

Edge-colored graphs as higher-dimensional maps

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Discretization of manifolds

- ▶ 2D discrete surfaces: triangulations, p -angulations and combinatorial maps
- ▶ 3D triangulations: gluings of tetrahedra

Discretization of manifolds

- ▶ 2D discrete surfaces: triangulations, p -angulations and combinatorial maps
- ▶ 3D triangulations: gluings of tetrahedra
- ▶ How to represent them in a suitable fashion for combinatorics?
- ▶ Equivalent of p -angulations?
- ▶ Enumeration?

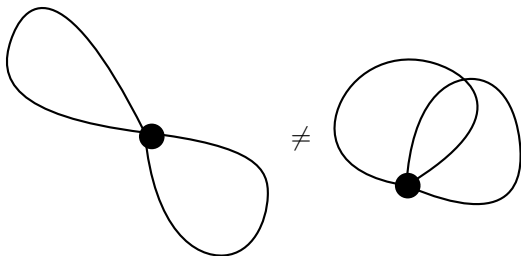
Colored triangulations and colored graphs

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Bijection with (stuffed colored Walsh) maps

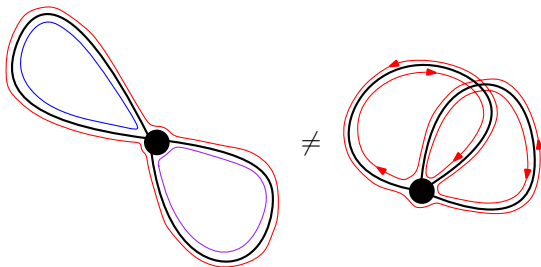
Combinatorial maps

Graph with cyclic ordering of edges incident to each vertex



Combinatorial maps

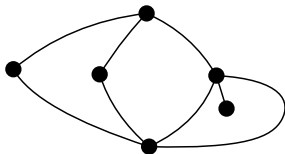
Graph with cyclic ordering of edges incident to each vertex



Cyclic ordering defines **faces**: follow the corners

$2p$ -angulation

- ▶ Faces of degree $2p$
- ▶ Duality: vertices of degree $2p$



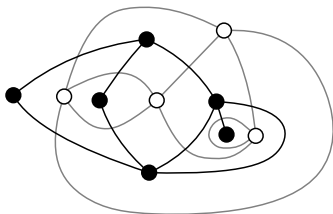
- ▶ Euler's relation with $E(M) = pV(M)$

$$F(M) - E(M) + V(M) = F(M) - (p-1)V(M) = 2 - 2g(M)$$

- ▶ $g(M) \geq 0 \Rightarrow$ bound on $F(M)$ linear in $V(M)$
- ▶ Maximizing $F(M)$ at fixed $V(M)$ equivalent to $g(M) = 0$

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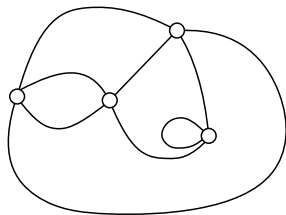
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What do we know?

Maps: from Tutte to today

- ▶ Enumeration [Tutte's equations, matrix models]
- ▶ Bijections [Cori-Vauquelin-Schaeffer, Bouttier-Di Francesco-Guitter]
- ▶ Topological recursion [Eynard]
- ▶ Continuum limit [Brownian sphere]
- ▶ More being developed nowadays [Hurwitz, integrable hierarchies, etc.]

What do we know?

Simplicial complexes in higher dim

- ▶ ??

What do we know?

Simplicial complexes in higher dim

- ▶ ??
- ▶ (Mostly) numerical works (Ambjorn, Jurkewic, Jonsson, Loll, etc.)
- ▶ Why? Combinatorics difficult to control!
- ▶ Most recent analytical attempts via
colored graphs
- ▶ Mostly by physicists
[Gurau, Krajewski, Rivasseau, Tanasa, Vignes-Tourneret and students]
- ▶ Try a more systematic combinatorial study
[Gurau-Schaeffer, Bonzom-Lionni and wip w/ Monteil]

