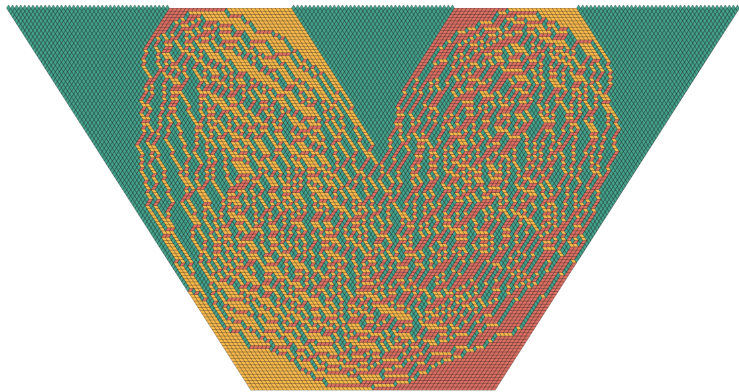


Figure: Half an  $n = 100$  Aztec diamond with final partition fixed  $\emptyset \prec \lambda(1) \succ' \dots \prec \lambda(n-1) \succ' \lambda(50) = 25^{50}$

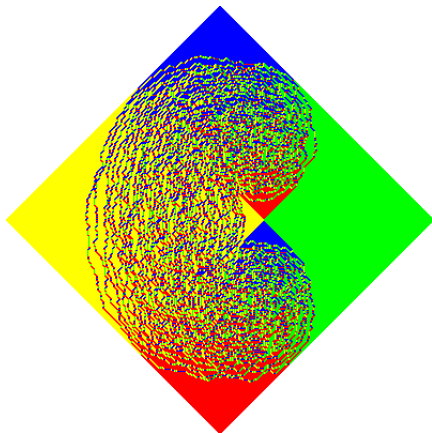
This corresponds to the Gelfand–Tsetlin polygons of Petrov 2012 (special case of Kenyon–Okounkov 2007). In our case:

$$\emptyset \prec \lambda(1) \succ' \dots \prec \lambda(n-1) \succ' \lambda(n) = \text{fixed } \lambda.$$

Compare with GT polygons – KO 2007 and P 2012

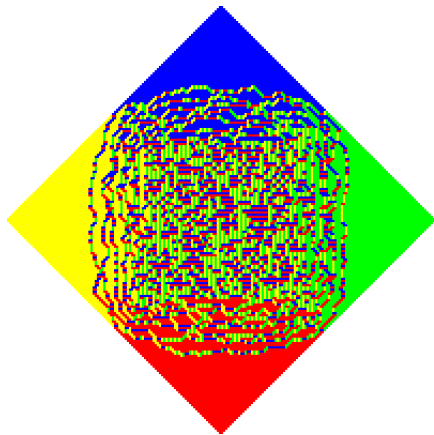


## Aztec diamond with frozen corner

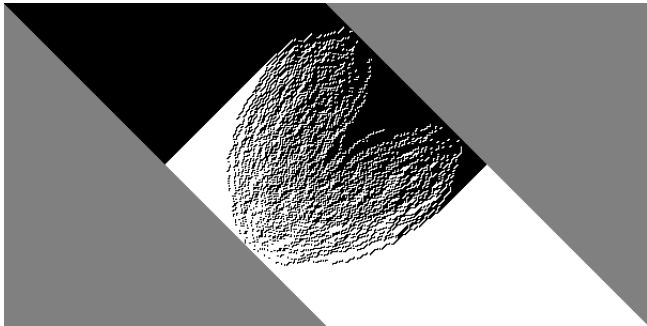


Work of Colomo–Sportiello, initially on the 6 vertex model. There is some Painlevé interpretation of the partition function here, a la Borodin–Arinkin 2009 (work of Pronko 2015 and Kniezal – 2015).

## 2-Periodic Aztec diamond



Studied by Chhita–Johansson 2014 and Chhita–Young 2013 using the inverse Kasteleyn matrix approach.



Thank you!