ASEP with open boundaries and Koornwinder polynomials

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Journées de combinatoire de Bordeaux
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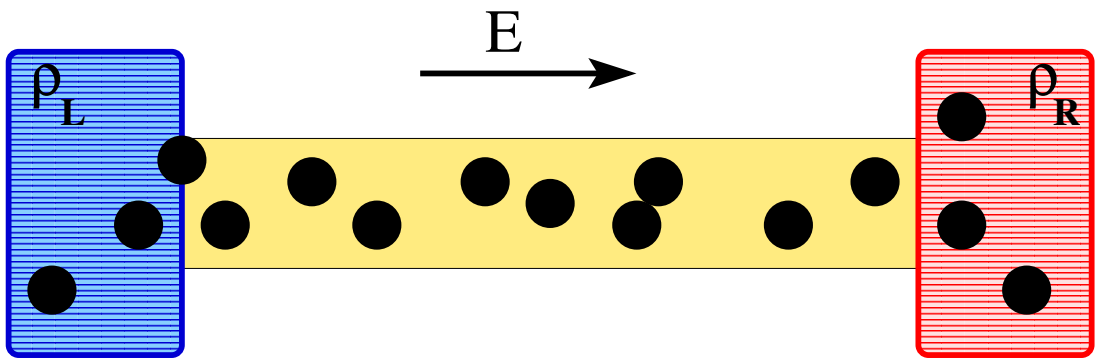
Introduction

Particles propagating under the effect of an external field and interacting with two reservoirs at different chemical potential ρ_L, ρ_R



No detailed balance: Macroscopic particle current

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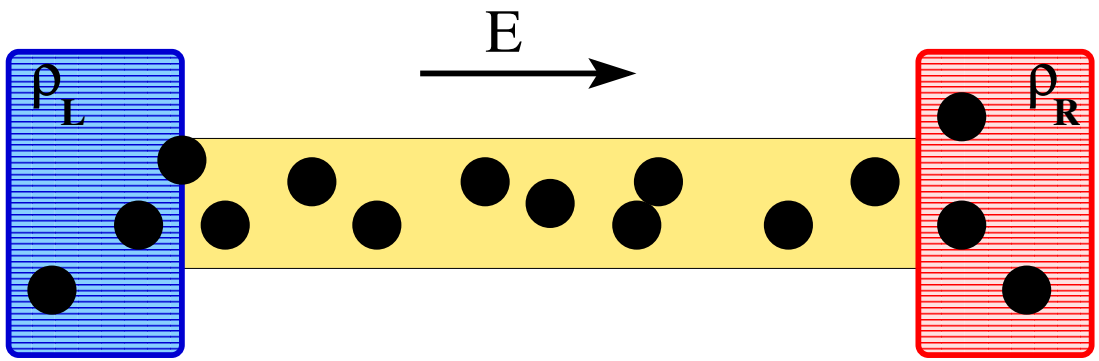