The physics

- ▶ Einstein's 2nd revolution

Gravitation = Geometry of space-time

- ▶ Quantum physics

Quantum = probabilistic, random

Gravitation and quantum together

Space-time metric is a random variable

Quantum gravity = random geometry

Discrete quantum gravity

Define quantum gravity at the discrete level

Two approaches

Discrete quantum gravity

Define quantum gravity at the discrete level

Two approaches

Regge calculus, LQG, Spin foams	Dynamical triangulations
Fix a discretization	Edges have fixed lengths
Geometry = edge lengths	Geometry = discretization
Quantization = $\int \prod_e d\ell_e$	Quantization
or $\sum_{\text{quantum numbers}}$	$\sum_{\text{Geometries}} = \sum_{\text{Triangulations}}$

In 2nd approach, $\sum_{\text{Triangulations}} \Rightarrow$ generating function!

How to represent triangulations?

Triangulations

- ▶ Gluing of simplices (tetrahedra, pentachora, etc.)
- ▶ Defined by **attaching maps**
- ▶ Ensemble of triangulations defined by constraints on attaching maps

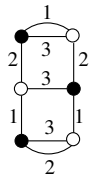
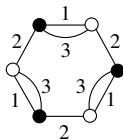
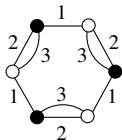
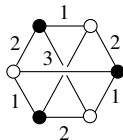
Various ensembles

- ▶ Various ensembles in topology (simplicial, CW, Δ -complexes, etc.)
- ▶ Not suitable for combinatorics (too wild)
- ▶ Digging through old work, found colored triangulations
[Italian school: crystallization, graph-encoded manifold]
- ▶ Represented by edge-colored graphs

Colored graphs

$(d + 1)$ -colored graphs

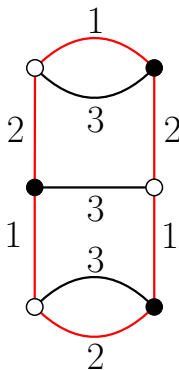
- ▶ Bipartite graphs
black and white vertices
- ▶ Edges colored with $d + 1$
possible colors
- ▶ Vertices of degree $d + 1$
- ▶ All colors incident exactly
once at each vertex



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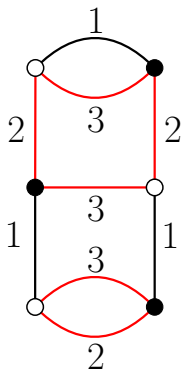


Faces are closed cycles with only **two** colors.

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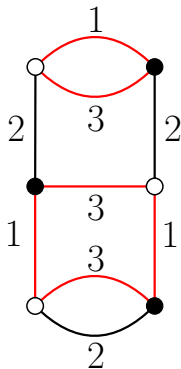


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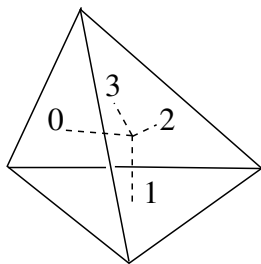


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Triangulations from colored graphs

$$\text{duality} \left\{ \begin{array}{l} \text{vertex} \rightarrow d\text{-simplex} \\ \text{edge} \rightarrow (d-1)\text{-simplex} \\ \text{face} \rightarrow (d-2)\text{-simplex} \\ k\text{-bubble} \rightarrow (d-k)\text{-simplex} \end{array} \right.$$

- ▶ Boundary triangles labeled by a color $c = 0, \dots, d$

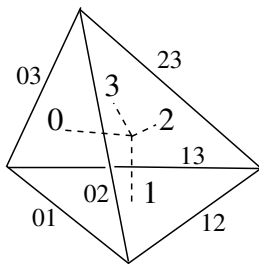


Colors identify all sub-simplices

Triangulations from colored graphs

duality	{	vertex	→	d -simplex
		edge	→	$(d - 1)$ -simplex
		face	→	$(d - 2)$ -simplex
		k -bubble	→	$(d - k)$ -simplex

- ▶ Boundary triangles labeled by a color $c = 0, \dots, d$
- ▶ Induced colorings
- ▶ Edges labeled by pair of colors

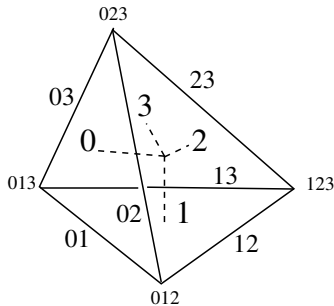


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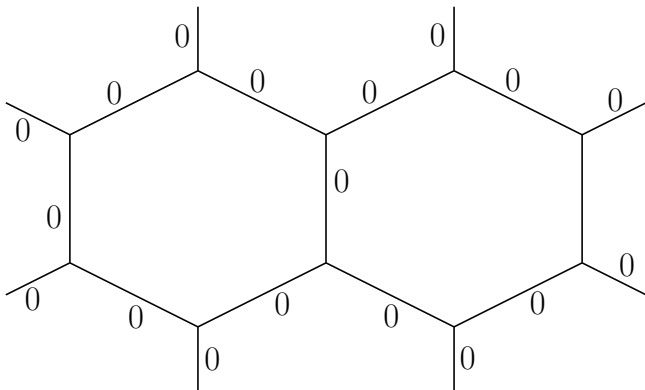
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- ▶ Boundary triangles labeled by a color $c = 0, \dots, d$
- ▶ Induced colorings
- ▶ Edges labeled by pair of colors
- ▶ Nodes labeled by three colors

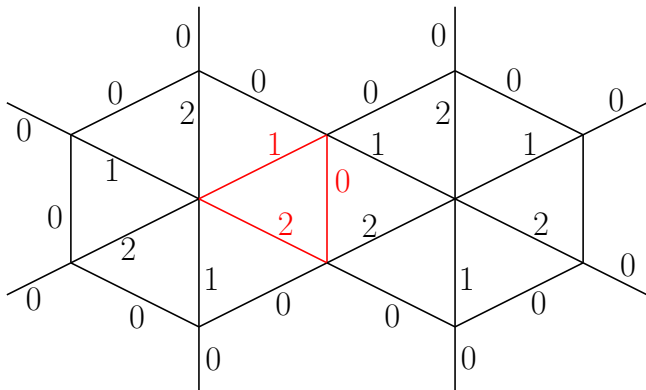


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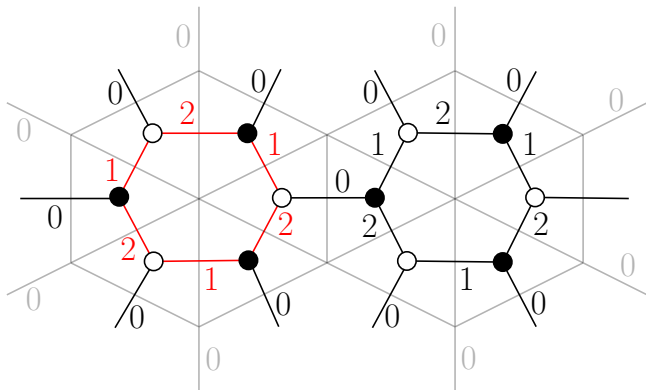
The 2D case