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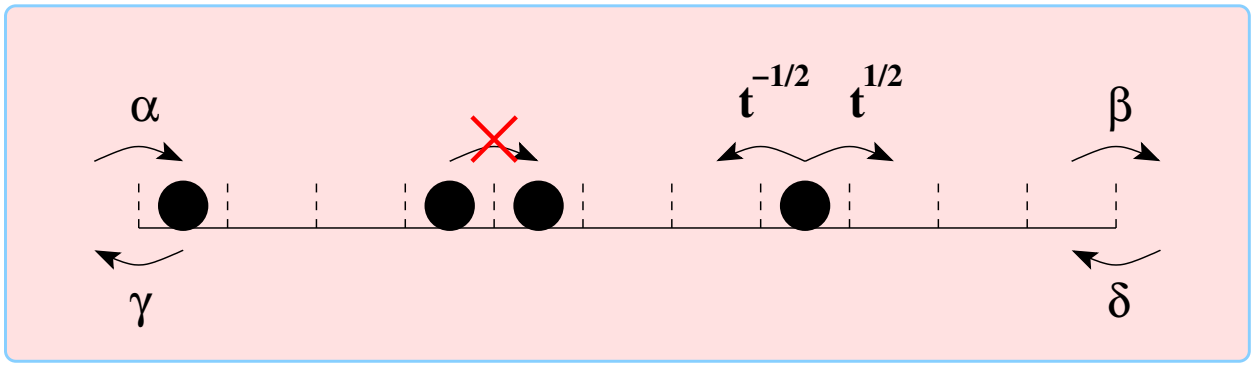
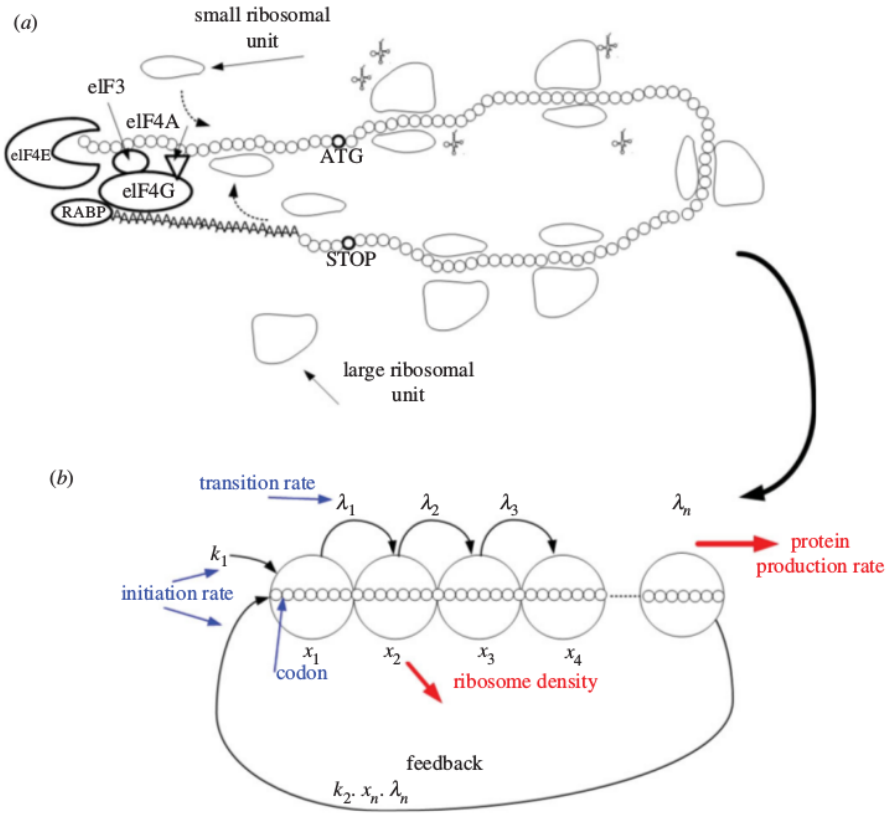
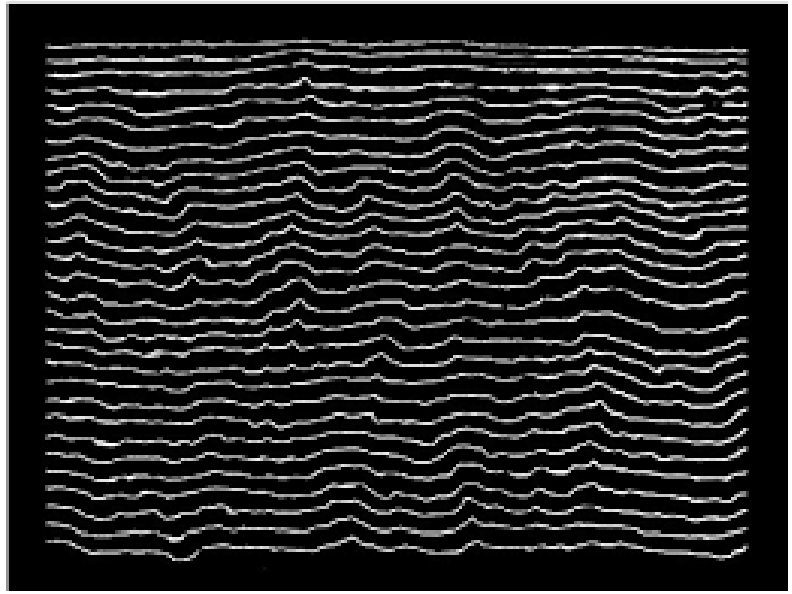
The Asymmetric Simple Exclusion Process (ASEP)