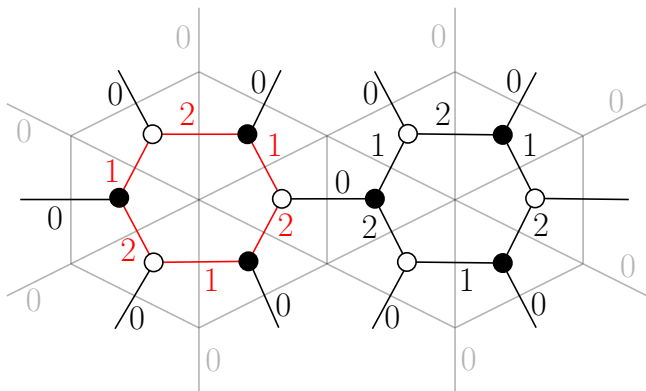
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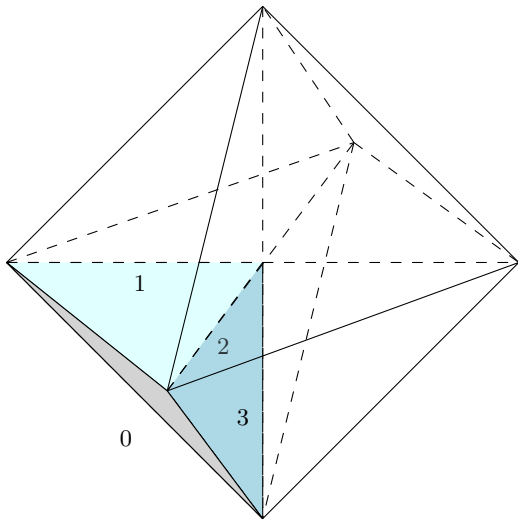
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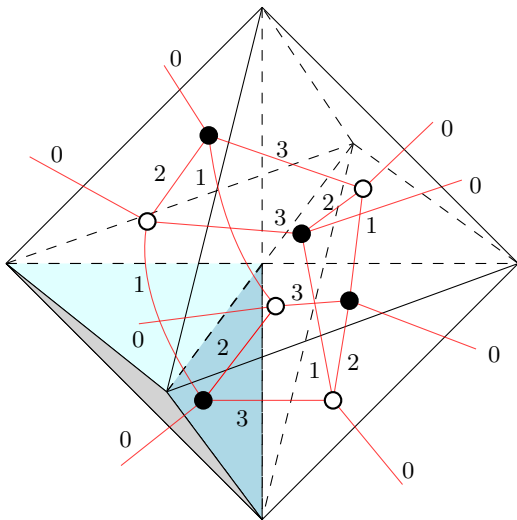
$2p$ -angle

- ▶ Gluing of $2p$ triangles with boundary of color 0
- ▶ Dually: Components with all colors but 0

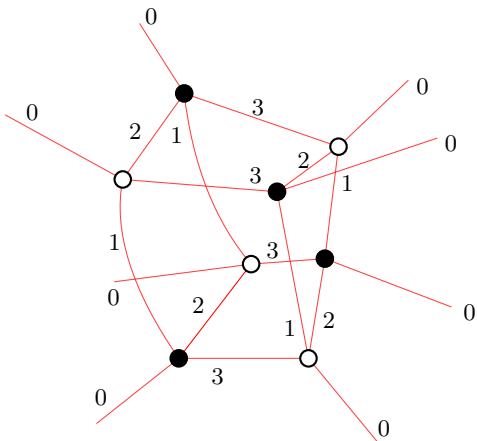
The 3D case



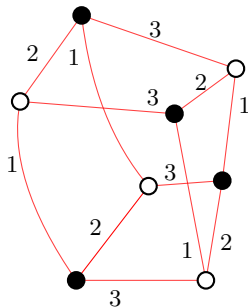
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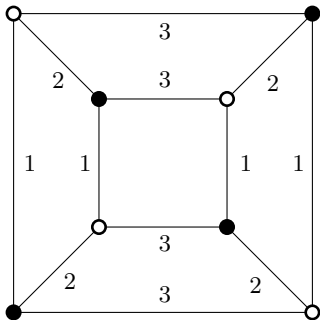
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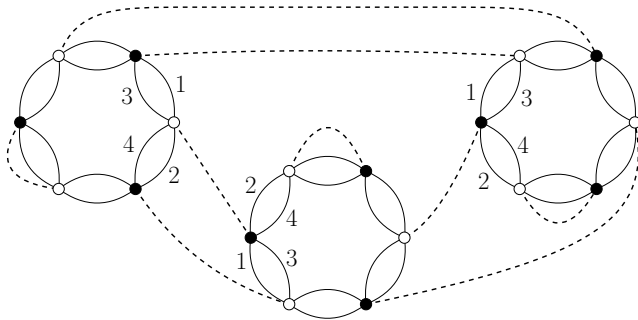


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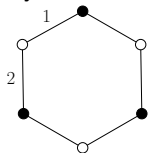
Bubbles

- ▶ Colored graph with colors $0, 1, \dots, d$ (triangulation in dim d)
- ▶ **Bubble**: connected piece with colors $1, \dots, d$
Obtained by removing the color 0
- ▶ All graphs obtained by gluing bubbles along edges of color 0
- ▶ $\mathcal{G}(B)$ set of $(d + 1)$ -colored graphs where all bubbles are B