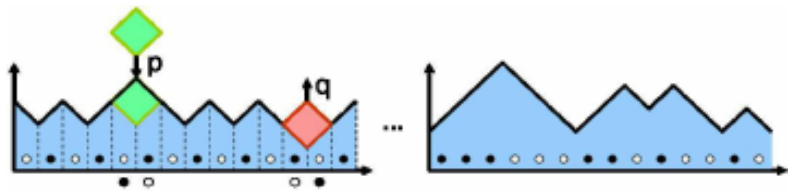


- One dimensional lattice
- Exclusion: at most one particle per site
- Asymmetric: jump rate to the right $t^{\frac{1}{2}}$, to the left $t^{-\frac{1}{2}}$
- Particles enter with rate α from left, with rate δ from right
- Particles leave with rate γ from left, with rate β from right

Applications

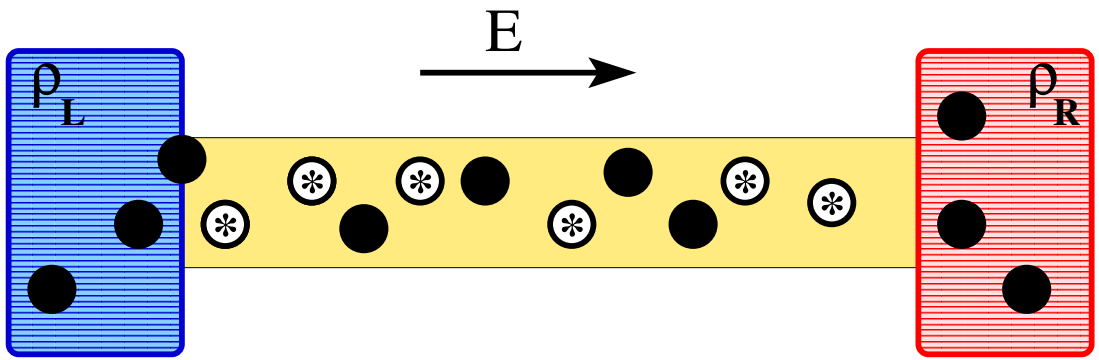




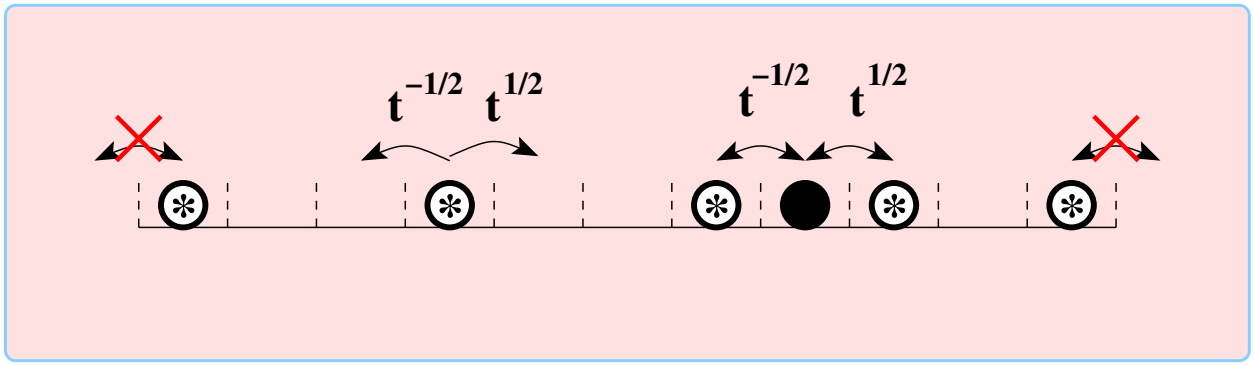


Impurities

Here we want to take into account the presence of impurities that are NOT EXCHANGED with the reservoirs

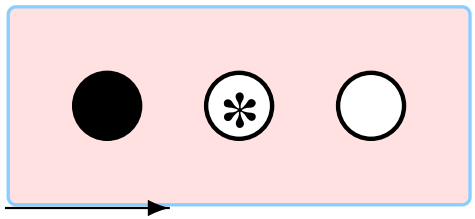


Two species ASEP



The impurity (*second-class particle*) has the same dynamics as a normal (*first-class particle*) , while the first-class particles treat it as a hole.

It is convenient to think at
first class, second class particles
and empty sites as
three kinds of particles



Master equation and Markov generator

Probabilities of the configurations $\{w\}$ evolve under a Master equation

$$\frac{d}{dt}P_w(t) = \sum_{w' \neq w} \mathcal{M}(w' \to w) P_{w'}(t) - \sum_{w' \neq w} \mathcal{M}(w \to w') P_w(t)$$

Using a vector representation for the probabilities

$$\mathcal{P}_{N,m}(t) = \sum_{w \in \mathcal{Q}(N,m)} P_w(t) w$$

Number of impurities

Length of the chain

The Master equation reads

$$\frac{d}{dt}\mathcal{P}(t) = \mathcal{M}\mathcal{P}(t)$$

Where the Markov generator is given by the sum of local terms

$$\mathcal{M} = \sum_{i=1}^{N-1} \mathbf{e}_i + \mathbf{f}_1 + \mathbf{f}_N.$$

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Stationary measure

In this talk I'll focus on the *Stationary measure*

$$\mathcal{MP}_{N,m} = 0$$

- Usually dealt with by the Matrix Product Ansatz [Derrida, Evans, Hakim, Pasquier]
- Rich combinatorics: rhombic alternative and rhombic staircase tableaux [Corteel, Williams, Mandelshtam, Viennot,...]
- Boundary induced phase transitions [Krug,...]
- Here I'll describe an algebraic approach based on:

Exchange/reflection equations.

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$\widehat{\mathcal{C}}_N$ Hecke Algebra

Define operators $T_0, T_1, \ldots, T_N$ as

$$T_0 = \alpha^{-\frac{1}{2}}\delta^{-\frac{1}{2}}\mathbf{f}_1 + \alpha^{\frac{1}{2}}\delta^{-\frac{1}{2}}\mathbf{1}$$

$$T_N = \beta^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}\mathbf{f}_N + \beta^{\frac{1}{2}}\gamma^{-\frac{1}{2}}\mathbf{1}$$

and for $1 \leq i \leq N-1$

$$T_i = \mathbf{e}_i + t^{-\frac{1}{2}}\mathbf{1}$$

They satisfy the commutation relations of the generators of the affine Hecke algebra $\hat{\mathcal{C}}_N$

$$\begin{aligned}
T_i - T_i^{-1} &= t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}} \\
T_i T_j &= T_j T_i \quad \text{if} \quad |i-j| > 1 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad \text{if} \quad i \neq 0, N-1 \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0 \\
T_N T_{N-1} T_N T_{N-1} &= T_{N-1} T_N T_{N-1} T_N
\end{aligned}$$

with $t_0^{\frac{1}{2}} = \alpha^{\frac{1}{2}}\delta^{-\frac{1}{2}}$, $t_N^{\frac{1}{2}} = \beta^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$ and $t_i = t$ for $1 \leq i \leq N-1$.