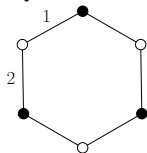
Bubbles II

- ▶ 2D: only bubbles with $2p$ vertices
Cycles of colors (1, 2)

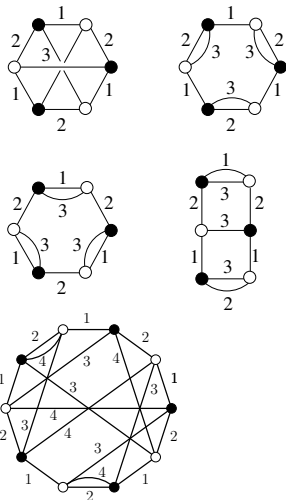


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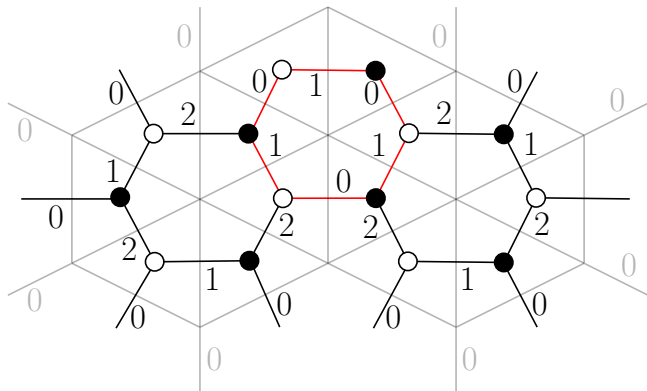
- ▶ 2D: only bubbles with $2p$ vertices
Cycles of colors (1, 2)



- ▶ Many more in higher dimensions
- ▶ Vast world to explore



Faces



Vertices: two types

- ▶ cycle with colors $(0, 1)$
- ▶ cycle with colors $(0, 2)$

The problem

- ▶ Set B a bubble, $G \in \mathcal{G}(B)$
- ▶ Enumerate w.r.t.
 - ▶ # bubbles $b(G)$
 - ▶ # subsimplices of codimension 2 which belong to bubble boundary
- ▶ Face of colors $(0, c)$: cycle with colors $(0, c)$

$$\text{Number of faces} \quad F(G) = \sum_{c=1}^d F_{0c}(G)$$

- ▶ Classify graphs according to $F(G)$ at fixed $b(G)$

$$\mathcal{G}_b(B) = \bigcup_F \mathcal{G}_b^{(F)}(B)$$

- ▶ Focus on $\mathcal{G}_b^{(F)}(B)$
How to maximize $F(G)$ at fixed number of bubbles $b(G)$?

Gurau's degree theorem

Bound on $F(G)$

There exists $\omega(G) \geq 0$

$$F(G) - (d - 1)(p(B) - 1)b(G) = d - \omega(G) \leq d$$

- ▶ $d = 2$
 $F(G) - (p(B) - 1)b(G) = 2 - \omega(G) \Rightarrow \omega(G) = 2g(G)$
- ▶ For $d \geq 3$, bound can be saturated only for certain type of bubbles
- ▶ Maximizing graphs (melonic) are series-parallel
 - ▶ Bijection with trees
 - ▶ Expected from numerics
- ▶ Need to investigate more generic bubbles

Colored triangulations and colored graphs

Bijection with (stuffed colored Walsh) maps

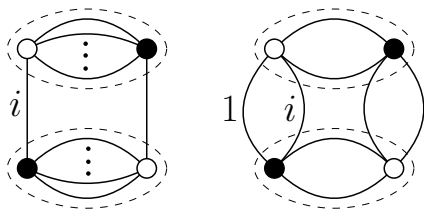
Quartic case

- ▶ How to control faces?