Integrability

The $\check{R}$ matrices

$$\check{R}_i(z) = \mathbf{1} + \frac{z-1}{t^{\frac{1}{2}}z - t^{-\frac{1}{2}}}\mathbf{e}_i.$$

The boundary scattering matrices

$$K_1(z|a,b) = \mathbf{1} + \frac{(z^2-1)}{(z-a)(z-b)}\delta^{-1}\mathbf{f}_1(\delta,\alpha),$$

$$K_N(z|c,d) = \mathbf{1} + \frac{(1-z^2)}{(cz-1)(dz-1)}\gamma^{-1}\mathbf{f}_N(\beta,\gamma).$$

Where we assume $t \neq 1$ and parametrize the boundary rates as

$$\alpha = \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})ab}{(a-1)(b-1)}, \qquad \delta = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{(a-1)(b-1)}$$

$$\beta = \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})cd}{(c-1)(d-1)}, \qquad \gamma = \frac{t^{-\frac{1}{2}} - t\frac{1}{2}}{(c-1)(d-1)}$$

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Stationary measure: exchange/reflection equations

We introduce a vector

$$\Psi_{N,m}(\mathbf{z}) = \sum_{w \in \mathcal{Q}(N,m)} \psi_w(\mathbf{z}) w$$

solution of the following exchange/reflection equations

$$\begin{aligned} \check{R}_i(z_i z_{i+1}^{-1}) \Psi_{N,m}(\mathbf{z}) &= \mathbf{s}_i \Psi_{N,m}(\mathbf{z}) \\ K_1(z_1) \Psi_{N,m}(\mathbf{z}) &= \mathbf{s}_0 \Psi_{N,m}(\mathbf{z}) \\ K_N(z_N) \Psi_{N,m}(\mathbf{z}) &= \mathbf{s}_N \Psi_{N,m}(\mathbf{z}). \end{aligned}$$

where

$$\begin{aligned} \mathbf{s}_i f(\dots, z_i, z_{i+1}, \dots) &= f(\dots, z_{i+1}, z_i, \dots) \\ \mathbf{s}_0 f(z_1, \dots) &= f(z_1^{-1}, \dots) \\ \mathbf{s}_N f(\dots, z_N) &= f(\dots, z_N^{-1}) \end{aligned}$$

Stationary measure: exchange/reflection equations

Consistency of the exchange/reflection equations is ensured by the Yang-Baxter equations

$$\check{R}_i(yz^{-1})\check{R}_{i+1}(xz^{-1})\check{R}_i(xy^{-1}) = \check{R}_{i+1}(xy^{-1})\check{R}_i(xz^{-1})\check{R}_{i+1}(yz^{-1})$$

$$\check{R}_1(xy^{-1})K_1(y)\check{R}_1(x^{-1}y^{-1}))K_1(x) = K_1(x)\check{R}_1(x^{-1}y^{-1})K_1(y)\check{R}_1(xy^{-1}),$$

$$\check{R}_{N-1}(xy^{-1})K_N(x)\check{R}_{N-1}(xy)K_N(y) = K_N(y)\check{R}_{N-1}(xy)K_N(x)\check{R}_{N-1}(xy^{-1}),$$

and the so called unitarity condition

$$\check{R}_i(z)\check{R}_i(z^{-1}) = \mathbf{1}.$$

$$K_1(x)K_1(x^{-1}) = \mathbf{1}, \qquad K_N(x)K_N(x^{-1}) = \mathbf{1}.$$

Stationary measure: exchange/reflection equations

Theorem

- The exchange equations have unique solution (up to multiplication by a function invariant under the action of $\mathbf{s}_0, \mathbf{s}_i, \mathbf{s}_N$)
- Under specialization $z_i = 1$ we get

$$\Psi_{N,m}(\mathbf{1}) \propto \mathcal{P}_{N,m}$$

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Exchange/reflection equations in components

$$\psi_{\ldots,\underbrace{\bullet,\bullet}_{i,i+1},\ldots}(\mathbf{z}) = \mathbf{s}_i \psi_{\ldots,\underbrace{\bullet,\bullet}_{i,i+1},\ldots}(\mathbf{z})$$

$$\psi_{*,\ldots}(\mathbf{z}) = \mathbf{s}_0 \psi_{*,\ldots}(\mathbf{z})$$

$$\psi_{\ldots,*}(\mathbf{z}) = \mathbf{s}_N \psi_{\ldots,*}(\mathbf{z})$$

$$\psi_{\ldots,\underbrace{\bullet,\bullet}_{i,i+1},\ldots}(\mathbf{z}) = t^{\frac{1}{2}} \widehat{T}_i \psi_{\ldots,\underbrace{\bullet,\bullet}_{i,i+1},\ldots}(\mathbf{z})$$

$$\psi_{\bullet,\ldots}(\mathbf{z}) = t_0^{\frac{1}{2}} \widehat{T}_0 \psi_{\circ,\ldots}(\mathbf{z})$$

$$\psi_{\ldots,\bullet}(\mathbf{z}) = t_N^{\frac{1}{2}} \widehat{T}_N \psi_{\ldots,\circ}(\mathbf{z})$$

Affine Hecke Again: Noumi representation

Noumi introduced a representation of $\widehat{\mathcal{C}}_N$ depending on 6 parameters a, b, c, d, t, q, acting on $\mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$

$$\widehat{T}_i = t^{\frac{1}{2}} - (t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1})\, \partial_i \tag{1}$$

$$\widehat{T}_0 = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}} \frac{(z_1 - a)(z_1 - b)}{z_1}\, \partial_0 \tag{2}$$

$$\widehat{T}_N = t_N^{\frac{1}{2}} - t_N^{-\frac{1}{2}} \frac{(cz_N - 1)(dz_N - 1)}{z_N} \partial_N, \tag{3}$$

where $t_0 = -q^{-1}ab$ $t_N = -cd$ and

$$\partial_i = \frac{1 - \mathbf{s}_i}{z_i - z_{i+1}}, \quad \partial_0 = \frac{1 - \mathbf{s}_0}{z_1 - q z_1^{-1}}, \quad \partial_N = \frac{1 - \mathbf{s}_N}{z_N - z_N^{-1}}.$$

Where $\mathbf{s}_i, \mathbf{s}_N$ are as before but

$$\mathbf{s}_0 f(z_1, \dots) = f(q z_1^{-1}, \dots)$$

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Non-symmetric Koornwinder [Sahi, Noumi, . . .]

The commutative sub-algebra $\mathcal{Y}_N$ generated by elements $Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}$ [Lusztig]

$$Y_i = (T_i \ldots T_{N-1})(T_N \ldots T_0)(T_1^{-1} \ldots T_{i-1}^{-1}).$$

Its common eigenfunctions are the *non-symmetric Koornwinder polynomials* $E_\alpha(\mathbf{z})$ $(\alpha \in \mathbb{Z}^N)$

$$E_\alpha(\mathbf{z}) = \mathbf{z}^\alpha + \sum_{\mathbf{z}^\beta \prec \mathbf{z}^\alpha} c_\beta \mathbf{z}^\beta,$$

$$\widehat{Y}_i E_\alpha(\mathbf{z}) = \omega_i(\alpha) E_\alpha(\mathbf{z}).$$

By using the exchange/reflection relations it is easy to show that

$$\psi_{\underbrace{\circ \cdots \circ}_{N-m}\underbrace{* \cdots *}_{m}}(\mathbf{z}) = E_{\underbrace{-1 \cdots -1}_{N-m}\underbrace{0 \cdots 0}_{m}}(\mathbf{z})$$

(Actually $E_{\underbrace{-1 \cdots -1}_{N-m}\underbrace{0 \cdots 0}_{m}}(\mathbf{z})$ doesn't depend on q).

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Symmetric Macdonald-Koornwinder polynomials [Koornwinder]

Koornwinder q-difference operator

$$\mathcal{D}_{q,t} = \sum_{i=1}^{N} \Phi_i(z_i)(T_{q,z_i} - 1) + \Phi_i(z_i^{-1})(T_{q,z_i}^{-1} - 1)$$

where T_{q,z_i} is the i-th q-shif operator

$$T_{q,z_i} f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, qz_i, \ldots, z_N)$$

and

$$\Phi_i(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{(1-tzz_j)(1-tzz_j^{-1})}{(1-zz_j)(1-zz_j^{-1})}$$