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- ▶ How to control faces? **Maps!**
- ▶ Same mechanism as Tutte's bijection between bipartite quadrangulations and generic maps, in the dual picture

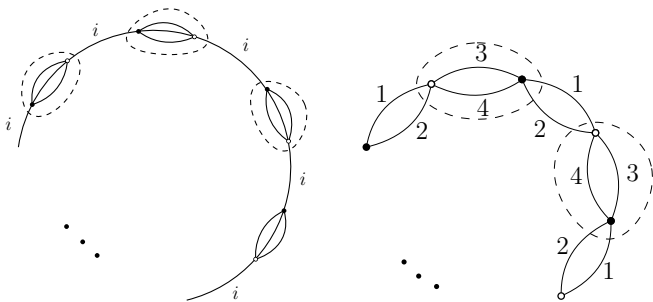
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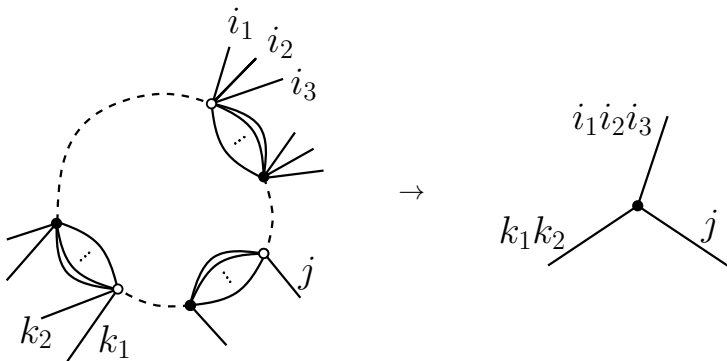


Universal part

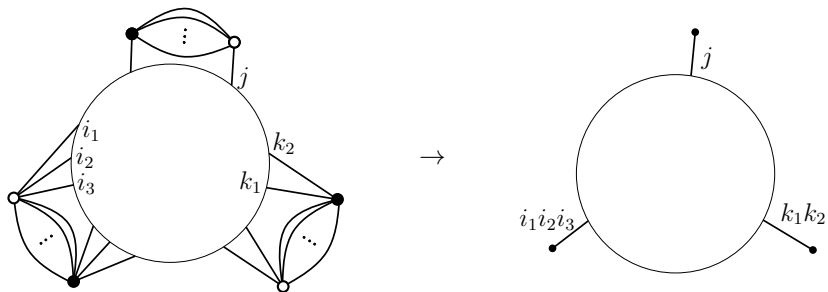
Decompose the graph into

- ▶ cycles of color 0 and pairs of vertices
- ▶ edges with colors in $\{1, \dots, d\}$ not contained in pairs

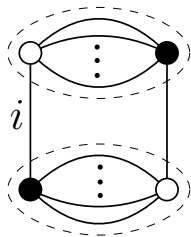
The universal part



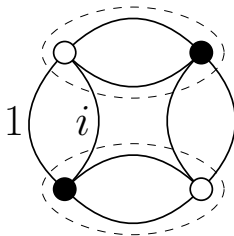
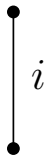
Non-universal part



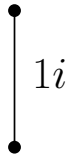
Non-universal part



\rightarrow



\rightarrow



The quartic case

- ▶ Generating function of (rooted) maps which maximize F at fixed b

$$f = \sum_M t^{\#\text{monocolored edges}} u^{\#\text{bicolored edges}}$$

- ▶ Monocolored edges are bridges
- ▶ Bicolored components are planar
- ▶ Edges of colors $1i$ and $1j$ meet on cut-vertices (similar to $O(n)$ model on planar maps)
- ▶ f is algebraic (explicit system)
- ▶ t dominates: square root singularity
- ▶ 1 type of bicolored edges and u dominates: exponent $3/2$ singularity
- ▶ (u_c, t_c) with exponent $2/3$ (proliferation of baby universes)
- ▶ For l types of bicolored edges
 - ▶ $l < 9/5$: same as planar maps
 - ▶ $l > 9/5$: back to square root singularity
 - ▶ $l = 9/5$: exponent $2/3$

Conclusion

- ▶ (At least some) Analytical results are achievable!
- ▶ More to be studied
- ▶ Harer–Zagier formula equivalent for unicellular maps?
- ▶ Topological recursion? Yes for quartic case!