Symmetric Macdonald-Koornwinder polynomials

The symmetric Macdonald-Koornwinder polynomials $P_\lambda(\mathbf{z})$

- Laurent polynomials in N variables
- Labeled by a partition λ, coefficient of $\mathbf{z}^\lambda$ in $P_\lambda(\mathbf{z})$ is 1
- Eigenfunctions of $\mathcal{D}_{q,t}$

$$\mathcal{D}_{q,t} P_\lambda(\mathbf{z}) = d_\lambda P_\lambda(\mathbf{z})$$

$$d_\lambda = \sum_{i=1}^{N} \left[q^{-1} abcdt^{2n-i-1}(q^{\lambda_i} - 1) + t^{i-1}(q^{-\lambda_i} - 1) \right]$$

- They are multivariables generalization of the Askey-Wilson polynomials

$$P_{\{m\}}(\mathbf{z}) \propto p_m(x; a, b, c, d|q).$$

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Symmetric Macdonald-Koornwinder polynomials

- **Orthogonality**

$$\langle P_\lambda, P_\mu \rangle = 0 \qquad \text{if} \qquad \lambda \neq \mu.$$

$$\langle p, q \rangle := \oint_{\mathcal{C}} \prod_{j=1}^{N} \frac{dz_j}{4\pi i z_j} \Delta_N(t, q; \mathbf{z}) p(\mathbf{z}) q(\mathbf{z}^{-1})$$

where the contour of integration $\mathcal{C}$ encircles the poles in aq^k, bq^k, cq^k, dq^k $(k \in \mathbb{Z}_+)$ and excludes all the others.

$$\Delta_N(t, q; \mathbf{z}) = \prod_{\substack{1 \leq i < j \leq N \\ \epsilon_1, \epsilon_2 = \pm 1}} \frac{(z_i^{\epsilon_1} z_j^{\epsilon_2}; q)_\infty}{(t z_i^{\epsilon_1} z_j^{\epsilon_2}; q)_\infty} \prod_{i=1}^{N} \frac{(z_i^{2\epsilon}; q)_\infty}{(a z_i^{\epsilon}, b z_i^{\epsilon}, c z_i^{\epsilon}, d z_i^{\epsilon}; q)_\infty},$$

Weighted partition function

Let $\bullet(w)$ be the number of first class particles in configuration w.
We define the *weighted partition function* as

$$Z_N(\xi; \mathbf{z}; a, b, c, d) := \sum_{w \in \mathcal{Q}(N,m)} \xi^{2\bullet(w)} \psi_w(\mathbf{z}).$$

Theorem

The generating function for the number of particles in the system is given by

$$Z_{N,m}(\xi; \mathbf{z}; a, b, c, d) = \xi^{N-m} P_{1^{N-m}0^m}(\xi\mathbf{z}|\xi\mathbf{z}; a_\xi, b_\xi, c_\xi, d_\xi)$$

with $a_\xi = \xi a, b_\xi = \xi b, c_\xi = \xi^{-1} c, d_\xi = \xi^{-1} d$.

Remark: the Macdonald-Koornwinder polynomials $P_{1^{N-m}0^m}(\mathbf{z})$ associated to single column diagrams $\lambda = 1^{N-m}0^m$ are independent of the parameter q.

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Current and density

We define an inhomogenous version of the current of particles

$$J_{N,m}(\mathbf{z}) := \frac{1}{Z_{N,m}(\xi = 1; \mathbf{z})} \sum_{w \in \mathcal{Q}(N,m)} \alpha \psi_{\circ w}(\mathbf{z}) - \gamma \psi_{\bullet w}(\mathbf{z}),$$

which under specialization $\mathbf{z} = \mathbf{1}$ reduces to the homogeneous case,

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Using the recursion relations we get

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- After specialization $\mathbf{z} = \mathbf{1}$, we obtain

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$$\langle \rho^{\bullet}_{N,m} \rangle = \left(\frac{1}{2N} \frac{\partial}{\partial \xi} \log Z_{N,m}(\xi; \mathbf{1}) \right) \Big|_{\xi=0},$$

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Integral representation

Using the Cauchy identity for Macdonald-Koornwinder polynomials [Mimachi] and assuming $t < 1$, we can write

$$Z_{N,m}(\xi;\mathbf{z}) = r_m^{-1}(a_\xi, b_\xi, c_\xi, d_\xi|t)\times$$
$$\xi^{N-m}\oint_{\mathcal{C}} \frac{dx}{4\pi i x}\Pi(\xi\mathbf{z}, x)w(x; a_\xi, b_\xi, c_\xi, d_\xi|t)p_m(x; a_\xi, b_\xi, c_\xi, d_\xi|t)$$

$p_m(x; a, b, c, d|t)$ is the m-th Askey-Wilson polynomial of base t in the variable $\frac{x+x^{-1}}{2}$,

$$w(x; a, b, c, d|t) = \frac{(x^2, x^{-2}; t)_\infty}{(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}, dx, dx^{-1}; t)_\infty},$$

$$\Pi(\mathbf{z}, \mathbf{x}) = \prod_{1\le i\le N}(z_i + z_i^{-1} - x - x^{-1})$$

$$r_m(a, b, c, d|t) = \frac{(abcdt^{2m}; t)_\infty}{(t^{m+1}, abt^m, act^m, adt^m, bct^m, bdt^m, cdt^m; t)_\infty}.$$

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Remarks

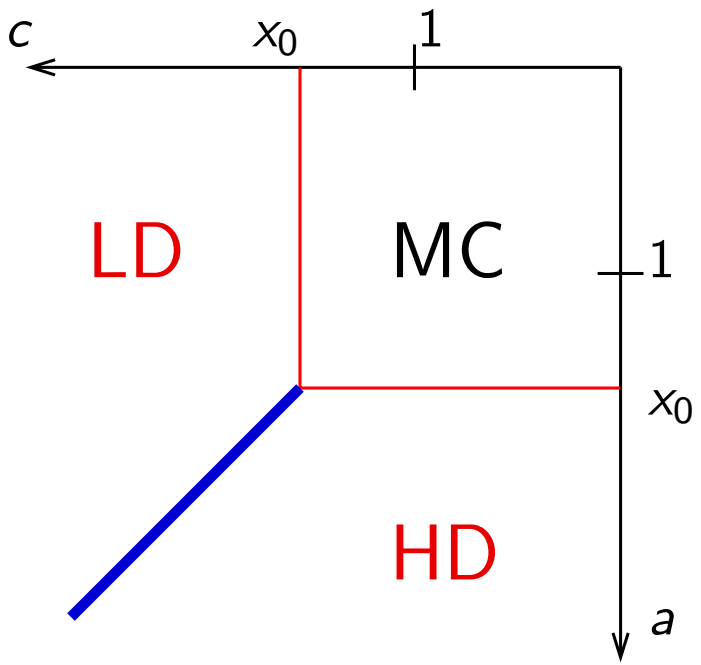
- This formula at $\mathbf{z} = \mathbf{1}$ generalizes the result of Uchiyama, Sasamoto and Wadati obtained for $m = 0$.
- At $\mathbf{z} = \mathbf{1}$ improves a much more complicated formula obtained by Uchiyama.
- Comparison with results of Corteel, Mandelshtam and Williams: combinatorial representations of P_w in terms of Rhombic Staircase Tableaux?

Phase diagram

$$x_0 < a, c: \quad J = \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})(1-\rho_*^2)}{4}, \quad \rho^\bullet = \frac{1-\rho_*}{2}$$

$$a < x_0, c: \quad J = \frac{a(t^{-\frac{1}{2}} - t^{\frac{1}{2}})}{(1-a)^2}, \quad \rho^\bullet = \frac{a}{a-1} - \rho_* > \frac{1-\rho_*}{2}$$

$$c < x_0, a: \quad J = \frac{c(t^{-\frac{1}{2}} - t^{\frac{1}{2}})}{(1-c)^2}, \quad \rho^\bullet = \frac{1}{1-c} < \frac{1-\rho_*}{2}$$



Conclusion

- The approach to the study of the stationary measure through exchange relations can be applied to any Yang-Baxter integrable stochastic process.
- Relations with multivariate orthogonal polynomials.
- Work backward: find Matrix Product representations of Macdonald-Koornwinder polynomials.
- Work with several species of particles $\longrightarrow$ general Koornwinder polynomials at $q = 1$.

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